MATH 173, FALL2022, HOMEWORK #3

ALEX IOSEVICH

1. Problems not in the book

Problem #1: Prove that if $\{a_j\}$, $\{b_j\}$ are sequences of real numbers (finite or infinite), $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left|\sum_{j=1}^{\infty} a_j b_j\right| \leq \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^{\infty} |b_j|^q\right)^{\frac{1}{q}}$$

by following the outlined given below.

Step 1: Prove that if $0 < x < y$ and $0 \leq t \leq 1$, then

$$e^{(1-t)x+ty} \leq (1-t)e^x + te^y.$$

Hint: Let $F(t) = (1-t)e^x + te^y - e^{(1-t)x+ty}$. Note that $F(0) = F(1) = 0$ and $F''(t) \leq 0$. Now use the Mean Value Theorem and obtain the desired conclusion.

Step 2: Prove (by using Step 1) that if x, y are positive real numbers and p, q are as above, then

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Step 3: Let $A = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}}$, $B = \left(\sum_{j=1}^{\infty} |b_j|^q\right)^{\frac{1}{q}}$, and rewrite the inequality you are trying to prove in the form

$$\left|\sum_{j=1}^{\infty} \frac{a_j}{A} \cdot \frac{b_j}{B}\right| \leq 1.$$

Hint: Prove this by applying Step 2 inside the summation on the left hand side.

2. Problems from the book

Section 2.1, problems 3, 4, 6

Section 2.2, problems 1, 2, 3, 5, 7, 8, 9