THE GEOMETRY OF PLANAR FOURIER EXPANSIONS

L. BRANDOLINI, L. COLZANI, A. IOSEVICH, AND G. TRAVAGLINI

ABSTRACT. Let A be an appropriate planar domain and let f be a piecewise smooth function on $\mathbb{R}^2$. We discuss the rate of convergence of

$$S_\lambda f(x) = \int_{\lambda A} \widehat{f}(\xi) \exp(2\pi i \xi \cdot x) d\xi$$

in terms of the interaction between the geometry of A and the geometry of the singularities of f. The most subtle case is when x belongs to the singular set of f and here Hilbert transform techniques play an important role.

The pointwise convergence of one-dimensional Fourier series of piecewise smooth functions is one of the best known topics in analysis. In higher dimensions the problem is harder and also of different nature. This is basically due to the failure of the Riemann localization principle, since the convergence at a given point does not depend exclusively on the regularity of the function in a neighborhood of the point. See [15] and [14, VII.4]. The purpose of this paper is to characterize the convergence properties of two-dimensional Fourier integrals of piecewise smooth functions in terms of certain natural geometric features. On this topic the classical reference is [1], but for more recent works see [2], [3], [5], [6], [8],[9], [10], [11], [12].

Define the Fourier transform and the spherical sums of Fourier integral of integrable functions as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(y) \exp(-2\pi i \xi \cdot y) dy,$$

$$S_\lambda f(x) = \int_{\{|\xi| < \lambda\}} \widehat{f}(\xi) \exp(2\pi i x \cdot \xi) d\xi.$$

In analogy with the one dimensional case one expects that the spherical sums of piecewise smooth functions converge everywhere to the

function expanded, but indeed the situation is slightly more compli-
cated. The simplest example to illustrate this fact is the character-
istic function of the n-dimensional unit ball centered at the origin
$B = B(0,1)$. Its Fourier transform can be expressed in terms of a
Bessel function, $\widehat{\chi}_B(\xi) = |\xi|^{-n/2} J_{n/2}(2\pi |\xi|)$, and it is possible to show
that the spherical sums $S_\lambda \chi_B(x)$ converge at every point $x \neq 0$ when
$\lambda \to +\infty$. This follows for example by an equiconvergence result
between Fourier-Bessel expansions and the classical one dimensional
trigonometric expansions. See [4]. On the other hand, the spherical
sums at the origin $x = 0$,

$$S_\lambda \chi_B(0) = |\{|\xi| = 1\}| \int_0^\lambda J_{n/2}(2\pi r) \, r^{n/2-1} dr$$

converge for $n = 2$, oscillate for $n = 3$, and are unbounded for $n > 3$.
More precisely, in the planar case $n = 2$, one can prove that when
$\lambda \to +\infty$,

$$S_\lambda \chi_B(x) = \begin{cases} 1 + \mathcal{O}\left(\lambda^{-1/2}\right) & \text{when } x = 0, \\[2mm] \chi_B(x) + \mathcal{O}\left(\lambda^{-1}\right) & \text{when } x \neq 0 \text{ and } |x| \neq 1, \\[2mm] 1/2 + \mathcal{O}\left(\lambda^{-1}\right) & \text{when } |x| = 1. \end{cases}$$

Observe that the worst rate of convergence $\lambda^{-1/2}$ takes place at the
origin, while at all other points the rate is λ^{-1}. In [11] there is a
more sophisticated example: The spherical sums of the characteristic
function of an ellipse converge with speed λ^{-1}, except that at points of
the evolute, an astroid. At the four vertexes of the astroid the speed of
convergence is $\lambda^{-3/4}$ and at the other points is $\lambda^{-5/6}$. What is suggested
by these examples is that the rate of convergence of $S_\lambda f(x)$ may not
be optimal if the geometry of the disc used to define the summation
operators interacts badly with the geometry of the singularities of the
function expanded, and this leads one to ask whether the situation
can be ameliorated by replacing the disc with a suitable domain, or
if the same phenomena occur. In particular, in this paper we define
summation methods more general than spherical sums and we apply
these summation methods to piecewise smooth functions.

Given a bounded planar domain A containing the origin, define the two dimensional partial sums of Fourier integral of integrable functions as

$$S_\lambda f(x) = \int_{\lambda A} \widehat{f}(\xi) \exp(2\pi i x \cdot \xi) d\xi.$$

Also define piecewise smooth functions as finite sums

$$f(x) = \sum_j g_j(x) \chi_{B_j}(x),$$

where the $g_j(x)$'s are smooth functions and the B_j's are bounded domains with smooth boundaries. In what follows it will suffice to consider each piece of these sums separately.

Our goal is to prove that under reasonable assumptions on A the partial sums $S_\lambda f(x)$ converge to $f(x)$ and also to estimate the speed of convergence. The most studied case is of course the spherical summability, that is when A is a disc, since in this case the kernel $K(y) = \widehat{\chi}_A(-y)$ associated to the operators S_λ is known explicitly. The case of a strictly convex body is similar in principle, because there are rather precise asymptotic estimates on $K(y)$. On the contrary, our analysis does not rely on explicit or asymptotic expressions of the kernel and we also avoid the convexity assumption. We only require that the domain A is bounded with smooth boundary and is strictly star-shaped with respect to the origin. More precisely, if $\xi(t)$ is a smooth parametrization of ∂A, we assume that the two vectors $\xi(t)$ and $\dot{\xi}(t)$ are always linearly independent. This assumption is quite natural because and we shall see that otherwise the convergence may fail.

The following are our main results.

THEOREM 1: *Let A be a bounded planar domain with smooth boundary and strictly star-shaped with respect to the origin. Also let $f(x) = g(x)\chi_B(x)$, with $g(x)$ smooth function and B bounded domains with smooth boundary. Then for some constants $c = c(x, A, B, g)$ and $\ell = \ell(x, A, B)$ we have*

$$
\begin{cases}
|S_\lambda f(x)| \le c\,\lambda^{-1/2} & if\ x \notin \overline{B}, \\[2ex]
|S_\lambda f(x) - g(x)| \le c\,\lambda^{-1/2} & if\ x \in B, \\[2ex]
|S_\lambda f(x) - \ell\,g(x)| \le c\,\lambda^{-1/2} & if\ x \in \partial B.
\end{cases}
$$

As we shall see the most subtle analysis occurs when x is on ∂B, since this leads us to consider singular integrals. When A is a disc, it is well known that $\ell(x, A, B) = 1/2$. In the general case this constant can be interpreted as the integral of the kernel $K(y) = \widehat{\chi}_A(-y)$ over a suitable half-plane. Indeed as $\lambda \to +\infty$ we obtain

$$
S_\lambda f(x) = \int_{\mathbb{R}^2} K(y) g(x - \lambda^{-1}y) \chi_B(x - \lambda^{-1}y) dy
$$

$$
\to g(x) \int_{H(x,B)} K(y) dy,
$$

where $H(x, B) = \lim_{\lambda \to +\infty} \lambda(x - B)$. Observe that the kernel $K(y)$ cannot be absolutely integrable since its Fourier transform is not continuous, hence the passage to the limit has be justified. Anyhow, when A is symmetric with respect to the origin also $K(y)$ is symmetric, and since the integral over the whole plane of this kernel equals $\chi_A(0) = 1$, one expects that the integral over a half plane through the origin is $1/2$.

The example of spherical sums of balls shows that the speed of convergence $\lambda^{-1/2}$ can be attained also at points where the function is smooth, and this suggests a lack of localization. However the phenomenon is in some sense exceptional, as the following result shows.

THEOREM 2: *There exists a set $E \subset \mathbb{R}^2$ of Hausdorff dimension at most one, such that if x is not in E then*

$$
|S_\lambda f(x) - f(x)| \le c\,\lambda^{-1}.
$$

As we said, the spherical sums of the characteristic function of an ellipse converge with speed λ^{-1}, except that at points of the evolute. Hence the optimal decay λ^{-1} may fail in a set of Hausdorff dimension one. On the other hand the worst possible decay $\lambda^{-1/2}$ in Theorem 1 is due to a perfect focusing of singularities and then it is quite exceptional.

In particular, for analytic curves the decay $\lambda^{-1/2}$ of the spherical means holds only if ∂B is a circle and x is its center. See [6].

The proofs of Theorem 1 and Theorem 2 are based on the study of certain oscillatory integrals. Since

$$S_\lambda f(x) = \lambda^2 \int_A \int_B g(y) \exp\left(2\pi i \lambda \xi \cdot (x - y)\right) d\xi dy,$$

a double application of the divergence theorem reduces $S_\lambda f(x) - f(x)$ essentially to an integral over the boundary,

$$I(\lambda, x) = \int_{\partial A} \int_{\partial B} \Psi(\xi, y) \exp\left(2\pi i \lambda \xi \cdot (x - y)\right) dy d\xi.$$

It is well known that the two integrals over ∂A and ∂B when taken separately are governed by curvature. However, the actual properties of the double integral are more subtle, do not depend only on ∂A and ∂B, but also on their interaction. Without any assumptions the above integral does not necessarily vanish as $\lambda \to +\infty$. In particular, if A is not star-shaped and ∂A contains a segment of the line $\{ta\}$, while $x - \partial B$ contains an orthogonal segment in $\{sb\}$, where t and s run over $\mathbb{R}$ and a and b are orthogonal vectors in $\mathbb{R}^2$, then the set of (ξ, y) on $\partial A \times \partial B$ where $\xi \cdot (x - y) = 0$ has positive measure and $I(\lambda, x)$ may have no decay in λ. On the other hand two non orthogonal segments give the decay $I(\lambda, x) \approx \lambda^{-1}$ and, more generally, this optimal decay takes place for most choices of ∂A and ∂B.

In suitable coordinates $I(\lambda, x)$ can be reduced to an oscillatory integral in $\mathbb{R}^2$ and one can apply the classical principle of the stationary phase, see [13, VIII.2]: If $\Phi(z)$ and $\Psi(z)$ are smooth functions on $\mathbb{R}^n$, if $\Psi(z)$ has compact support and if $|\partial^\alpha/\partial z^\alpha \Phi(z)| > \varepsilon > 0$ in the support of $\Psi(z)$, then

$$\left| \int_{\mathbb{R}^n} \Psi(z) \exp\left(2\pi i \lambda \Phi(z)\right) dz \right| \leq c\lambda^{-1/|\alpha|}.$$

Moreover, if $\Phi(z)$ has only non-degenerate critical points, that is the Hessian $\left[\partial^2/\partial z_j^2 \Phi(z)\right]$ is invertible at every point where the gradient $\left[\partial/\partial z_j \Phi(z)\right]$ vanishes, then

$$\left| \int_{\mathbb{R}^n} \Psi(z) \exp\left(2\pi i\lambda \Phi(z)\right) dz \right| \le c\lambda^{-n/2}.$$

In order to apply these results we need to study the phase function $\xi \cdot (x - y)$ on $\partial A \times \partial B$. In particular, if all critical points of the phase are non-degenerate the decay is at least λ^{-1}. When there are degenerate critical points the decay can be worse, however we shall see that at degenerate critical points the curvatures of ∂A and ∂B do not vanish, so that some second order derivatives of the phase are different from zero and this gives at least the decay $\lambda^{-1/2}$. We shall also see that for most points x the phase has only non degenerate critical points, hence while the decay λ^{-1} is generic, lower orders of decay are in a sense exceptional.

Finally, it is possible to give a geometric description of the set of points at which the convergence of $S_\lambda f(x)$ is slow. Let $\xi(t)$ be a parametrization of ∂A with $\xi(t)$ and $\dot{\xi}(t)$ linearly independent. The polar curve ∂A^* is the curve $z(t)$ defined by the system of linear equations

$$\begin{cases} \xi(t) \cdot z(t) = 1, \\ \dot{\xi}(t) \cdot z(t) = 0. \end{cases}$$

Observe that, being $\xi(t)$ and $\dot{\xi}(t)$ linearly independent, the system has exactly one smooth solution. Also observe that when A is strictly convex, this polar curve ∂A^* is the boundary of the polar set $A^* = \{z : \forall \xi \in A, \ z \cdot \xi < 1\}$.

The decay of $|S_\lambda f(x) - f(x)|$ is determined by the nature of critical points of the phase function $\xi(t) \cdot (y(s) - x)$ and our last result relates this decay to the interaction between x, ∂B, and ∂A^*.

THEOREM 3: *Let A be a smooth star-shaped domain containing the origin and let $\xi(t)$ and $z(t)$ be parametrizations of ∂A and ∂A^*. Finally let $y(s)$ be a parametrization of the boundary of a smooth domain ∂B. Then the following hold:*

a) The phase $\Phi(t, s, x) = \xi(t) \cdot (y(s) - x)$ has a critical point at (t, s) when $\partial B - x$ is tangent to a dilate of ∂A^*,

$$\begin{cases} \mu z(t) = y(s) - x, \\[2mm] \dot{z}(t) = \nu \dot{y}(s). \end{cases}$$

b) *At this critical point the determinant of the Hessian matrix vanishes if and only if the two curves ∂B and $x + \mu \partial A^*$ have the same curvature,*

$$\ddot{y}(s) = \mu^{-1} \left(\dot{z}(t) \cdot \dot{z}(t) \right)^{-2} \ddot{z}(t) - \mu^{-1} \left(\dot{z}(t) \cdot \dot{z}(t) \right)^{-4} \left(\dot{z}(t) \cdot \ddot{z}(t) \right) \dot{z}(t).$$

It turns out that the exact order of decay of $|S_\lambda f(x) - f(x)|$ is related to the order of contact between $x + \mu \partial A^*$ and ∂B. When A is the disc $\{ |\xi| < 1 \}$, then $\partial A^* = \partial A$ and the condition that $x + \mu \partial A^*$ and ∂B have order of contact higher than two defines the evolute of B. We thus recover the classical results on spherical means.

Let us now present the proof of our results.

Proofs of Theorems 1 and 2: Let $m(\xi)$ be a smooth radial function, with support in a small disc $B(0, 2\varepsilon)$ and equal to 1 in $B(0, \varepsilon)$. Decompose $S_\lambda f(x)$ into

$$S_\lambda f(x) = \int_{\mathbb{R}^2} m(\lambda^{-1}\xi) \widehat{f}(\xi) \exp(2\pi i x \cdot \xi) d\xi$$

$$+ \int_{\mathbb{R}^2} \left(\chi_A(\lambda^{-1}\xi) - m(\lambda^{-1}\xi) \right) \widehat{f}(\xi) \exp(2\pi i x \cdot \xi) d\xi$$

$$= M_\lambda f(x) + R_\lambda f(x).$$

The operators M_λ are smoothed analogues of the S_λ, and it is easy to show that they have the required summability properties.

LEMMA 1: *Let $m(\xi)$ be a smooth radial function with compact support and with $m(0) = 1$, and define*

$$M_\lambda f(x) = \int_{\mathbb{R}^2} m(\lambda^{-1}\xi) \widehat{f}(\xi) \exp(2\pi i x \cdot \xi) d\xi.$$

Let also $f(x) = g(x)\chi_B(x)$ be piecewise smooth. Then if $x \notin \partial B$,

$$|M_\lambda f(x) - f(x)| \le c\lambda^{-2},$$

and if x *is in* ∂B,

$$\left| M_\lambda f(x) - \frac{1}{2}g(x) \right| \le c\lambda^{-1}.$$

Proof: Let

$$M(y) = \int_{\mathbb{R}^2} m(\xi) \exp(2\pi i y \cdot \xi) d\xi.$$

This kernel is radial, rapidly decreasing, with integral one. Writing $|y| \, M(y) = H(|y|)$ and integrating in polar coordinates one obtains

$$\int_{\mathbb{R}^2} m(\lambda^{-1}\xi)\widehat{f}(\xi) \exp(2\pi i x \cdot \xi) d\xi - f(x)$$

$$= \int_{\mathbb{R}^2} M(y) \left(f(x - \lambda^{-1}y) - f(x) \right) dx$$

$$= \int_0^{+\infty} H(t) \int_{\{|\sigma|=1\}} \left(f(x - \lambda^{-1}t\sigma) - f(x) \right) d\sigma dt.$$

Since at every point x not in ∂B,

$$\left| \int_{\{|\sigma|=1\}} \left(f\left(x - \lambda^{-1}t\sigma\right) - f(x) \right) d\sigma \right| \le c\lambda^{-2}t^2,$$

we have

$$\int_0^{+\infty} |H(t)| \left| \int_{\{|\sigma|=1\}} \left(f(x - \lambda^{-1}t\sigma) - f(x) \right) d\sigma \right| dt \le c\lambda^{-2} \int_0^{+\infty} t^2 |H(t)| \, dt.$$

When x is in ∂B the proof is the same. It is enough to observe that

$$\left| \int_{\{|\sigma|=1\}} g(x - \lambda^{-1}t\sigma)\chi_B(x - \lambda^{-1}t\sigma)d\sigma - \pi g(x) \right| \le c\lambda^{-1}t.$$

Actually if $m(\xi) = 1$ in a neighborhood of $\xi = 0$ and if x is not in ∂B, a much better estimate than λ^{-2} hold, but this rough estimate is more than enough in what follow.

■

We now consider $R_\lambda f(x)$. Since

$$\int_{\mathbb{R}^2} \left(\chi_A(\lambda^{-1}\xi) - m(\lambda^{-1}\xi) \right) \widehat{f}(\xi) \exp(2\pi i x \cdot \xi) d\xi$$

$$= \lambda^2 \int_A \int_B (1 - m(\xi))\, g(y) \exp\left(2\pi i \lambda \xi \cdot (x - y)\right) dy d\xi,$$

we need to study an integral of the type

$$\lambda^2 \int_A \int_B F(\xi, y) \exp\left(2\pi i \lambda \xi \cdot (x - y)\right) dy d\xi,$$

where $F(\xi, y)$ is a smooth function with $F(\xi, y) = 0$ if $|\xi| \le \varepsilon$. Using the divergence theorem one can replace this integral over $A \times B$ with one over $\partial A \times \partial B$. Since

$$\operatorname{div}_y \left(\exp(2\pi i \lambda \xi \cdot (x - y)) F(\xi, y)\xi \right)$$

$$= \exp(2\pi i \lambda \xi \cdot (x - y))\xi \cdot \nabla_y F(\xi, y)$$

$$-2\pi i \lambda |\xi|^2 F(\xi, y) \exp(2\pi i \lambda \xi \cdot (x - y)),$$

the divergence theorem in the y-variable gives

$$\lambda^2 \int_A \int_B F(\xi, y) \exp(2\pi i \lambda \xi \cdot (x - y)) dy d\xi$$

$$= -\frac{i\lambda}{2\pi} \int_A \int_B |\xi|^{-2} \xi \cdot \nabla_y F(\xi, y) \exp(2\pi i \lambda \xi \cdot (x - y)) dy d\xi$$

$$+ \frac{i\lambda}{2\pi} \int_A \int_{\partial B} |\xi|^{-2} \xi \cdot n(y) F(\xi, y) \exp(2\pi i \lambda \xi \cdot (x - y)) dy d\xi,$$

where $n(y)$ is the unit normal to ∂B at the point y. By our assumptions $F(\xi, y) = 0$ when $|\xi| \leq \varepsilon$, so that $|\xi|^{-2} \xi \cdot \nabla_y F(\xi, y)$ is smooth and the integral with this term over $A \times B$ is of the same type as the one we started, with a better power of λ in front. It is therefore enough to study the second integral over $A \times \partial B$. Write $i(2\pi)^{-1} |\xi|^{-2} \xi \cdot n(y) F(\xi, y) = G(\xi, y)$. By the divergence theorem in the ξ-variable,

$$\lambda \int_A \int_{\partial B} G(\xi, y) \exp(2\pi i \lambda \xi \cdot (x - y)) dy d\xi$$

$$= \frac{i}{2\pi} \int_A \int_{\partial B} |x - y|^{-2} (x - y) \cdot \nabla_\xi G(\xi, y) \exp(2\pi i \lambda \xi \cdot (x - y)) dy d\xi$$

$$+ \frac{i}{2\pi} \int_{\partial A} \int_{\partial B} |x - y|^{-2} (x - y) \cdot n(\xi) G(\xi, y) \exp(2\pi i \lambda \xi \cdot (x - y)) dy d\xi.$$

Introducing a sort of polar coordinates on A one can see the second integral over $A \times \partial B$ as superpositions of integrals over $\partial A \times \partial B$. It then suffice to consider the last integral over $\partial A \times \partial B$. Observe that if x is not in ∂B, $|x - y|^{-2} (x - y)$ is non-singular and the integral over ∂B is well defined. On the contrary, if x is in ∂B the integral has to be defined in the principal value sense. To summarize, we have reduced our partial sums operators to an oscillatory integral over $\partial A \times \partial B$ with phase $\xi \cdot (x - y)$. In order to go on, we need to study in some details the singularities of this phase.

LEMMA 2: *Assume A star-shaped, ∂A and ∂B smooth. Let $\xi(t)$ and $y(s)$ be parametrization by arc-length of ∂A and ∂B, and define $\Phi(t, s, x) = \xi(t) \cdot (y(s) - x)$. This phase function $(t, s) \rightarrow \Phi(t, s, x)$ has the following properties:*

a) For a given s there are at most two t at which the phase has a critical point.

b) For every x there exists $\varepsilon > 0$ such that the phase has only non-degenerate critical points with $|y(s) - x| < \varepsilon$.

c) The phase $(t, s) \rightarrow \Phi(t, s, x)$ has a degenerate critical point at (t, s) only if ∂A has non-vanishing curvature at $\xi(t)$ and ∂B has non-vanishing curvature at $y(s)$.

d) The set of points x in $\mathbb{R}^2$ for which the phase has degenerate critical points is union of at most countably many images of smooth maps from $\mathbb{R}$ to $\mathbb{R}^2$. In particular this set has Hausdorff dimension at most one.

Proof: The point (t, s) is a critical point of the phase $\xi(t) \cdot (y(s) - x)$ if the gradient vanish and it is a degenerate critical point if also the Hessian determinant is zero,

$$(*) \quad \begin{cases} \xi(t) \cdot \dot{y}(s) = 0, \\[2mm] \dot{\xi}(t) \cdot (y(s) - x) = 0, \\[2mm] (\xi(t) \cdot \ddot{y}(s))\left(\ddot{\xi}(t) \cdot (y(s) - x)\right) - \left(\dot{\xi}(t) \cdot \dot{y}(s)\right)^2 = 0. \end{cases}$$

Since A is star-shaped and $\dot{y}(s) \neq 0$, in $(*)$ the first equation $\xi(t) \cdot \dot{y}(s) = 0$ for every s has exactly two solutions t. This proves (a).

In (b) we may assume that $x \in \partial B$ and $x = y(0)$, otherwise $|y(s) - x| \geq \varepsilon$ and there is nothing to prove. When $s = 0$ the system $(*)$ reduces to $\xi(t) \cdot \dot{y}(0) = 0$ and $\dot{\xi}(t) \cdot \dot{y}(0) = 0$, but this contradicts the fact that $\xi(t)$ and $\dot{\xi}(t)$ are linearly independent. Hence $s = 0$ is not a solution and, since the set of solutions is compact, the system has no solutions with $|y(s) - x| < \varepsilon$ if ε is sufficiently small. This proves (b).

If $\ddot{\xi}(t)$ or $\ddot{y}(s)$ is zero, the first and third equations in $(*)$ become $\xi(t) \cdot \dot{y}(s) = 0$ and $\dot{\xi}(t) \cdot \dot{y}(s) = 0$. But again this contradicts the fact that $\xi(t)$ and $\dot{\xi}(t)$ are linearly independent, hence at degenerate critical points the curvatures of ∂A and ∂B do not vanish. This proves (c).

It remains to estimate the dimension of the set of degenerate critical points. Since $\xi(t) \cdot \dot{y}(s) = 0$ imply that $\dot{\xi}(t) \cdot \dot{y}(s) \neq 0$, in $(*)$ the first equation $\xi(t) \cdot \dot{y}(s) = 0$ defines two smooth curves $t = t(s)$. Let the matrix $M(s)$ and vector $V(s)$ be defined by

$$M(s) = \begin{bmatrix} \dot{\xi}(t(s)) \\ (\xi(t(s)) \cdot \ddot{y}(s)) \ddot{\xi}(t(s)) \end{bmatrix},$$

$$V(s) = \begin{bmatrix} \dot{\xi}(t(s)) \cdot y(s) \\ (\xi(t(s)) \cdot \ddot{y}(s)) \left(\ddot{\xi}(t(s)) \cdot y(s) \right) - \left(\dot{\xi}(t(s)) \cdot \dot{y}(s) \right)^2 \end{bmatrix}.$$

Then we can write the second and the third equation in $(*)$ as $M(s) \cdot x = V(s)$. Let $\bigcup_j (a_j, b_j)$ be the set of s where the determinant of $M(s)$ is different from zero. If $a_j < s < b_j$ the matrix $M(s)$ is invertible and $x = M^{-1}(s) \cdot V(s)$.

In order to complete the proof of the lemma, it suffices to show that at points (t, s, x) solutions of $(*)$ the determinant of $M(s)$ is non zero. First observe that $\xi(t(s)) \cdot \ddot{y}(s) \neq 0$ and $\ddot{\xi}(t(s)) \cdot (y(s) - x) \neq 0$, otherwise the third equation in becomes $\dot{\xi}(s) \cdot \dot{y}(t) = 0$ and this contradicts $\xi(t) \cdot \dot{y}(s) = 0$. Also, $\dot{\xi}(t(s))$ and $\ddot{\xi}(t(s))$ are non-zero and orthogonal. Hence the matrix $M(s)$ is non singular.
∎

The meaning of the following lemma is that one can isolate the critical points of the phase function.

LEMMA 3: *There exists a smooth and finite partition of unity of* $\partial A \times \partial B,$

$$\sum_j P_j(t, s) + \sum_j Q_j(t, s) = 1,$$

with the following properties:
a) $|\ddot{y}(s)| > \varepsilon > 0$ *in the each of the supports of* $\{P_j(t, s)\}.$
b) In each of the supports of $\{Q_j(t, s)\}$ *there is at most one critical point of the phase* $\Phi(t, s, x),$ *and this critical point is non-degenerate.*

Proof: Denote by C the set of degenerate critical points of the phase function. This set is compact and the continuous function $|\ddot{y}(s)|$ does not vanish on it, hence it takes a positive minimum, $|\ddot{y}(s)| \geq 2\varepsilon > 0$ on C. Let

$$\{(t,s): \ |\ddot{y}(s)| > \varepsilon\} = \bigcup_j \{(t,s): \ a_j < s < b_j\}.$$

¿From this covering of C we can extract a finite covering, which we denote with the same notation. Then we can construct smooth functions $P_j(t,s)$, with supports in $\{(t,s): \ a_j < s < b_j\}$ and with $P_j(t,s) = 1$ if (t,s) is also in a small neighborhood of C. Since non-degenerate critical points are isolated, there are only finitely many non-degenerate critical points outside $\bigcup_j \{(t,s): \ P_j(t,s) = 1\}$ and we can complete the partition of unity by defining $\sum_j Q_j(t,s) = 1 - \sum_j P_j(t,s).$
■

We need to estimate the integral

$$I(\lambda, x) = \int_{\partial A} \int_{\partial B} |x-y|^{-2} (x-y) \cdot n(\xi) G(\xi, y) \exp(2\pi i \lambda \xi \cdot (x-y)) dy d\xi.$$

We first consider the non singular case $x \notin \partial B$. Using the previous lemma we can decompose $I(\lambda, x)$ into a sum of integrals of type

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi(t,s) \exp(-2\pi i \lambda \Phi(t,s,x)) ds dt,$$

where $\Psi(t,s)$ is smooth with compact support. If in this support the phase $\Phi(t,s,x)$ has at most a non-degenerate critical point, then the two-dimensional method of stationary phase gives the estimate λ^{-1}. The other possibility is that $|\ddot{y}(s)| > \varepsilon > 0$ and in this case $\left| \frac{\partial}{\partial s} \Phi(t,s,x) \right| + \left| \frac{\partial^2}{\partial s^2} \Phi(t,s,x) \right| > \delta > 0$, because $\xi(t) \neq 0$ and $\dot{y}(s)$ is orthogonal to $\ddot{y}(s)$. The one-dimensional method of stationary phase applied to the s-integral then gives the estimate $\lambda^{-1/2}$. Observe that, by Lemma 2, for most x the phase $\Phi(t,s,x)$ has only non-degenerate points and this imply that while the estimate λ^{-1} is the norm, the estimate $\lambda^{-1/2}$ is the exception. In particular we have proved Theorem 2.

It remains to consider the case $x \in \partial B$ and in this case the integral $I(\lambda, x)$ is singular. With a smooth partition of unity we can cut away the part of ∂B far from x, since in this part the integral is non singular and it can be estimated as before. Therefore we can assume that $G(\xi, y) = 0$ if $|y - x| \geq \varepsilon$ and, by Lemma 2, we can also assume that

in the support of $G(\xi, y)$ the phase has at most one non-degenerate critical point. To summarize, by a change of variables we are reduced to studying an integral of the type

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Psi(t, s)}{s} \exp(-2\pi i \lambda \Phi(t, s, x)) ds dt,$$

where $\Psi(t, s)$ is smooth with compact support, and in this support either the phase $\Phi(t, s, x)$ has no critical points, or it has only a non-degenerate critical point at $(0, 0)$.

Suppose first that $\Phi(t, s, x)$ has no critical point at $(0, 0)$ and $\Phi(t, 0, x) = 0$. Since $\dfrac{\partial}{\partial t} \Phi(t, 0, x) = 0$, we have $\dfrac{\partial}{\partial s} \Phi(t, 0, x) \neq 0$ and the change of variables $u = \Phi(t, s, x)$ yields for a suitable smooth $\Theta(u, s, x)$,

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Psi(t, s)}{s} \exp(-2\pi i \lambda \Phi(t, s, x)) ds dt$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Theta(u, s, x)}{s} \exp(-2\pi i \lambda u) ds du$$

$$= \int_{-\infty}^{+\infty} \left(\int_{0}^{+\infty} \frac{\Theta(u, s, x) - \Theta(u, -s, x)}{s} ds \right) \exp(-2\pi i \lambda u) du.$$

A repeated integration by parts in the u-variables then shows that the integral is dominated by λ^{-k} for every k.

Now suppose that $\Phi(t, s, x)$ has a critical point at $(0, 0)$ and $\Phi(t, 0, x) = 0$. We may write $\Phi(t, s, x) = s\Omega(t, s, x)$ and the change of variables $u = \Omega(t, s, x)$ yields for a suitable smooth $\Theta(u, s, x)$,

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Psi(t, s)}{s} \exp(-2\pi i \lambda \Phi(t, s, x)) ds dt$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Theta(u, s, x)}{s} \exp(-2\pi i \lambda s u) ds dt.$$

Let $\varphi(s)$ be a smooth function with compact support and with $\varphi(s) = 1$ when $\Theta(u, s, x) \neq 0$. Then we can write

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Theta(u, s, x)}{s} \exp(2\pi i \lambda u s) ds du$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Theta(u, 0, x) \frac{\varphi(s)}{s} \exp(2\pi i \lambda u s) ds du$$

$$+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Theta(u, s, x) - \Theta(u, 0, x)}{s} \varphi(s) \exp(2\pi i \lambda u s) ds du.$$

Since $\dfrac{\Theta(u, s, x) - \Theta(u, 0, x)}{s} \varphi(s)$ is smooth and the phase us is non-degenerate, the second integral is dominated by λ^{-k}. Finally, for the first integral we have

$$\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \frac{\varphi(s)}{s} \exp(2\pi i \lambda u s) ds \right) \Theta(u, 0, x) du$$

$$= \int_{-\infty}^{+\infty} \left(i\pi + \mathcal{O}\left((1 + \lambda |u|)^{-k} \right) \right) \Theta(u, 0, x) du$$

$$= i\pi \int_{-\infty}^{+\infty} \Theta(u, 0, x) du + \mathcal{O}\left(\lambda^{-k} \right).$$

The proof of Theorem 1 and Theorem 2 is then complete.
■

Proof of Theorem 3: We collect some properties of the polar curve. First observe that even if ∂A has arc-length parametrization $\xi(t)$, the parametrization $z(t)$ of ∂A^* is not necessarily regular, in particular $\dot{z}(t) = 0$ when $\ddot{\xi}(t) = 0$. Even if ∂A is nice, ∂A^* may present singularities, nevertheless we want to prove that at points of interest for our problem these singularities do not occur.

LEMMA 4: *Assume A star-shaped and let $\xi(t)$ be parametrization by arc-length of ∂A. Define the polar curve ∂A^* by*

$$\begin{cases} \xi(t) \cdot z(t) = 1, \\ \dot{\xi}(t) \cdot z(t) = 0. \end{cases}$$

Then the following properties hold:
a) $\xi(t) \cdot \dot{z}(t) = 0$.
b) $\dot{\xi}(t) \cdot z(t) = \xi(t) \cdot \ddot{z}(t) = -\dot{\xi}(t) \cdot \dot{z}(t)$.
c) If $\ddot{\xi}(t) \neq 0$*, then* $\dot{z}(t) \neq 0$ *and* $\ddot{z}(t) \neq 0$.
d) At regular points $\dot{z}(t) \neq 0$ *the curvature of the curve* $z(t)$ *is*

$$(\dot{z}(t) \cdot \dot{z}(t))^{-1} \ddot{z}(t) - (\dot{z}(t) \cdot \dot{z}(t))^{-2} (\dot{z}(t) \cdot \ddot{z}(t)) \dot{z}(t).$$

Proof: Differentiating $\xi(t) \cdot z(t) = 1$ and using $\dot{\xi}(t) \cdot z(t) = 0$ we obtain (a). Differentiating $\dot{\xi}(t) \cdot z(t) = 0$ and (a) we obtain (b).

Assume that $\ddot{\xi}(t) \neq 0$. Since $\dot{\xi}(t)$ and $\ddot{\xi}(t)$ are orthogonal and $\dot{\xi}(t) \cdot z(t) = 0$, we have $\ddot{\xi}(t) \cdot z(t) \neq 0$. Hence (c) follows from (b).

Finally, differentiating twice the vector $z(t)$ with respect to the arclength $\int |\dot{z}(t)| \, dt$ we obtain (d).

∎

The critical points of the phase $\xi(t) \cdot (y(s) - x)$ are the solutions of the system

$$\begin{cases} \xi(t) \cdot \dot{y}(s) = 0, \\ \dot{\xi}(t) \cdot (y(s) - x) = 0. \end{cases}$$

Comparing $\dot{\xi}(t) \cdot z(t) = 0$ with $\dot{\xi}(t) \cdot (y(s) - x) = 0$ we deduce that at a critical point $y(s) - x$ is proportional to $z(t)$. Comparing $\xi(t) \cdot \dot{z}(t) = 0$ with $\xi(t) \cdot \dot{y}(s) = 0$ we deduce that $\dot{z}(t)$ is proportional to $\dot{y}(s)$, and since $|\dot{y}(s)| = 1$ the constant of proportionality is $\nu = \pm |\dot{z}(t)|$. This proves first part of the theorem.

Letting $\gamma = -\mu^{-1} (\dot{z}(t) \cdot \dot{z}(t))^{-4} (\dot{z}(t) \cdot \ddot{z}(t))$, we can write the curvature of $x + \mu z(t)$ as $\mu^{-1} \nu^{-2} \ddot{z}(t) + \gamma \dot{z}(t)$. If in the Hessian determinant of $\xi(t) \cdot (y(t) - x)$ we replace $y(s)$ with $x + \mu z(t)$ and $\dot{y}(t)$ with $\nu^{-1} \dot{z}(s)$, using the lemma we obtain

$$(\xi(t) \cdot \ddot{y}(s)) \left(\ddot{\xi}(t) \cdot (y(s) - x) \right) - \left(\dot{\xi}(t) \cdot \dot{y}(s) \right)^2$$

$$= \mu \left(\xi(t) \cdot \ddot{y}(s) \right) \left(\ddot{\xi}(t) \cdot z(t) \right) - \nu^{-2} \left(\dot{\xi}(t) \cdot \dot{z}(s) \right)^2$$

$$= \mu \left(\xi(t) \cdot \ddot{y}(s) \right) \left(\ddot{\xi}(t) \cdot z(t) \right) - \nu^{-2} \left(\ddot{\xi}(t) \cdot z(t) \right) (\xi(t) \cdot \ddot{z}(t))$$

$$= \mu \left(\ddot{\xi}(t) \cdot z(t) \right) (\xi(t) \cdot (\ddot{y}(s) - \mu^{-1} \nu^{-2} \ddot{z}(t)))$$

$$= \mu \left(\ddot{\xi}(t) \cdot z(t) \right) (\xi(t) \cdot (\ddot{y}(s) - \mu^{-1} \nu^{-2} \ddot{z}(t) - \gamma \dot{z}(t))) \,.$$

Now observe that since $\dot{\xi}(t) \cdot z(t) = 0$, we have $\ddot{\xi}(t) \cdot z(t) \neq 0$. Since at a critical point $\xi(t) \cdot \dot{y}(s) = 0$, $\xi(t)$ is parallel to $\ddot{y}(s)$. Since $\dot{y}(s)$ is parallel to $\dot{z}(t)$, also $\ddot{y}(s)$ is parallel to $\mu^{-1} \nu^{-2} \ddot{z}(t) + \gamma \dot{z}(t)$. Hence the Hessian determinant is zero if an only if the curvatures of the two curves $y(s)$ and $x + \mu z(t)$ are equal, $\ddot{y}(s) = \mu^{-1} \nu^{-2} \ddot{z}(t) + \gamma \dot{z}(t)$.
∎

We conclude with some remarks.

REMARK 1: In our theorems we assumed ∂B smooth, but similar results should hold when ∂B is only piecewise smooth. We also assumed A star-shaped, however in the proofs the sets A and $B - x$ enter symmetrically, hence it is possible to move the hypotheses from one set to the other.

REMARK 2: The proof of Theorem 1 in the case $x \in \partial B$ is essentially a result on the boundedness of the Hilbert transform along a dilated of ∂B. We point out that there is no contradiction between our result and the counterexample in [7, p. 742], where a suitable Hilbert transform is shown to be unbounded.

REMARK 3: It is natural to conjecture that Theorem 2 has an n-dimensional extension. Indeed, as in the proof of Theorem 1, a double application of the divergence theorem gives

$$S_\lambda f(x) - f(x)$$

$$\approx \lambda^n \int_A \int_B F(\xi, y) \exp(2\pi i \lambda \xi \cdot (x - y)) dy d\xi$$

$$\approx \lambda^{n-2} \int_{\partial A} \int_{\partial B} G(\xi, y) \exp(2\pi i \lambda \xi \cdot (x - y)) dy d\xi.$$

The point (ξ, y, x) varies in $\partial A \times \partial B \times \mathbb{R}^n$. If for a given x the phase $(\xi, y) \to \xi \cdot (y - x)$ has only non degenerate critical points, then the $2n - 2$-dimensional oscillatory integral over $\partial A \times \partial B$ gives a decay λ^{-n-1} and $|S_\lambda f(x) - f(x)| \leq c \lambda^{-1}$. Now observe that the manifold $\partial A \times \partial B \times \mathbb{R}^n$ has dimension $3n - 2$ and the phase $(\xi, y) \to 2\pi i \lambda \xi \cdot (x - y)$ as degenerate critical points if $2n - 1$ equations are satisfied. This suggests that the set of x with degenerate phase has at most dimension $n - 1$.

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DIPARTIMENTO DI INGEGNERIA, UNIVERSITÀ DEGLI STUDI DI BERGAMO, VIALE G.MARCONI 5, 24044 DALMINE (BG), ITALY.

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DI MILANO-BICOCCA, VIA BICOCCA DEGLI ARCIMBOLDI 8, 20126 MILANO, ITALY.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI–COLUMBIA, COLUMBIA MO, USA.

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DI MILANO-BICOCCA, VIA BICOCCA DEGLI ARCIMBOLDI 8, 20126 MILANO, ITALY.