# SPHERICAL MEANS AND THE RESTRICTION PHENOMENON

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Abstract. Let $\Gamma$ be a smooth compact convex planar curve with arc length $dm$ and let $d\sigma = \psi dm$ where $\psi$ is a cutoff function. For $\Theta \in SO(2)$ set $\sigma_\Theta(E) = \sigma(\Theta E)$ for any measurable planar set $E$. Then, for suitable functions $f$ in $\mathbb{R}^2$, the inequality

$$\left\{ \int_{SO(2)} \left[ \int_{\mathbb{R}^2} \left| \widehat{f}(\xi) \right|^2 d\sigma_\Theta(\xi) \right]^{s/2} d\Theta \right\}^{1/s} \leqslant c \left\| f \right\|_p$$

represents an average over rotations, of the Stein-Tomas restriction phenomenon. We obtain best possible indices for the above inequality when $\Gamma$ is any convex curve and under various geometric assumptions.

## 1. Introduction

Let $S$ be a smooth compact hypersurface in $\mathbb{R}^n$. For suitable functions $f$ let $\mathcal{R}f$ denote the restriction of the Fourier transform to $S$, i.e. $\mathcal{R}f = \widehat{f}|_S$, where

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x)\, dx.$$

By a well known argument due to Stein and Tomas, (see e.g. [25]),

$$\left| \widehat{d\sigma}(\xi) \right| \leq c \left| \xi \right|^{-r}, \tag{1.1}$$

$r > 0$, implies that

$$\mathcal{R} : L^p(\mathbb{R}^n) \to L^2(S) \quad \text{with} \quad p = \frac{2(r+1)}{r+2}, \tag{1.2}$$

where $d\sigma = \psi dm$, $dm$ denotes the Lebesgue measure on $S$ and $\psi$ is a suitable cutoff fuction. If the Gaussian curvature of $S$ does not vanish, then $r = \frac{n-1}{2}$, and a homogeneity argument due to Knapp shows that the exponent $p$ cannot be improved. If the hypersurface is convex and finite type then (1.2) implies (1.1). See [13]. On the other hand, (1.2) does not hold for a line segment. One way of thinking about this is

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that $\widehat{f}$ may be concentrated in the direction orthogonal to the segment. This idea is captured by the fact that (1.1) in this case does not hold for any $r > 0$. However, it is reasonable to ask whether (1.2) holds for a "typical" line segment, or, in other words, whether (1.2) holds on the average for a certain range of exponents. We shall see below that in the appropriate sense this is in fact the case.

More precisely we study the restriction problem for the Fourier transform after averaging over rotations. We consider a smooth convex arc $\Gamma$ in the two dimensional plane and we study, for suitable functions $f$ on $\mathbb{R}^2$, estimates of the form

$$\left\{ \int_{SO(2)} \left[ \int_{\mathbb{R}^2} \left| \widehat{f}(\xi) \right|^2 d\sigma_\Theta(\xi) \right]^{s/2} d\Theta \right\}^{1/s} \leqslant c \left\| f \right\|_p \tag{1.3}$$

where $s \geqslant 1$, $1 \leqslant p \leqslant 2$, $\Theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \in SO(2)$, $d\sigma = \psi dm$, where $dm$ denotes the Lebesgue measure on $\Gamma$ and $\psi$ is a suitable cutoff fuction, while $\sigma_\Theta$ is defined by $\sigma_\Theta(E) = \sigma(\Theta E)$.

As we have seen above, when $s = \infty$ the results depend on the isotropic decay of the Fourier transform of $d\sigma$, which in turn depends very precisely on the local curvature properties of $\Gamma$. On the other hand, when $s < \infty$ we shall see that the results depend on a number of interesting factors. Among them are the average decay of the Fourier transform, the location of the center of rotation, and the degree to which the arc in question is different from a circle. After all, if the arc coincides locally with a circle, and if the point of rotation is the center of the circle, then the process of averaging over the rotations does not help and we are left with the result given by (1.2) above. We shall prove the following sharp results:

*i)* If $\Gamma$ is any convex arc then (1.3) holds if $p = \frac{6}{5}$ and $s = 6$.

*ii)* If $\Gamma$ is any convex arc and the origin is the center of curvature for no point in $\Gamma$, then (1.3) holds if $p = \frac{4}{3}$ and $s = 4$ (observe that this case includes the segment).

*iii)* If $\Gamma$ is any convex arc with order of contact at most 4 with any circle centered at the origin, then (1.3) holds if $p = \frac{4}{3}$ and $s = 2$. Here the typical example is the parabola.

The case $s = 2$ is of particular interest since certain known operators come out naturally. For example, if $\Gamma$ is a segment and the point of rotation is not on $\Gamma$ then the case $s = 2$ of (1.3) is controlled using estimates of Bochner-Riesz means of negative order. This is in contrast to the case when the point of rotation is on the segment, where the analogous estimate can be controlled using results for fractional integrals.

We also obtain some results where the Lebesgue measure is replaced by a fractal measure. The resulting exponents in (1.3) depend on the fractal dimension. In fact, for some ranges of exponents the upper Minkowski dimension is the main factor in (1.3). (See also [10]).

## 2. Average decay of the Fourier transform

One of the main techniques used in this paper is the average decay of the Fourier transform of the surface carried measure. More precisely, consider the estimate

$$\left\{ \int_{SO(2)} \left| \widehat{d\sigma}(\Theta \xi) \right|^q d\Theta \right\}^{1/q} \leq c \, |\xi|^{-\alpha} \tag{2.1}$$

A result due to Podkorytov (see [20] and also below for an alternative proof) says that if $q = 2$, then (2.1) holds with $\alpha = \frac{1}{2}$ for any convex $\Gamma$, including the case of a segment.

The Lemma 1, (see below), explicitly relates (2.1) and (1.3) and this is enough to prove $i)$ above. In the same way one can prove $ii)$ for the case of a segment. However this argument does not prove the whole statement and a more sophisticated technique taking into account the geometry and the location of $\Gamma$ is required.

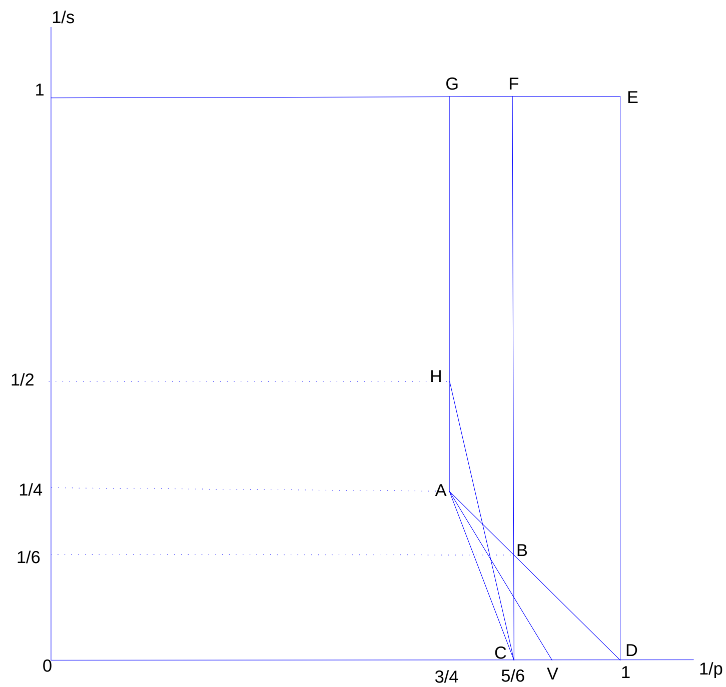
For the interested reader we point out that the average decay of the Fourier transform appears in a natural way in many other problems in harmonic analysis. The idea of using $L^q$ average decay in the explicit way appears to have been originated by Randol (see below), though similar estimates have appeared much earlier in analytic number theory. It has since been used in many problems such as the irregularity of distribution (see e.g. [2], [19] and [6]), estimates on the number of lattice points in randomly positioned domains (see e.g. [21], [6], [7] and [14]), Radon transforms, (see e.g. [22]), summation of Fourier series, see e.g. ( [5] and [8]), non-isotropic maximal averaging operators, (see [17] and [18]), tiling and spectral problems (see e.g. [15] and [16]).

## 3. Statement of results and outline of the paper

The following picture illustrates the results stated below.

In the above figure $A = \left(\frac{3}{4}, \frac{1}{4}\right)$, $B = \left(\frac{5}{6}, \frac{1}{6}\right)$, $H = \left(\frac{3}{4}, \frac{1}{2}\right)$, $V = \left(\frac{2\gamma+1}{2\gamma+2}, 0\right)$.

Here are the statements of the main results in this paper.




**Theorem 1.** *Suppose the pair* $\left(\frac{1}{p}, \frac{1}{s}\right)$ *belongs to the closed trapezoid* $BDEF$*. Then, for any convex planar arc* $\Gamma$ *there exists a positive constant* $c$ *such that for every test function* $f$ *(1.3) holds. The result is sharp in the sense that if* $\left(\frac{1}{p}, \frac{1}{s}\right)$ *is out of* $BDEF$ *there exists* $\Gamma$ *such that (1.3) fails.*

The above range of exponenents can be improved if we assume suitable geometric properties of $\Gamma$.

**Theorem 2.** *Suppose the pair* $\left(\frac{1}{p}, \frac{1}{s}\right)$ *belongs to the closed pentagon* $HCDEG$*. Then, for any convex planar arc* $\Gamma$ *with strictly positive curvature and order of contact at most* $4$ *with any circle in the plane there exists a positive constant* $c$ *such that for every test function* $f$ *(1.3) holds. The result is sharp in the sense that if* $\left(\frac{1}{p}, \frac{1}{s}\right)$ *is out of* $HCDEG$ *there exists* $\Gamma$ *such that (1.3) fails.*

**Theorem 3.** *Suppose the pair* $\left(\frac{1}{p}, \frac{1}{s}\right)$ *belongs to the closed trapezoid* $ADEG$*. Then, for any segment* $\Gamma$ *there exists a positive constant* $c$

*such that for every test function $f$ (1.3) holds. The result is sharp in the sense that if $\left(\frac{1}{p}, \frac{1}{s}\right)$ is out of $ADEG$ there exists $\Gamma$ such that (1.3) fails.*

The previous result on the segments is a particular case of the following theorem, where the position of $\Gamma$ with respect to the point of rotation is considered.

**Theorem 4.** *Suppose the pair $\left(\frac{1}{p}, \frac{1}{s}\right)$ belongs to the closed trapezoid $ADEG$. Then, for any convex planar arc $\Gamma$, such that the origin is the center of curvature for no point in $\Gamma$, there exists a positive constant $c$ such that for every test function $f$ (1.3) holds. The result is sharp in the sense that if $\left(\frac{1}{p}, \frac{1}{s}\right)$ is out of $ADEG$ there exists a segment $\Gamma$ such that (1.3) fails.*

**Theorem 5.** *Suppose the pair $\left(\frac{1}{p}, \frac{1}{s}\right)$ belongs to the closed pentagon $AVDEG$. Let $B(x,\delta) = \{y \in \Gamma : \mathrm{dist}\,(y, T_x(\Gamma)) \leq \delta\}$, where $T_x(\Gamma)$ denotes the tangent line to $\Gamma$ at $x$. Suppose that $\sigma(B(x,\delta)) \leq C\delta^{\beta}$, $\beta \leq \frac{1}{\gamma}$ and suppose that $\Gamma$ satisfies the assumptions of Theorem 4 (the graph of the function $y = x^{\gamma}$, $\gamma$ even positive integer, $|x| \leq 1$ is the typical example). Then there exists a positive constant $c$ such that for every test function $f$ (1.3) holds. The result is sharp in the sense that if $\left(\frac{1}{p}, \frac{1}{s}\right)$ is out of $AVDEG$ there exists $\Gamma$ such that (1.3) fails.*

When $s = 2$ we are reduced to studying the inequality

$$\left\{\int_{SO(2)} \int_{\mathbb{R}^2} \left|\widehat{f}(\xi)\right|^2 d\sigma_\Theta(\xi)\, d\Theta\right\}^{1/2} \leqslant c\,\|f\|_p \tag{3.1}$$

and we prove the following result.

**Theorem 6.** *i) Suppose the origin belongs to $\Gamma$, then (3.1) holds iff $p \leq 4/3$.*

*ii) Suppose the origin does not belong to $\Gamma$, but there exists a point in $\Gamma$ whose normal line passes through the origin. We have two subcases.*

*iia) $\Gamma$ contains an arc in a circle centered at the origin, then the rotation plays no role and (3.1) holds iff $p \leq 6/5$.*

*iib) $\Gamma$ has order of contact $N$ with an arc in a circle centered at the origin, then (3.1) holds iff $p \leq \frac{6N}{5N-2}$ .*

*iii) Suppose that we are not in the cases (i) or (ii). Then (3.1) holds iff $p \leq 2$.*

We now outline the proofs of the results. The details will be given in the next section.

The positive part of Theorem 1 is the direct consequence of the following two results.

**Lemma 1.** *Let $\Gamma$ be a convex planar arc and let $\mu$ be a finite measure with support in $\Gamma$. Assume that for some $1 \leq q \leq +\infty$*

$$\left\{ \int_{SO(2)} \left| \widehat{d\mu_\Theta}(\xi) \right|^q d\Theta \right\}^{1/q} \leq c \, |\xi|^{-\alpha} \tag{3.2}$$

*where $\alpha \geq 0$ and $c$ is independent of $\xi$ and $\mu_\Theta(E) = \mu(\Theta E)$. Then*

$$\left\{ \int_{SO(2)} \left| \int_{\mathbb{R}^2} \left| \widehat{f}(\xi) \right|^2 d\mu_\Theta(\xi) \right|^{q(\alpha+1)} d\Theta \right\}^{1/2q(\alpha+1)} \leqslant c \, \|f\|_{2(\alpha+1)/(\alpha+2)} \, . \tag{3.3}$$

The proof of the above lemma uses a complex interpolation argument which partially resembles the one in [22].

**Lemma 2.** *Let $\Gamma$ and $d\sigma$ be as before. Then*

$$\left\{ \int_0^{2\pi} \left| \widehat{d\sigma}(\rho(\cos\theta, \sin\theta)) \right|^2 d\theta \right\}^{1/2} \leq c \, \rho^{-1/2}.$$

The above lemma has been proved by Podkorytov in [20]. We shall give two different proofs below.

The main point in the proof of Theorem 2 is a particular case of a family of results for the case $s = 2$. See below in this section.

We now outline the proof of Theorem 4. The proof is based on the direct analytic interpolation argument. Let $t$ denote the arclength parameter on $\Gamma$. We essentially consider the analytic family $t^z dt$ and interpolate between the endpoints $\mathrm{Re}\,(z) = 1$ and $\mathrm{Re}\,(z) = -1$. Two different cases come up in establishing the bound for $\mathrm{Re}\,(z) = 1$. If the point of rotation is on $\Gamma$, then we are led to consider fractional integration, whereas if the point of rotation is not on $\Gamma$, we are led to consider Bochner-Riesz means of negative order. We first prove the inequality for the functions appropriately localized on the Fourier transform side, and then use Littlewood-Paley theory to recover the result for the full operator.

The positive part of Theorem 5 can be proved as follows. A well known argument, (see e.g. [9]) gives the uniform decay of the Fourier transform of the surface carried measure of order at worst $-\frac{1}{\gamma}$, so the

Stein-Tomas argument (see the first paragraph of the introduction) implies the restriction theorem with $p = \frac{2(\gamma+1)}{2\gamma+1}$, $s = \infty$. By interpolation we obtain the segment $A - V$ in the diagram above.

We now outline the case $s = 2$, which already contains most of the ideas used in the paper. In this case we are interested in estimating

$$\left\{ \int_0^{2\pi} \int_{\mathbb{R}^2} \left| \widehat{f}(\Theta(v, h(v))) \right|^2 d\sigma d\theta \right\}^{1/2}$$

where $\Gamma$ is the graph of a smooth convex function $h$ supported on $[-1, 1]$ and

$$\Theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Observe that two different cases arise. If $\Gamma$ passes through the origin, then setting

$$\xi = (v, h(v))\,\Theta$$

we get $d\sigma d\theta \approx \frac{1}{|\xi|} d\xi$, which amounts to fractional integration. On the other hand if $\Gamma$ has order of contact at most 2 with any circle centered at the origin and it does not pass through the origin (assume it passes through the point $(0, 1)$), then setting

$$\xi = \Theta(v, 1 + h(v))$$

we get $d\sigma d\theta \approx \left| |\xi|^2 - 1 \right|^{-1/2} d\xi$, which resembles a Bochner-Riesz mean of negative order.

## 4. Proofs

**Proof of Lemma 1.** Write $r = q(\alpha+1)$ and let $r'$ be the conjugate index. We take the square of the L.H.S. in (3.3):

$$\left\{\int_{SO(2)}\left|\int_{\mathbb{R}^2}\left|\widehat{f}(\xi)\right|^2 d\mu_\Theta(\xi)\right|^r d\Theta\right\}^{1/r}$$

$$= \sup_{\|g\|_{r'}=1}\int_{SO(2)}\int_{\mathbb{R}^2}\left|\widehat{f}(\xi)\right|^2 d\mu_\Theta(\xi)g(\Theta)d\Theta$$

$$= \sup_{\|g\|_{r'}=1}\int_{SO(2)}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(y)e^{-2\pi iy\cdot\xi}dy\int_{\mathbb{R}^2}\overline{f(x)}e^{2\pi ix\cdot\xi}dx\, d\mu_\Theta(\xi)g(\Theta)d\Theta$$

$$= \sup_{\|g\|_{r'}=1}\int_{\mathbb{R}^2}\overline{f(x)}\int_{SO(2)}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(y)e^{-2\pi i(y-x)\cdot\xi}dyd\mu_\Theta(\xi)g(\Theta)d\Theta\, dx$$

$$\leq c\,\|f\|_{2(\alpha+1)/(\alpha+2)}$$

$$\times \sup_{\|g\|_{r'}=1}\left\|\int_{\mathbb{R}^2} f(y)\int_{SO(2)}\widehat{d\mu_\Theta}(y-x)g(\Theta)d\Theta\, dy\right\|_{2(\alpha+1)/\alpha}.$$

We have to prove that

$$\mathfrak{K}_g : L^{2(\alpha+1)/(\alpha+2)} \to L^{2(\alpha+1)/\alpha}, \tag{4.1}$$

with an operator norm bounded indipendently of $g$, where $\mathfrak{K}_g f = K_g * f$ and

$$K_g(x) = \int_{SO(2)}\widehat{d\mu_\Theta}(x)g(\Theta)d\Theta.$$

Following an idea in [22] we define

$$K_g^z(x) = \int_{SO(2)}\widehat{i_z}(x_1\sin\theta + x_2\cos\theta)\widehat{d\mu}(\Theta^{-1}x)g^{1+(r'-1)z}(\Theta)d\Theta.$$

where $i_z$ denotes the distribution such that

$$< i_z, \psi >= \frac{1}{\Gamma(z)}\int_0^{+\infty} t^{z-1}\psi(t)dt.$$

We also point out that $g$ can be chosen non negative and that the number $(x_1\sin\theta + x_2\cos\theta)$ is the second component of $\Theta^{-1}x$.

We now consider the analytic family of operators

$$\mathfrak{K}_g^z : f \mapsto K_g^z * f.$$

Since $\widehat{i_0} \equiv 1$ then $K_g^0 = K_g$ and $\mathfrak{K}_g^0 = \mathfrak{K}_g$. For every $\lambda \in \mathbb{R}$ we shall prove the following bounds

$$\mathfrak{K}_g^{1+i\lambda} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \tag{4.2}$$

and

$$\mathfrak{K}_g^{-\alpha+i\lambda} : L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2) \tag{4.3}$$

(which imply (4.1) and therefore (3.3)).

In order to prove (4.2) we have to show that $\widehat{K_g^{1+i\lambda}} \in L^\infty$ with norm independent of $g$. This follows from the fact that $i_{1+i\lambda}$ is bounded and $g^{r'+(r'-1)i\lambda}$ is integrable. To prove (4.3) we need to show that $K_g^{-\alpha+i\lambda} \in L^\infty$ independently of $g$. This is easy since $\left|\widehat{i_{-\alpha+i\lambda}}(t)\right| \le c\,|t|^\alpha$, where $c$ is independent of $t$ and $\lambda$. Therefore, being $g \in L^{r'}$, and $q(\alpha+1) = r$ and assuming (3.2) we have

$$\begin{aligned}\left|K_g^{-\alpha+i\lambda}(x)\right| &\le c\,|x|^\alpha \left|\int_{SO(2)} \widehat{\mu}(\Theta^{-1}x) g^{1-(r'-1)\alpha+i(r'-1)\lambda}(\theta)d\theta\right| \\ &\le c\,|x|^\alpha \left\{\int_{SO(2)} \left|\widehat{\mu}(\Theta^{-1}x)\right|^q d\theta\right\}^{1/q} \le c.\end{aligned}$$

■

**Proof of Lemma 2.** We can split the curve into pieces so that we can assume $\Gamma$ to be the graph of a function $u : [0,a] \to \mathbb{R}$ satisfying $u(0) = u'(0) = 0$ and $|u'(t)| < \frac{1}{10}$. Then

$$\begin{aligned}&\int_0^{2\pi} \left|\widehat{d\sigma}(\rho\cos\theta, \rho\sin\theta)\right|^2 d\theta \\ &= \int_0^{2\pi} \int_{\mathbb{R}^2} e^{2\pi i x\cdot\rho(\cos\theta,\sin\theta)} d\sigma(x) \int_{\mathbb{R}^2} e^{-2\pi i y\cdot\rho(\cos\theta,\sin\theta)} d\sigma(y) d\theta \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_0^{2\pi} e^{2\pi i(x-y)\cdot\rho(\cos\theta,\sin\theta)} d\theta d\sigma(x) d\sigma(y) \\ &= 2\pi \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} J_0(2\pi\rho\,|x-y|) d\sigma(x) d\sigma(y) \\ &= 2\pi \int_0^a \int_0^a J_0\left(2\pi\rho\sqrt{(t_1-t_2)^2 + (u(t_1)-u(t_2))^2}\right) \\ &\qquad \times \sqrt{1+|u'(t_1)|^2}\psi(t_1,u(t_1))\sqrt{1+|u'(t_2)|^2}\psi(t_2,u(t_2)) dt_1 dt_2 \\ &= 2\pi \int_0^a \int_0^a J_0\left(2\pi\rho\sqrt{(t_1-t_2)^2 + (u(t_1)-u(t_2))^2}\right) \gamma(t_1)\gamma(t_2) dt_1 dt_2.\end{aligned}$$

By simmetry we can assume $t_1 \ge t_2$. We change variable in the inner integral setting

$$z = \sqrt{(t_1-t_2)^2 + (u(t_1)-u(t_2))^2}$$

and we reduce to estimating

$$\int_0^a \int_0^{c(t_2)} J_0\left(2\pi\rho z\right)\varphi(z,t_2)dzdt_2.$$

Since the Jacobian of the transformation is

$$\frac{1+\frac{u(t_1)-u(t_2)}{t_1-t_2}u'(t_1)}{\sqrt{1+\left(\frac{u(t_1)-u(t_2)}{t_1-t_2}\right)^2}}$$

$\varphi$ and $\frac{\partial\varphi}{\partial z}$ are uniformly bounded. After putting $\rho z = w$ we obtain

$$\rho^{-1}\int_0^a \int_0^{\rho c(t_2)} J_0\left(2\pi w\right)\varphi(\frac{w}{\rho},t_2)dwdt_2 \leq c\rho^{-1},$$

either integrating by parts and appealing to the bound

$$\left|\int_0^x J_0(2\pi w)dw\right| \leq c,$$

or using the asymptotics of $J_0$.

■

**Proof of Theorem 1**. The positive part follows from Lemma 1 and Lemma 2. To show that the result is sharp we take $\Gamma$ containing an arc of a circle centered at the origin and a segment. The presence of the arc forces $p \leq 6/5$ (see e.g. [25]). The necessity of the bound $p \leq s'$ will be proved later.

■

**Proof of Theorem 2.** By interpolation the relevant points are $C$ and $H$ in the figure. The positive part follows from the standard restriction theorem and from the Theorem 6.

To prove the necessity, assume $\Gamma$ is the graph of a function $y = h(x)$, with $h(0) = 1$, $h'(0) = 0$, $h''(0) = -1$. Consider, for a small $\delta > 0$, the rectangle $R = [-\delta,\delta] \times [-\delta^2+1, \delta^2+1]$. Since

$$h(x) - \sqrt{1-x^2} \approx x^4$$

we have $\sigma_\Theta(R) \approx \delta$ for $|\theta| \leq \delta^{1/2}$. Now let $f$ satisfy $\widehat{f} = \chi_R$. Then $\|f\|_p = c\delta^{3-3/p}$ and therefore the truth of (1.3) implies $\frac{1}{2}+\frac{1}{2s} \geq 3-\frac{3}{p}$, i.e.

$$\frac{1}{s} \geq 5 - \frac{6}{p}.$$

This is the line passing through the points $H$ and $C$ in the figure. On the other hand, suppose $\Gamma$ passes through the origin and choose

$g$ satisfying $\widehat{g} = \chi_{B_\varepsilon}$ where $B_\varepsilon$ is the disc of radius $\varepsilon$ centered at the origin. Then (1.3) implies $\frac{1}{2} \geq 2 - \frac{2}{p}$, i.e. $\frac{1}{p} \geq \frac{3}{4}$. ■

**Proof of Theorem 4**. It is enough to prove the case $p = 4/3$, $s = 4$ (the point $A$ in the figure):

$$\left\{ \int_{SO(2)} \left[ \int_{\mathbb{R}^2} \left| \widehat{f}(\xi) \right|^2 d\sigma_\Theta(\xi) \right]^2 d\Theta \right\}^{1/4} \leqslant c \, \|f\|_{4/3} \tag{4.4}$$

We first handle the case when $\Gamma$ passes through the origin. Let $f_k$ be defined by $\widehat{f_k}(\xi) = \widehat{f}(\xi)\beta\left(2^k |\xi|\right)$, where $\beta$ is the usual Littlewood-Paley cutoff function. Then

$$\begin{aligned}
&\left\{ \int_{SO(2)} \left[ \int_{\mathbb{R}^2} \left| \widehat{f}(\xi) \right|^2 d\sigma_\Theta(\xi) \right]^2 d\Theta \right\}^{1/2} \\
&\leq \left\{ \int_{SO(2)} \left[ \int_{\mathbb{R}^2} \sum_k \left| \widehat{f_k}(\xi) \right|^2 d\sigma_\Theta(\xi) \right]^2 d\Theta \right\}^{1/2} \\
&= \left\{ \int_{SO(2)} \left[ \sum_k \int_{\mathbb{R}^2} \left| \widehat{f_k}(\xi) \right|^2 d\sigma_\Theta(\xi) \right]^2 d\Theta \right\}^{1/2} \\
&\leq \sum_k \left\{ \int_{SO(2)} \left[ \int_{\mathbb{R}^2} \left| \widehat{f_k}(\xi) \right|^2 d\sigma_\Theta(\xi) \right]^2 d\Theta \right\}^{1/2} .
\end{aligned}$$

Assume (4.4) holds with $f$ replaced by $f_k$ and $c$ independent of $k$. Then the last term is bounded by

$$c \sum_k \|f_k\|_{4/3}^2 \leq \left\| \left( \sum_k |f_k|^2 \right)^{1/2} \right\|_{4/3}^2 \leq c \, \|f\|_{4/3}^2 ,$$

where we have used Minkowski inequality and the Littlewood-Paley theorem.

We must prove that

$$\left\{ \int_{SO(2)} \left[ \int_{\mathbb{R}^2} \left| \widehat{f_k}(\xi) \right|^2 d\sigma_\Theta(\xi) \right]^2 d\Theta \right\}^{1/4} \leqslant c \, \|f_k\|_{4/3} . \tag{4.5}$$

Observe that

$$
\begin{aligned}
(4.6)\qquad & \sup_{\Theta}\left\{\int_{\mathbb{R}^2}\left|\widehat{f}_k(\xi)\right|^2 d\sigma_\Theta(\xi)\right\}^{1/2} \\
&= \sup_{\Theta}\left\{\int_{\operatorname{supp}\widehat{f}_k}\left|\widehat{f}_k(\xi)\right|^2 d\sigma_\Theta(\xi)\right\}^{1/2} \\
&\leq \left\|\widehat{f}_k(\xi)\right\|_\infty \sup_{\Theta}\left\{\int_{\operatorname{supp}\widehat{f}_k} d\sigma_\Theta(\xi)\right\}^{1/2} \\
&\leq c\left\|f_k\right\|_1 2^{-k/2},
\end{aligned}
$$

while

$$
\begin{aligned}
& \left\{\int_{SO(2)}\int_{\mathbb{R}^2}\left|\widehat{f}_k(\xi)\right|^2 d\sigma_\Theta(\xi)d\Theta\right\}^{1/2} \\
(4.7)\qquad & \leq c\left\{\int_{\mathbb{R}^2}\left|\widehat{f}_k(\xi)\right|^2 \frac{d\xi}{|\xi|}\right\}^{1/2} \\
& \leqslant c2^{k/2}\left\|f_k\right\|_2 .
\end{aligned}
$$

We used the fact that when $\Gamma$ passes through the origin $d\sigma_\Theta d\Theta \approx \frac{1}{|\xi|}d\xi$. The estimate (4.5) follows by real interpolation.

We now turn to the case when $\Gamma$ does not pass through the origin. Since the origin is not a center of curvature we can identify $\Gamma$ with the graph of a function $h(v)$ with $|v|\leq 1$, $h(0)=1$, $h'(0)=0$, $h''(v)\neq -1$. The argument is similar to the previous one, but the Littlewood-Paley cutoff is $\beta\left(2^k\left(|\xi|-1\right)\right)$. Again it suffices to prove the estimate (4.5) using the reduction above.

To prove an anolog of (4.6) under the assumption that $\Gamma$ does not pass through the origin, we observe that

$$
\int_{\operatorname{supp}\widehat{f}_k} d\sigma_\Theta(\xi) \approx 2^{-k/2}
$$

because of the geometric assumption on $\Gamma$. Consequently, we have

$$
\sup_{\Theta}\left\{\int_{\mathbb{R}^2}\left|\widehat{f}_k(\xi)\right|^2 d\sigma_\Theta(\xi)\right\}^{1/2} \leq c\left\|f_k\right\|_1 2^{-k/4}.
$$

in place of the estimate (4.6).

To prove an appropriate analog of (4.7), let $\xi = \Theta\left((v,h(v)\right)$. The Jacobian of the transformation is $|v+h(v)h'(v)| \approx |v|$ because of the assumptions on $h$. Since $|\xi|^2 = v^2+h(v)^2$ the left hand side of (4.7) is bounded by

$$\left\{\int_{SO(2)}\int_{\mathbb{R}^2}\left|\widehat{f}_k(\xi)\right|^2 d\sigma_\Theta(\xi)d\Theta\right\}^{1/2}$$
$$\leq c\left\{\int_{\{|\xi|\geq 1\}}\left|\widehat{f}_k(\xi)\right|^2\left|\xi^2-1\right|^{-1/2}d\xi\right\}^{1/2}$$
$$\leqslant c2^{k/4}\left\|f_k\right\|_2 .$$

We now prove that our estimates are sharp. Suppose $\Gamma$ is a segment centered at the origin. Let $R(\varepsilon,\delta)=[-\delta,\delta]\times[-\varepsilon,\varepsilon]$ with $\varepsilon\leq\delta$. Then, for $p>1$, $\left\|\widehat{\chi}_{R(\varepsilon,\delta)}\right\|_{L^p}\approx\varepsilon^{1-1/p}\delta^{1-1/p}$. Let $f(x)=\widehat{\chi}_{R(\varepsilon,\delta)}(x)$. Then, when $s>2$ ,

$$\left\{\int_{SO(2)}\left|\int_{\mathbb{R}^2}\left|\widehat{f}(\xi)\right|^2 d\mu_\Theta(\xi)\right|^{s/2}d\Theta\right\}^{1/s}\approx\varepsilon^{1/s}\delta^{1/2-1/s}.$$

If we want $\varepsilon^{1/s}\delta^{1/2-1/s}\leq c\varepsilon^{1-1/p}\delta^{1-1/p}$ for any $0<\varepsilon\leq\delta\leq 1$ we find the two conditions

$$p\leq 4/3 \quad , \qquad p\leq\frac{s}{s-1}.$$

■

**Proof of Theorem 3**. This is a particular case of the previous theorem, since a segment has no centers of rotation. It also follows directly by Lemma 1 and the estimates in [6] or [7].

■

**Proof of Theorem 5.** The endpoints $D$ and $G$ are trivial. The endpoint $V$ is the consequence of Stein-Tomas restriction theorem in the form stated in the introduction (see (1.2)) and the well known fact that under the assumptions of this theorem one has isotropic decay of the Fourier transform of order $-\frac{1}{\gamma}$. The endpoint $A$ is proved in Theorem 4. This completes the proof of the positive part of the Theorem.

To see that the result is sharp let $\Gamma$ be, say, the curve

$$\{(s,s^\gamma):-1\leq s\leq 1\}$$

(we recall that $\gamma$ is a positive even integer). Let $\widehat{f_\delta}$ denote the Fourier transform of the characteristic function of the $\delta$ by $\delta^\gamma$ rectangle with the long side on the $x$ -axis. Then $\left\|f_\delta\right\|_p\approx\delta^{\frac{\gamma+1}{p'}}$, while the left hand side of (1.3) $\approx\delta^{\frac{\gamma-1}{s}}\delta^{\frac{1}{2}}$. It follows that

$$\frac{\gamma+1}{p'}\leq\frac{\gamma-1}{s}+\frac{1}{2},$$

which proves the sharpness of the segment $AV$. The condition $p \leq \frac{4}{3}$ is proved above.

■

## 5. The case $s = 2$

When $s = 2$ the L.H.S. of (1.3) becomes

$$\left\{ \int_{SO(2)} \int_{\mathbb{R}^2} \left| \widehat{f}(\xi) \right|^2 d\sigma_\Theta(\xi) d\Theta \right\}^{1/2} = \left\{ \int_{\mathbb{R}^2} \left| \widehat{f}(\xi) \right|^2 d\mu(\xi) \right\}^{1/2}$$

where $d\mu = \int_{SO(2)} d\sigma(\Theta(\xi)) d\Theta$, and we therefore have to study the familiar restriction theorem considered for the radial measure $\mu$. Actually in most cases $\mu$ is a function and the restriction theorem depends on its singularities, which in turn depend on the position of $\Gamma$ with respect to the origin (i.e. the center of rotation). This is made clear in the following theorem where the technically simpler case of a segment is considered. The general case is basically a corollary of the method.

**Theorem 7.** *Let $\Gamma = \{(0, y) \in \mathbb{R}^2 : 0 \leq y \leq 1\}$ and for every $A = (a_1, a_2) \in \mathbb{R}^2$ set $\Gamma_A = A + \Gamma$. For every test function $f$ define $T_A f(s, \Theta) = \widehat{f}|_{\Theta(\Gamma_A)}(s)$, where $\Theta \in SO(2)$ and $\Theta(\Gamma_A)$ is the rotated segment. We consider three different cases.*

*i) Suppose $a_1 = 0$ and $-1 \leq a_2 \leq 0$, i.e. the origin belongs to $\Gamma$, then $T_A$ maps $L^p(\mathbb{R}^2)$ into $L^2(\Gamma_A \times SO(2))$ iff $p \leq 4/3$.*

*ii) Suppose $a_1 \neq 0$ and $-1 \leq a_2 \leq 0$, i.e. the origin belongs to the closed strip consisting of the lines perpendicular to $\Gamma$ and intersecting $\Gamma$, then $T_A$ maps $L^p(\mathbb{R}^2)$ into $L^2(\Gamma_A \times SO(2))$ iff $p \leq 3/2$.*

*iii) Suppose $a_2 < -1$ or $a_2 > 0$, i.e. we are neither in case (i) nor in case (ii), then $T_A$ maps $L^p(\mathbb{R}^2)$ into $L^2(\Gamma_A \times SO(2))$ iff $p \leq 2$.*

**Proof.** $(i)$ follows from Theorem 3, however we shall give the following alternative proof. We assume $a_2 = 0$, since otherwise we consider the two segments separately, then

$$\begin{aligned}
&\int_{\gamma_A} \int_{SO(2)} |T_A(f(s, \Theta)|^2 \, ds d\Theta \\
&= \int_{|\xi| \leq 1} \left| \widehat{f}(\xi) \right|^2 |\xi|^{-1} \, d\xi \leq \int_{\mathbb{R}^2} \left| \widehat{f}(\xi) \, |\xi|^{-1/2} \right|^2 d\xi \\
&= \int_{\mathbb{R}^2} \left| I_{1/2}(f)(\xi) \right|^2 d\xi \leq \left\{ \int_{\mathbb{R}^2} |f(x)|^{4/3} \, dx \right\}^{3/2}
\end{aligned}$$

where $I_{1/2}$ is the fractional integral operator and the last inequality follows from a result of Hardy, Littlewood and Sobolev (see e.g. [24]). Furthermore this bound is sharp because of the counterexamples for fractional integration and the fact that $\int_{|\xi|\geq 1}\left|\widehat{f}(\xi)\right|^2|\xi|^{-1}\,d\xi$ is trivially bounded (or directly by considering $f$ equal to the charactersitic function of a disc of radius $R$).

We now prove $(ii)$. Again we may assume that $a_2=0$. We may also assume $a_1=1$. Through a change of variables we have

$$\begin{aligned}&\int_{\gamma_A}\int_{SO(2)}\left|T_A(f(s,\Theta)\right|^2 ds d\Theta\\&=\int_{1\leq|\xi|\leq\sqrt{2}}\left|\widehat{f}(\xi)\right|^2|\xi|^{-1}\left(|\xi|^2-1\right)^{-1/2}d\xi\\&\approx\int_{1\leq|\xi|\leq\sqrt{2}}\left|\widehat{f}(\xi)\left(|\xi|^2-1\right)^{-1/4}\right|^2 d\xi.\end{aligned}$$

We now claim that the last term is bounded by $c\left\|f\right\|_{L^{3/2}}$ and it cannot be bounded by $c\left\|f\right\|_{L^p}$ if $p>3/2$. The idea is to reduce the problem to one on Bochner-Riesz means of negative order (although we are outside of the unit disc). We have

$$\begin{aligned}&\int_{1\leq|\xi|\leq\sqrt{2}}\left(|\xi|^2-1\right)^{-1/4}e^{-2\pi i\xi\cdot x}\,d\xi\\&=\int_1^{\sqrt{2}}\left(\rho^2-1\right)^{-1/4}\int_{\Sigma_1}e^{-2\pi i\rho\omega\cdot x}\,d\omega\,\rho d\rho\\&=2\pi\int_1^{\sqrt{2}}\left(\rho^2-1\right)^{-1/4}J_0(2\pi\rho\,|x|)\rho d\rho\\&=2\pi\int_0^{\sqrt{2}-1}\rho^{-1/4}\left(\rho+2\right)^{-1/4}J_0(2\pi(\rho-1)\,|x|)(\rho+1)d\rho\\&=\pi\int_0^{\sqrt{2}-1}\rho^{-1/4}\left(\rho+2\right)^{-1/4}\\&\times\left\{\left((\rho-1)\,|x|\right)^{-1/2}\left(e^{i(2\pi(\rho-1)|x|-\pi/4)}+e^{-i(2\pi(\rho-1)|x|-\pi/4)}\right)\right.\\&\left.+\;O\left(\rho^{-3/2}\,|x|^{-3/2}\right)\right\}\,d\rho.\end{aligned}$$

It is enough to deal with

$$\begin{aligned}&|x|^{-1/2}\,e^{-i2\pi|x|}\int_0^{\sqrt{2}-1}\rho^{-1/4}e^{i2\pi\rho|x|}\,d\rho\\&=|x|^{-5/4}\,e^{-i2\pi|x|}\int_0^{\left(\sqrt{2}-1\right)|x|}\rho^{-1/4}e^{i2\pi\rho}\,d\rho\\&=|x|^{-5/4}\,e^{-i2\pi|x|}B(|x|).\end{aligned}$$

Now we are in position to appeal to the proof of Theorem 1 in [1], which yields boundedness from $L^{3/2}$ to $L^2$. Alternatively, we could prove the $p = \frac{3}{2}$, $s = 2$ bound directly by using a method similar to Bak's. Indeed, using the measure $(z+1)\ t_+^z dt$ on $\Gamma$, we get a $s = 2$, $p = 2$ bound by Plancherel's theorem if $\operatorname{Re}(z) = 1$. If $\operatorname{Re}(z) = -1$, we just get the $L^2$ norm of the restriction to the circle, so the $p = \frac{6}{5}$, $s = 2$ bound follows by Stein-Tomas. By analytic interpolation we recover $p = \frac{3}{2}$, $s = 2$ for $z = 0$.

The sharpness follows from the argument in [3, p. 231] or from the proof of the next theorem.

In order to prove $(iii)$ we point out that $\int_{\gamma_A} \int_{SO(2)} |T_A(f(r,\Theta)|^2\, dr d\Theta$ approximately equals the $L^2$-norm of $f * \widehat{\chi}_D$ where $D$ is the annulus

$$\left\{ \sqrt{a_1^2 + a_2^2} \leq |\xi| \leq \sqrt{a_1^2 + (a_2+1)^2} \right\}.$$

Then the positive result follows from Plancherel theorem, while the necessity follows from a lemma of Hörmander concerning translation invariant operators on $\mathbb{R}^n$ ([12]).

■

**Proof of Theorem 6.** The first case reduces to case $(i)$ in the previous theorem. Indeed we may assume

$$\Gamma = \left\{ (v, h(v)) \in \mathbb{R}^2 : 0 \leq v \leq 1,\ h(0) = h'(0) = 0 \right\}$$

and we note that

$$\int_\Gamma \int_{SO(2)} |T_\gamma(f(v,\Theta)|^2\, dv d\Theta \approx \int_{|\xi|\leq 1} \left| \widehat{f}(\xi) \right|^2 |\xi|^{-1}\, d\xi$$

since $|\xi| \approx v$.

$iia)$ is the familiar restriction restriction theorem (see [26] or [25])

As for $(iib)$ we write $\xi = \Theta \cdot (v, h(v))$ and the Jacobian of this transformation is $\approx v^{N-1}$, while $|\xi|^2 - 1 \approx v^N$. Then there exist two positive constants $c_1$ and $c_2$ such that

$$\int_\Gamma \int_{SO(2)} |T_\Gamma(f(v,\Theta)|^2\, dv d\Theta \approx \int_{c_1 \leq |\xi| \leq c_2} \left| \widehat{f}(\xi) \left( |\xi|^2 - 1 \right)^{\frac{1-N}{2N}} \right|^2 d\xi$$
$$\leq \|f\|_{L^{6N/(5N-2)}(\mathbb{R}^2)}$$

by the same argument as in part $(ii)$ of the previous proof. The bound is sharp by the result in [3] or by the following argument.

Consider, for a small $\delta > 0$, the rectangles of size $[-\delta, \delta] \times [-\delta^2, \delta^2]$ with the midpoints on the circle $\left\{ \left( x, \sqrt{1 - x^2} \right) \right\}$ and the long sides parallel to the tangents to the circle at the above points. By the assumptions on $h$, the intersections of these rectangles with $\Gamma$ has length

$\approx \delta$ if $|x| \leq c\delta^{2/N}$. This implies

$$\frac{1}{2} + \frac{1}{N} \geq 3\left(1 - \frac{1}{p}\right)$$

i.e. $p \leq \frac{6N}{5N-2}$.

(*iii*) is similar to part (*iii*) in the previous theorem.

■

**Remark 1.** *In particular, the above theorem shows that if* $\Gamma$ *is a segment and the point of rotation is not on the segment, then (1.3) holds with* $s = 2$, $p = \frac{3}{2}$. *If the point of rotation is on the segment we get (1.3) with* $s = 4$, $p = \frac{4}{3}$. *Finally, if the point of rotation is not contained in the strip perpendicular to* $\Gamma$, *we proved that (1.3) holds with* $s = 2$, $p = 2$. *In light of this, it is reasonable to ask if the* $s = 2$, $p = \frac{3}{2}$ *estimate can be improved to, say,* $s = 3$, $p = \frac{3}{2}$ *estimate, since then the points* $\left(\frac{3}{4}, \frac{1}{4}\right)$, $\left(\frac{2}{3}, \frac{1}{3}\right)$, *and* $\left(\frac{1}{2}, \frac{1}{2}\right)$ *would lie on a line. The following counterexample shows that this is not the case, or, more precisely, that if* $p = \frac{3}{2}$, *then* $s \leq 2$.

*Indeed, let* $\widehat{f_\delta}$ *denote the characteristic function of the* $\delta$ *by* $\delta^2$ *rectangle containing the segment* $\Gamma$ *in its interior, with the long side of the rectangle running parallel to* $\Gamma$. *Then* $\|f_\delta\|_p \approx \delta^{\frac{3}{p'}}$, *while the left hand side of (1.3)* $\approx \delta^{\frac{1}{s}}\delta^{\frac{1}{2}}$. *(Note that the calculation uses the fact that the point of rotation is not on* $\Gamma$ *and that the circle has contact of order* 2 *with a line segment, hence the dimensions of the rectangle). It follows that*

$$\frac{3}{p'} \leq \frac{1}{s} + \frac{1}{2},$$

*and since* $p = \frac{3}{2}$, *it follows that* $s \leq 2$.

The following result extends the $s = 2$ theorem for $\Gamma$ a segment, with the point of rotation on $\Gamma$, to the case when $\Gamma$ is a fractal subset of a segment.

**Theorem 8.** *Let* $E$ *be a closed subset of* $(0, 1)$. *Let* $N(E, \delta)$ *denote the minimal number of intervals of length* $2^{-l}$ *needed to cover* $E$. *Suppose that the upper Minkowski dimension of* $E$ *is* $\alpha$, *i.e*

$$\lim_{\delta \downarrow 0} \sup \frac{\log(N(E, \delta))}{-\log(\delta)} = \alpha,$$

*let* $\Gamma = E$ *equipped with Hausdorff measure* $d\mu$, *and suppose that the center of rotation is on* $\Gamma$. *Then (1.3) holds for* $p \leq \frac{4}{4-\alpha}$.

**Proof.** There is no harm in assuming that the center of rotation is the origin and that the segment is of the form $\{(v,0),0\leq v\leq 1\}$. It suffices to show that

$$\frac{2^j}{N(E,2^{-j})}\int\int_{E_j}\left|\widehat{f}\left(v\cos(\theta),v\sin(\theta)\right)\right|^2 dvd\theta\leq C\left\|f\right\|^2_{\frac{4}{4-\alpha}}$$

with constants independent of $j$, where $E_j$ denotes the union of consecutive diadic intervals that contain at least one point of $E$. After a change of variables $\xi=(v\cos(\theta),v\sin(\theta))$, replacing $|\xi|^{-1}$ by $|\xi|^{-2+\alpha}$, (recall that $\alpha\leq 1$), followed by the change of variables $\nu=2^j\xi$, we get

$$\frac{1}{N(E,2^{-j})}\int_{|\xi|\in 2^jE_j}\left|\widehat{f_j}(\xi)\right|^2|\xi|^{-2+\alpha}d\xi,$$

where $f_j(x)=2^{2j}f(2^jx)$. Using fractional integration and the identity $\|2^{2j}f(2^j\cdot)\|_p^2=2^{4j}2^{-\frac{4j}{p}}\|f\|_p^2$, we get

$$\frac{1}{N(E,2^{-j})}2^{4j}2^{-\frac{4j}{p}}\|f\|_p\approx 2^{-j\alpha}2^{4j}2^{-\frac{4j}{p}}\|f\|_p.$$

Since $p=\frac{4}{4-\alpha}$ by fractional integration, the quantity above is uniformly controlled by $\|f\|_p$ and the proof is complete.

■

**Remark 2.** *At least in the case when* $\alpha\leq\frac{1}{2}$, *it is not hard to show that the* $L^2$ *average decay of the Fourier transform of the fractal measure* $d\mu$ *above is of order* $-\frac{\alpha}{2}$. *However this information alone is not enough to get the sharp result since the average decay lemma only gives us* $v=2\alpha+4$, $p=\frac{2(\alpha+2)}{\alpha+4}$.

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