ON AVERAGING OPERATORS ASSOCIATED WITH CONVEX HYPERSURFACES OF FINITE TYPE

Alexander Iosevich   Eric Sawyer   Andreas Seeger

1. Introduction

Let Σ be a smooth convex hypersurface in $\mathbb{R}^d$, $d \geq 3$, and denote surface measure on Σ by $d\sigma$. Let χ be a compactly supported C^∞ function and let

$$\widetilde{\Sigma} = \text{supp } \chi \cap \Sigma.$$

For $t > 0$ define the convolution operator $\mathcal{A}_t$ by

$$\mathcal{A}_t f(x) = \int f(x - ty')\chi(y')d\sigma(y') \tag{1.1}$$

and an associated maximal function

$$\mathcal{M}f(x) = \sup_{t>0} |\mathcal{A}_t f(x)|. \tag{1.2}$$

The main issues in this paper are the L^p boundedness of the maximal operator $\mathcal{M}$ and the regularity properties of the averaging operator $\mathcal{A} \equiv \mathcal{A}_1$.

Stein [22] showed that if Σ is a $(d-1)$-dimensional sphere in $\mathbb{R}^d$, $d \geq 3$, then $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^d)$ for $p > d/(d-1)$ and unbounded for $p \leq d/(d-1)$. Greenleaf [11] proved similar results under the conditions on the decay of the Fourier transform $\widehat{d\sigma}$. In particular if Σ is a hypersurface and the Gaussian curvature of Σ does not vanish, one obtains the same result as for the sphere. The two dimensional version of Stein's result was proved by Bourgain [1].

If the Gaussian curvature is allowed to vanish one would like to determine the best possible value of p_0 such that L^p boundedness holds for $p > p_0$. Cowling and Mauceri [7] showed that there are surfaces where $p_0 \in (2, \infty)$ and Sogge and Stein [21] showed that such $p_0 < \infty$ exists if the Gaussian curvature is assumed to vanish of only finite order. The extension of Bourgain's result to plane curves of finite type was obtained in [12] using scaling; this method does not readily apply in higher dimensions.

In this paper we consider a *convex* surface Σ of *finite line type* in $\mathbb{R}^d$, $d \geq 3$, *i.e.* it is assumed that each tangent line has finite order of contact. Bruna, Nagel and Wainger [2] expressed the decay of the Fourier transform $\widehat{d\sigma}$ using the caps

$$B(x, \delta) = \{y \in \Sigma : \text{dist}(y, H_x(\Sigma)) < \delta\};$$

here $H_x(\Sigma)$ denotes the tangent plane at $x \in \Sigma$ (considered as an affine subspace of $\mathbb{R}^d$ passing through x). The estimate is

$$|\widehat{d\sigma}(\xi)| \leq C\big[|B(x_+, |\xi|^{-1})| + |B(x_-, |\xi|^{-1})|\big]$$

Research supported in part by NSF grants.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

where $x_\pm$ are the points on Σ for which ξ is a normal vector and $|B|$ denotes the surface measure of B. The behavior of the maximal operator $\mathcal{M}$ is not just determined by the size of the balls of given height δ, but also by the number of balls of height δ and fixed diameter $\gg \delta^{(d-1)/2}$. Taking this into account Nagel, Wainger and the third author [18] proved maximal theorems on $\mathbb{R}^d$, $d \geq 3$, using the quantity

$$\Gamma_r(\delta) = \Big(\int_{\widetilde{\Sigma}} |B(x,\delta)|^{r-1} d\sigma(x)\Big)^{1/r} \tag{1.3}$$

for $r > 1$. Note that if $\sup_x |B(x,\delta)| = O(\delta^a)$ then $\Gamma_r(\delta) = O(\delta^{a(1-1/r)})$; however if $a < (d-1)/2$ then $\Gamma_r(\delta)$ tends to be significantly smaller. The first theorem in [18] addresses the case $p > 2$. Suppose that

$$\int_0^1 \delta^{-1/p} \Gamma_{\frac{p}{p-2}}(\delta) \frac{d\delta}{\delta} \leq A < \infty \quad \text{and} \quad p > 2 \tag{1.4}$$

then $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^d)$.

Another theorem was proved by the first two authors in [14] and [13], completely settling the case $p > 2$. Namely let $d(y, H_x(\Sigma))$ be the distance of $y \in \Sigma$ to the tangent plane $H_x(\Sigma)$ through x; then the maximal operator $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^d)$, for $p > 2$, if

$$d(\cdot, H_x(\Sigma))^{-1/p} \in L^1(\widetilde{\Sigma}) \tag{1.5}$$

for every $x \in \widetilde{\Sigma}$; conversely, the condition (1.5) at points with $\chi(x) \neq 0$ is necessary for L^p boundedness. In §4 we shall use a variant of the argument in [14] to show that the sufficiency of (1.5) actually follows from the sufficiency of (1.4). It follows a posteriori that for $p > 2$ the L^p boundedness of $\mathcal{M}$, the finiteness of the integral (1.4) and the condition (1.5) are equivalent if Σ is closed and $\chi \equiv 1$.

We remark that the hypothesis (1.4) implies L^p boundedness for a class of convex hypersurfaces, with the L^p bounds depending only on A and certain admissible constants (for the definition of admissibility see §2). On the other hand, for a single convex body the assumption (1.5) is often easier to verify.

The analogue of (1.4) for $p < 2$ is the condition

$$\int_0^1 [\log(1+\delta^{-1})]^{\frac{1}{p}-\frac{1}{2}} \delta^{-\frac{1}{p}} \Gamma_{\frac{p}{2-p}}(\delta) \frac{d\delta}{\delta} < \infty; \tag{1.6}$$

if $p < 2$ and (1.6) is satisfied then $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^d)$. This statement is (implicitly) contained in [18] (*cf.* Theorem 2.5 below). Note that if the curvature does not vanish then $|B(x,\delta)| \approx \delta^{(d-1)/2}$ and $\Gamma_{\frac{p}{2-p}}(\delta) \approx \delta^{(d-1)(1-1/p)}$ so the integral (1.6) converges if and only if $p > d/(d-1)$, which is Stein's maximal theorem. The nonvanishing of the curvature is not necessary; as one can see by checking (1.6) for surfaces of the form

$$x_d = -c + \sum_{i=1}^{d-1} |x_i|^{a_i}, \qquad 2 \leq a_1 \leq \cdots \leq a_{d-1}, \tag{1.7}$$

where the a_i are even integers. In this case L^p boundedness holds for $p > d/(d-1)$ if $a_i \leq d$ for $i = 1, \dots, d-1$. In §2 a related result will be deduced from (1.6) in §2; namely L^p boundedness holds if the Gaussian curvature belongs to $L^\gamma(\widetilde{\Sigma})$ for all $\gamma < 1/(d-2)$.

It is not presently known whether for $p < 2$ the condition (1.6) always gives the correct range of L^p boundedness up to endpoints. Moreover it is not known precisely how (1.7) relates to the notions of type and multitype. One purpose of this paper is to prove some partial results in this direction and obtain a fairly complete picture in three dimensions.

In order to formulate our results we now review the definitions of type and multitype. For convex hypersurfaces in $\mathbb{R}^d$ a natural notion of multitype has been implicitly introduced by Schulz [20]. Various related and more general notions of multitype had been previously formulated in complex analysis, see in particular Catlin's paper [3]; later Yu [26] has given a simple formulation of Catlin's multitype condition for convex domains in $\mathbb{C}^n$, building on the results in [20].

We first consider a smooth real valued function Φ defined in a neighborhood of the origin in an n-dimensional vector space $\mathbb{E}_n$ so that $\Phi(0) = \nabla\Phi(0) = 0$. We say that a vector v in $\mathbb{E}^n$ has contact of order $m+1$ if

$$\Phi(sv) = O(s^{m+1}) \quad \text{if} \quad s \to 0.$$

Let

$$S^m = \{v \in \mathbb{E}^n : \ v \text{ has contact of order } m+1.\} \tag{1.8}$$

It is shown in [20] that S_m is a linear subspace of $\mathbb{E}^n$ and that there are even integers $m_1, \dots, m_k$ so that $m_1 < \dots < m_k$, $1 \le k \le n$ and $m_0 := m_1 - 1 \ge 1$ and

$$0 = S^{m_k} \subsetneq \cdots \subsetneq S^{m_0} := \mathbb{E}^n;$$

and the sequence is maximal, *i.e.*

$$S^m = S^{m_k} \text{ if } m_{k-1} < m \le m_k.$$

The largest number m_k is the *type* of Φ at 0. Let $\dim S^{m_i} = n_i$, so that $n_0 = n$ and $n_k = 0$. For $i = 1, \dots, n$ let

$$a_i = m_j \quad \text{if} \quad n - n_{j-1} < i \le n - n_j, \quad j = 1, \dots, k;$$

the n-tuple $\mathfrak{a} = (a_1, \dots, a_n)$ is then called the *multitype* of Φ at 0. Clearly this definition is independent of the *linear* coordinate system on $\mathbb{E}_n$.

Now let Σ be a convex hypersurface in $\mathbb{R}^d$ and let $P \in \Sigma$. Then near P the surface is a graph over its tangent plane at P. For a suitable choice of the unit normal vector n_P at P the surface can be parametrized by

$$\begin{aligned} &T_P\Sigma \to \mathbb{R} \\ &v \mapsto P + v + \Phi(v) n_P \end{aligned} \tag{1.9}$$

where Φ is a convex function vanishing of second order at the origin. We say that Σ is of multitype $\mathfrak{a} = (a_1, \dots, a_{d-1})$ at P if Φ has multitype $\mathfrak{a}$ at the origin. This notion is invariant under affine transformations in $\mathbb{R}^d$. Moreover, if Σ is given as a graph $w_n = \Psi(w')$ then it is easy to see that the multitype at $P = (w', \Psi(w'))$ is equal to the multitype of the function

$$y' \mapsto \Psi(w' + y') - \Psi(w') - \langle y', \nabla_{w'} \Psi(w') \rangle.$$

Calculations in [18] on examples of the form (1.7) suggest the following

Conjecture: Let Σ be a convex surface in $\mathbb{R}^{d-1}$, let $P \in \Sigma$ and let $\mathfrak{a} = (a_1, \dots, a_{d-1})$ be the multitype at P. Define ν_k by

$$\nu_k = \sum_{j=k}^{d-1} \frac{1}{a_j}, \quad k = 1, \dots, d-1; \qquad \nu_d = 0. \tag{1.10}$$

We conjecture that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^d)$ if the support of χ is contained in a sufficiently small neighborhood of P and if

$$p > \max_{k=1,\dots,d} \frac{k}{k-1+\nu_k}. \tag{1.11}$$

Note that among the numbers (1.11) only the one corresponding to $k=1$ can be ≥ 2 so that the condition for L^p boundedness for $p>2$ reduces to

$$p > \sum_{j=1}^{d-1} \frac{1}{a_i}. \tag{1.12}$$

As observed in [14] the condition (1.12) is equivalent to the integrability condition (1.5) for $x = P$ so that the equivalence of (1.5) and (1.4) mentioned above amounts to the equivalence of (1.12) and (1.4). More generally, one may also conjecture that L^p boundedness holds if for every $l \in \{0, 1, \dots, d-1\}$ and for every l-plane E through P the function $x \mapsto [\mathrm{dist}(x, E)]^{-1}$ belongs to $L^{(d-l)/p}$ (here the 0-plane through P is just $\{P\}$).

In the present paper we shall concentrate on the simplest case, $d = 3$.

Theorem 1.1. *Let Σ be a smooth convex hypersurface of finite line type in $\mathbb{R}^3$. Let $P \in \Sigma$, let $\mathfrak{a} = (a_1, a_2)$ be the multitype at P and let $K(x)$ be the Gaussian curvature at x.*

Let $\mathcal{M}$ be the maximal operator as defined in (1.2). There is a neighborhood U of P in Σ so that the following statements hold if χ is supported in U.

(i) Suppose that $a_1 > 2$. Then $\mathcal{M}$ is bounded if and only if $p > (\frac{1}{a_1} + \frac{1}{a_2})^{-1}$.

(ii) Suppose that $a_1 = 2$, $0 < \gamma < 1$ and $K^{-\gamma} \in L^1(U)$. Then $\mathcal{M}$ is bounded if $p > \frac{2a_2(1-\gamma)+2+4\gamma}{a_2(1-\gamma)+2+2\gamma}$.

(iii) If $a_1 = 2$ then $\mathcal{M}$ is bounded for $p > \max\{\frac{3}{2}, \frac{2a_2}{a_2+1}\}$.

We note that (i) is already contained in [14], but we shall give a different proof in §4 by deducing it from (1.4). Also note that (i) and (iii) together verify the above conjecture in three dimensions; however there are cases where (ii) gives a better result (see §4). Statement (iii) follows from statement (ii) by using

Theorem 1.2. *Let Σ be a smooth convex hypersurface of finite line type $\leq m$ in $\mathbb{R}^3$, and let K be the Gaussian curvature function on Σ. If $\gamma < (m-2)^{-1}$ then $K^{-\gamma}$ is locally integrable on Σ.*

We now discuss the regularity properties of the averaging operator $\mathcal{A} = \mathcal{A}_1$. A positive and apparently quite precise result for Besov spaces[1] $B^p_{\alpha,q}$ and Sobolev spaces L^p_α can be formulated in terms of the balls $B(x,\delta)$, using a condition similar to (1.4), (1.6).

Theorem 1.3. *Suppose $\Sigma \subset \mathbb{R}^d$ is convex, smooth and of finite line type. Suppose that $1 \leq p \leq 2$ and suppose that*

$$\sup_{\delta>0} \delta^{\frac{1}{q}-\frac{1}{p}-\alpha} \Big(\int_{\tilde{\Sigma}} |B(x,\delta)|^{\frac{2q(p-1)}{p+q-pq}} d\sigma(x) \Big)^{\frac{1}{p}+\frac{1}{q}-1} < \infty \tag{1.13}$$

holds for some (p,q) with $p \leq q$. Then $\mathcal{A}$ maps the Besov space $B^p_{\beta,r}$ boundedly to $B^q_{\beta+\alpha,r}$.

[1]Recall that $\|f\|_{B^p_{\beta,r}} \approx (\sum_{k=0}^\infty [2^{k\beta}\|\mathcal{L}^k f\|_p]^r)^{1/r}$ with suitable Littlewood-Paley cutoffs $\mathcal{L}^k$ localizing frequencies to annuli $|\xi| \approx 2^k$ if $k > 0$.

Moreover, if $1 < p \le 2$, $p \le q < \infty$, then $\mathcal{A}$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q_\alpha(\mathbb{R}^d)$ if $q \ge 2$ and bounded from $L^p(\mathbb{R}^d)$ to $L^{q-\varepsilon}_\alpha(\mathbb{R}^d)$ if $p \le q \le 2$.

Clearly the second assertion about Sobolev estimates is a consequence of the first assertion for Besov spaces, by standard embedding theorems (*i.e.* Littlewood-Paley inequalities).

Again one can try to relate the condition (1.13) to the multitype. Consider the model example (1.7) where $a_1 \le \cdots \le a_{d-1}$ are even integers, ν_k as in (1.10). We note that for this example a complete description of the $L^p \to L^q$ estimates for $\mathcal{A}$ has been given by Ferreyra, Godoy and Urciuolo [10] (without the restriction that the a_i are even integers), see also the paper by Sang Hyuk Lee [16]. Both proofs relied on a method introduced by Christ [5].

A calculation for the model example shows that (1.13) is satisfied when

$$\alpha \le \min_{1\le k\le d} \left[\nu_k + k - 1 - \frac{k+\nu_k}{p} + \frac{\nu_k+1}{q}\right], \tag{1.14}$$

see §3. For $\alpha = 0$ this becomes $\frac{1}{q} \ge \frac{\nu_k+k}{\nu_k+1}\frac{1}{p} - \frac{\nu_k+k-1}{\nu_k+1}$; this is the condition given in [10]. Concerning the case $p = q$ one obtains (for the model example) that $\mathcal{A}$ is bounded from $B^p_{\beta,r}$ to $B^p_{\beta+\alpha,r}$ and from $B^{p'}_{\beta,r}$ to $B^{p'}_{\beta+\alpha,r}$ provided that $\alpha \le \nu_{k+1} + k/p$, if $a_k \le p \le a_{k+1}$.

To formulate a conjecture for $L^p_\beta \to L^q_{\beta+\alpha}$ regularity (or related Besov-type estimates) in the general case one simply replaces $(a_1, \dots, a_{d-1})$ in the model example by the multitype at P and assumes that χ has small support near P. Then (1.14) should imply the $L^p \to L^q_\alpha$ for the averaging operator, if $p < q$, and $1 < p \le 2$. Clearly by duality the boundedness region is symmetric with respect to the diagonal $1/p + 1/q = 1$, so it suffices to consider the case $p \le 2$. One expects that at least for the case $p = q$ boundedness may fail at the vertices of the boundedness region, see [6] for counterexamples in two dimensions. We note that complete $L^p \to L^q$ results in two dimensions are in [19], [5].

In three dimensions we prove the conjecture up to certain endpoint results.

Theorem 1.4. *Let Σ be a smooth compact convex hypersurface of finite line type in $\mathbb{R}^3$, let $P \in \Sigma$ and let $\mathfrak{a} = (a_1, a_2)$ be the multitype of Σ at P. Let $\nu_1 = a_1^{-1} + a_2^{-1}$, $\nu_2 = a_2^{-1}$ and let $\mathcal{T}(P)$ be the set of all $(\frac{1}{p}, \frac{1}{q}, \alpha)$ with $p \le q$ satisfying the conditions*

$$\alpha \le \nu_1 - \frac{1+\nu_1}{p} + \frac{1+\nu_1}{q} \tag{1.15.1}$$

$$\alpha < \nu_2 + 1 - \frac{2+\nu_2}{p} + \frac{1+\nu_2}{q} \tag{1.15.2}$$

$$\alpha \le 2 - \frac{3}{p} + \frac{1}{q} \tag{1.15.3}$$

and

$$\alpha \le \nu_1 + \frac{1+\nu_1}{q} - \frac{1+\nu_1}{p} \tag{1.16.1}$$

$$\alpha < \nu_2 + \frac{2+\nu_2}{q} - \frac{1+\nu_2}{p} \tag{1.16.2}$$

$$\alpha \le \frac{3}{q} - \frac{1}{p} \tag{1.16.3}$$

Then there is a neighborhood U of P such that $\mathcal{A}$ is bounded from $B^p_{\beta,r}(\mathbb{R}^3)$ to $B^q_{\beta+\alpha,r}(\mathbb{R}^3)$ if $\operatorname{supp} \chi \in U$ *and $(1/p, 1/q, \alpha)$ belongs to $\mathcal{T}(P)$.*

Moreover $\mathcal{A}$ is bounded from $L^p_\beta(\mathbb{R}^3)$ to $L^q_{\beta+\alpha}(\mathbb{R}^3)$ if $(1/p, 1/q, \alpha)$ belongs to the interior of $\mathcal{T}(P)$.

Remark 1.5. If $p \le 2 \le q$ and the $B^p_{0,r} \to B^q_{\alpha,r}$ estimate holds for a given p, q with $p \le 2 \le q$ then the $L^p_\beta \to L^q_{\alpha+\beta}$ estimate follows; this yields partial endpoint results for the Sobolev estimates.

Remark 1.6. (i) A natural conjecture for $L^p \to L^q$ estimates is given in terms of distances to tangent lines and planes. Let $\Delta_j(p,q) = 1/p - j/q$ and $\rho_l(d,p,q) = d - l - 1 + \Delta_{d-l}/(1-\Delta_1)$. Suppose that for $l = 0, 1, \dots, d-1$ and for all l-planes E through P the functions $x \mapsto [\text{dist}(x, E)]^{-1}$ belong to $L^\rho(\Sigma)$ for $\rho = \rho_l(d,p,q)$. One may conjecture that $\mathcal{A}$ maps $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ (provided, of course, that χ is supported in a sufficiently small neighborhood of P).

If $d = 3$ then the description of multitype together with estimates in §3 can be used to show that the above conditions are equivalent with the conditions given in Theorem 1.4.

(ii) It is easy to see that the condition for $l = d-1$ in (i) is necessary, by testing $\mathcal{A}$ on characteristic functions of cylinders with base $B(P, \delta)$ and height δ.

(iii) Analogously, one can formulate a conjecture for the L^p boundedness of the maximal operator in terms of distances to tangent planes and lines. The conjecture is that $\mathcal{M}$ maps $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ if for $l = 0, \dots, d-1$ and for all l-planes E through P the functions $x \mapsto [\text{dist}(x, E)]^{-1}$ belong to $L^{\frac{d-l}{p}}(\Sigma)$.

The paper is organized as follows. In §2 we shall derive estimates for operators associated to certain classes of convex functions, emphasizing uniformity of these estimates. In §3 we shall discuss various properties of the multitype and the associated scaling; in particular we prove versions of Theorem 1.2. The proofs of Theorems 1.1 and 1.4 are contained in §4, and some examples are considered in §5.

2. Operators associated to convex functions of finite line type

In this section we collect facts which are either immediate consequences of estimates for classes of convex functions of finite type in [2], [9] or [18], or can be obtained by modifications of arguments in those papers.

Let $B_T \subset \mathbb{R}^n$ denote the open ball of radius T centered at 0. In what follows it is always assumed that $T \le 1$. For $0 < b \le M$, $N \in \mathbb{Z}^+$, $2 \le m < N$, let $\mathcal{S}^n_T(b, M, m, N)$ be the class of all $C_N(\overline{B_T})$ functions g with the property that for all $x \in B_T$

$$
\begin{aligned}
&g(0) = \nabla g(0) = 0 \\
&\frac{d^2}{(dt)^2} g(x + t\theta)\big|_{t=0} \ge 0 \text{ for all } \theta \in S^n \\
&\max_{2 \le j \le m} \Big| \big(\frac{d}{dt}\big)^j g(x + t\theta)\big|_{t=0} \Big| \ge b \text{ for all } \theta \in S^n \\
&\max_{|\alpha| \le N} \Big| \big(\frac{\partial}{\partial x}\big)^\alpha g(x) \Big| \le M
\end{aligned}
\tag{2.1}
$$

Next let $\mathfrak{a} = (a_1, a_2, \dots, a_n)$ an n-tuple with even integers so that $2 \le a_1 \le \dots \le a_n$. We define $\mathcal{S}^n_T(b, M, \mathfrak{a}, N)$ to be the class of all functions in $\mathcal{S}^n_T(b, M, a_n, N)$ with the property that

$$
\max_{2 \le j \le a_i} \Big| \big(\frac{\partial}{\partial x_i}\big)^j g(x) \Big| \ge b. \tag{2.2}
$$

We also set

$$
\nu_k = \sum_{j=k}^n \frac{1}{a_j}, \quad k = 1, \dots, n, \quad \text{and} \quad \nu_{n+1} = 0. \tag{2.3}
$$

We note that if Σ is convex and of finite line type and if $P \in \Sigma$ is of multitype $\mathfrak{a}$ then there is a neighborhood of P in Σ where Σ can be parametrized by (1.9) and so that $\Phi \circ L \in \mathcal{S}_T^{d-1}(b, M, \mathfrak{a}, N)$ for a rotation L and suitable constants T, b, M.

Constants in estimates which will depend only on the parameters n, b, M, m or $\mathfrak{a}$, N are called *admissible*. All constants in this section will be admissible, but statements involving the multitype in §3 and §4 below will contain "nonadmissible" constants.

Notice that if $\Phi \in \mathcal{S}_{2T}^n(b, M, m, N)$ the functions

$$w \mapsto \Phi(y+w) - \Phi(y) - \langle w, \nabla\Phi(y)\rangle$$

belong to the class $\mathcal{S}_T^n(b, 3M, m, N)$ for all $|y| \le T$. A similar remark applies to the class $\mathcal{S}_{2T}^n(b, M, \mathfrak{a}, N)$.

We now recall an important inequality from [2] (see also variants in [9], [18]). Let $|w| \le T$ and let

$$P_{w,y}(s) = \sum_{j=2}^{m} \frac{1}{j!}\langle w, \nabla\rangle^j \Phi(y) \frac{s^j}{j!} + M\frac{s^{m+1}}{(m+1)!}$$

$$\widetilde{P}_{w,y}(s) = \sum_{j=2}^{m} \frac{1}{j!}|\langle w, \nabla\rangle^j \Phi(y)| \frac{s^j}{j!} + M\frac{s^{m+1}}{(m+1)!}$$

Then there exists an admissible constants C_1, so that for $|y| \le T$, $|w| \le T$, $0 \le s \le 1$,

$$C_1^{-1}\widetilde{P}_{w,y}(s) \le \Phi(y+sw) - \Phi(y) - \langle w, \nabla\Phi(y)\rangle \le C_1 P_{w,y}(s). \tag{2.4}$$

Notice that by (2.4) there exists an admissible constant $c_0 > 0$ so that for all

$$\delta \le c_0 T^m =: \delta_0 \tag{2.5}$$

the sets

$$\mathcal{B}(x,\delta) = \{y : \ |y| \le T; \ |\Phi(y) - \Phi(x) - \langle \nabla\Phi(x), y-x\rangle| \le \delta\} \tag{2.6}$$

are contained in $\{|x| \le 2T\}$. If $\Sigma = \text{graph}(\Phi)$ then these sets are comparable to projections of the balls $B(y, \delta)$ defined in the introduction.

Proposition 2.1. *Let $\Phi \in \mathcal{S}_{2^nT}^{d-1}(b, M, \mathfrak{a}, N)$, $m = a_n$, $N \ge m+1$. There are admissible constants $C_1, \dots, C_5$, $\mu_1 \ge 1$, $C_0 > c_0^{-1}$, so that the following statements hold.*

(i) Let $1 \le l \le n$ and let E be an l-plane through the origin. Let $\delta \le C_0^{-1}T^m$, $\mu_1 \le \mu \le C_1^{-1}\delta^{-1/m}$. Then for all $|w| \le T$ the set $\mathcal{B}(w, \mu\delta)$ is contained in $\{|w| \le 2T\}$. Moreover if $V_E(x, w_0, \mu\delta)$ is the l-dimensional volume of the cross sections $(x+E) \cap \mathcal{B}(w_0, \mu\delta)$, then for $w_1, w_2 \in \mathcal{B}(w_0, \delta)$ one has

(2.7)
$$C_2^{-1}\Big(\frac{\mu}{\mu_1}\Big)^{l/m} V_E(w_1, w_0, \mu_1\delta) \le V_E(w_1, w_0, \mu\delta) \le C_3 V_E(w_2, w_0, \mu\delta) \le C_2C_3\Big(\frac{\mu}{\mu_1}\Big)^{l/2} V_E(w_2, w_0\mu_1\delta).$$

(ii) Let $\delta \le C_0^{-1}T^m$, and let $\mathcal{B}(x,\delta)$ be as in (2.6), ν_k as in (2.3). Then for $|x| \le T$,

$$|\mathcal{B}(x,\delta)| \le C_4\delta^{\nu_1}. \tag{2.8}$$

(iii) For $k = 1, \dots, n$ let

$$K_k(x) = \det \begin{pmatrix} \Phi_{x_1x_1} & \dots & \Phi_{x_1x_k} \\ \vdots & & \vdots \\ \Phi_{x_kx_1} & \dots & \Phi_{x_kx_k} \end{pmatrix}. \tag{2.9}$$

Then for $\delta \le C_0^{-1}T^m$

$$|\mathcal{B}(x,\delta)| \le C_5 \delta^{\frac{k}{2}+\nu_{k+1}} \big[\sup_{y\in\mathcal{B}(x,\delta)} |K_k(y)| \big]^{-1/2}. \tag{2.10}$$

Proof. The chain of inequalities (2.7) is an easy consequence of [18, Corollary 2.6] which in turn was based on (2.4). Inequality (2.8) is proved by induction over the dimension. It is true for $n = 1$ by (2.4). Let $n > 1$. Then again by (2.4) one sees that the set

$$\mathcal{J}(\delta) = \{x_n : \text{ there is } x' \in \mathbb{R}^{n-1} \text{ so that } (x', x_n) \in \mathcal{B}(x,\delta)\} \tag{2.11}$$

is contained in an interval of length $\le C\delta^{1/a_n}$. The functions $y' \mapsto \Phi(y, y_n)$ belong to $\mathcal{S}^{n-1}_{2^{n-1}T}(b, M, \mathfrak{a}', N)$, with $\mathfrak{a}' = (a_1, \dots, a_{n-1})$. By the induction hypothesis the $n-1$-dimensional slices through $\mathcal{B}(x,\delta)$ at height $y_n \in I$ have volume $\le C\delta^{1/a_1+\dots1/a_{n-1}}$. The assertion follows by integrating over $\mathcal{J}(\delta)$.

We now turn to the estimate (2.10), and consider first the case $k = n$. In [9] it is shown for arbitrary polynomials of degree $\le q+1$ that

$$\max_{|u|\le n} |\det P''(u)| \le C_{n,q} \max_{|u|\le 1} |P(u)|^n \tag{2.12}$$

where $C_{n,q}$ is an absolute constant. Now by estimates for functions in $\mathcal{S}^n_{2T}(b, M, m, N)$ ([2, §3]) there are constants c_0, C_0 and a polynomial $P_{\delta,x}$ of degree $\le m$, vanishing of second order at x, so that

$$\{y : P_{\delta,x}(y) \le c_0\delta\} \subset \mathcal{B}(x,\delta) \subset \{y : P_{\delta,x}(y) \le C_0\delta\}; \tag{2.13}$$

here the constants c_0, C_0 do not depend on x and δ. Following [9] we apply a result of John to wit there is a translation τ_{-x} and a symmetric positive definite linear transformation T so that $B(1) \subset T(\tau_{-x}\mathcal{B}(x,\delta) \subset B(n)$ where $B(1)$ and $B(n)$ denote the balls of radii 1 and n, centered at the origin. By (2.13)

$$\det T^{-1} \max_{|u|\le n} |\det P''_{\delta,x}(x_0 + T^{-1}u)|^{1/2} \le C_{n,m} \max_{|u|\le 1} |P_{\delta,x}(x_0 + T^{-1}u)|^{n/2}$$

and since $\det T^{-1}$ is comparable with the measure of $\mathcal{B}(x,\delta)$ the assertion follows for $k = n$.

To show (2.10) we argue by induction on n, the case $k = n$ is already taken care of. Let $n > k$. Pick $z \in \mathcal{B}(x,\delta)$ so that $K_k(z) \le 2\min_{y\in B(x,\delta)} K_k(y)$. Let $V_x(y_n,\delta)$ be the $n-1$ dimensional slice of $\mathcal{B}(x,\delta)$ at height y_n. Then by the induction hypothesis

$$\begin{aligned} |V_x(z_n, \mu_1\delta)| &\le C\delta^{k/2+\sum_{i=k+1}^{n-1} a_i^{-1}} \big[\max_{z':(z',z_n)\in\mathcal{B}(x,\mu_1\delta)} K_k((z', z_n)) \big]^{-1/2} \\ &\le C\delta^{k/2+\sum_{i=k+1}^{n-1} a_i^{-1}} \big[\max_{z:z\in\mathcal{B}(x,\delta)} K_k(z) \big]^{-1/2}; \end{aligned}$$

in this formula the sum in the exponent is not present when $k = n-1$. By (2.7)

$$V_x(y_n,\delta) \le V_x(y_n, \mu_1\delta) \le CV_x(z_n, \mu_1\delta)$$

and integrating over $y_n \in J(\delta)$ yields another factor of δ^{1/a_n}, as in the proof of (2.8). $\square$

We now let $n = d-1$ and consider the regularity properties of the following integral operator acting on functions in $\mathbb{R}^d$,

$$\mathcal{A}_t f(x) = \int f(x'-y', x_d - t(\Phi(y') + c_d))\chi(y')dy'. \tag{2.14}$$

Here $\Phi \in \mathcal{S}_r^{d-1}(b, M, m, N)$, and the smooth cutoff function χ is supported in $\{x' : |x'| \leq T\}$, $T < 2^{-d+1}r$. We shall not try to minimize smoothness and therefore always assume that N is large; by "large" we mean $N \geq 10dm$, which is assumed in the remainder of this section.

Our first result is an estimate for $\mathcal{A}_1$ after a localization in frequency space. Let $\delta > 0$ be small, and let $\beta \in C_0^\infty(\mathbb{R}^d)$ be supported in $\{\xi : 1/2 < |\xi| \leq 2\}$. Define $\mathcal{L}_\delta$ by

$$\widehat{\mathcal{L}_\delta f}(\xi) = \beta(\delta\xi)\widehat{f}(\xi). \tag{2.15}$$

Proposition 2.2.

Suppose that $1 \leq p \leq 2$ and $1/r = 1/p + 1/q - 1$. Then

$$\|\mathcal{L}_\delta \mathcal{A}_1 f\|_q \leq C\delta^{\frac{1}{q}-\frac{1}{p}} \Big(\int_{|w|\leq T} |\mathcal{B}(w,\delta)|^{r(1-\frac{1}{p}+\frac{1}{q})-1} dw\Big)^{1/r} \|f\|_p \tag{2.16}$$

for all $f \in L^p(\mathbb{R}^d)$.

Proof. The proof follows a pattern of [18] and we shall be brief. Observe $\mathcal{A}_1 f = d\mu * f$ where $d\mu$ is a smooth density on Σ. We split $d\mu = \sum_j d\mu_j$ where each $d\mu_j$ is supported in a cap $\mathcal{B}_j$ of height $\approx \delta$ and the caps (or "balls") have finite overlap. This splitting is done by using a partition of unity subordinated to the $\mathcal{B}_j$, see [2] for the metric properties of the caps and [18] for the necessary quantitative bounds for the partition of unity.

For sequences $\gamma = \{\gamma_j\}$ consider the bilinear operator

$$T_\delta[\gamma, f] = \sum_j \gamma_j L_\delta[d\mu_j * f].$$

The inequality (2.16) follows by choosing $\gamma = (1, 1, 1, \dots)$ from from the following more general estimate, valid for $p \leq 2$:

$$\|T_\delta[\gamma, f]\|_q \leq C\delta^{\frac{1}{q}-\frac{1}{p}} \Big(\sum_j [|\gamma_j|\, |B_j|^{1-\frac{1}{p}+\frac{1}{q}}]^r\Big)^{1/r} \|f\|_p, \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1. \tag{2.17}$$

Indeed (2.17) is clear for $p = 1 = q$, and also for $p = 1$, $q = \infty$ (where $r = \infty$). The nontrivial part is the case $p = 2 = q$ (again then $r = \infty$); but this estimate is a consequence of Theorem 2.2 in [18]. The general case follows by interpolation. $\square$

The next result is an immediate consequence, and also proves Theorem 1.3.

Corollary 2.3. *Suppose that $1 \leq p \leq 2$ and suppose that*

$$\sup_{\delta>0} \delta^{\frac{1}{q}-\frac{1}{p}-\alpha} \Big(\int_{\{|w|\leq T\}} |\mathcal{B}(w,\delta)|^{\frac{2q(p-1)}{p+q-pq}} dw\Big)^{\frac{1}{p}+\frac{1}{q}-1} \leq A < \infty \tag{2.18}$$

holds for some (p,q) with $p \le q$. Then $\mathcal{A}$ maps the Besov space $B^p_{\beta,r}(\mathbb{R}^d)$ boundedly to $B^q_{\beta+\alpha,r}(\mathbb{R}^d)$

Remark 2.4. For the model example (1.7), *i.e.* $x_d = c_d - \sum_{j=1}^{d-1} |x_j|^{a_j}$ one has

$$\int |\mathcal{B}(w,\delta)|^\eta dw \le C \max_{1\le k\le d} \delta^{(1+\eta)\nu_k + \frac{(k-1)\eta}{2}};$$

see [18, formula (5.2)]. From this the sharp estimates for the maximal operator have been deduced in [18]; moreover Corollary 2.3 implies that the averaging operator maps $B^p_{\beta,r}$ to $B^q_{\beta+\alpha,r}$ if $p \le 2$ and (1.14) is satisfied. Concerning $L^p \to L^q$ estimates this gives an extension to some endpoints of estimates in [], namely in the cases where $p \le 2 \le q$.

In order to prove the maximal Theorem 1.1 we shall rely on the following result implicitly in [18].

Theorem 2.5. *Let $\mathcal{A}_t f$ be as in (2.14) and define the associated maximal function by $\mathcal{M}f(x) = \sup_{t>0} |\mathcal{A}_t f(x)|$. Suppose that*

$$\Gamma_{\frac{p}{2-p}}(\delta) = \Big(\int_{\tilde\Sigma} |\mathcal{B}(w,\delta)|^{\frac{2p-2}{2-p}} dw\Big)^{\frac{2-p}{p}},$$

and the inequality

$$\int_0^1 [\log(1+\delta^{-1})]^{\frac1p - \frac12} \delta^{-1/p} \Gamma_{\frac{p}{2-p}}(\delta) \frac{d\delta}{\delta} \le A < \infty$$

holds. Then $\mathcal{M}$ is bounded on L^p; the operator norm is dominated by $CA\|\chi\|$ where C is admissible and $\|\chi\|$ is a suitable Sobolev norm of χ.

Proof. Let $H_{\delta,t}(x) = t^n \mathcal{L}_\delta[d\mu](tx)$ where $\mathcal{L}_\delta$ is as in the proof of Proposition 2.2. By [18, (4.4)]

$$\big\| \sup_{t>0} |H_{\delta,t} * f\big\|_p \le C[\log \delta^{-1}]^{\frac1p - \frac12} \delta^{-1/p} \Gamma_{\frac{p}{2-p}}(\delta)$$

for small δ and the statement of the theorem follows by introducing a dyadic decomposition for large frequencies and summing the estimates for the operators corresponding to the pieces. $\square$

A consequence of Theorem 2.5 and Proposition 2.1 is

Proposition 2.6. *Let $\Phi \in \mathcal{S}^{d-1}_{2T}(b, M, \mathfrak{a}, N)$, $N > m+1$ and let $\mathcal{B}(w,\delta)$ be defined as in (2.6). Let $k \in \{1,\dots,d-1\}$, $\beta > 0$ and $\eta > 1/2$. Suppose that*

$$|\mathcal{B}(w,\delta)| \le C\delta^\eta \tag{2.20}$$

(in particular we can choose $\eta = \nu_1$ if $\nu_1 > 1/2$) and

$$\int K_k(x)^{-\beta} d\sigma(x) \le A. \tag{2.21}$$

Then the following statements hold.

(i) $\mathcal{M}$ is L^p bounded for $p > \frac{1+2\eta}{2\eta}$.

(ii) If $\beta \ge \frac{1}{k-1+2\nu_{k+1}}$ then $\mathcal{M}$ is L^p bounded for $p > \frac{k+1+2\nu_{k+1}}{k+2\nu_{k+1}}$.

(iii) If $\beta < \frac{1}{k-1+2\nu_{k+1}}$ then $\mathcal{M}$ is L^p bounded for $p > p_0(\beta,\eta,k) = \frac{1+2\eta-2\beta(k+2\nu_{k+1}-2\eta)}{2\eta-\beta(k+2\nu_{k+1}-2\eta)}$.

Proof. First note that the restriction $\eta > 1/2$ implies that $\frac{k+1+2\nu_{k+1}}{k+2\nu_{k+1}} < 2$ and $p_0(\beta,\eta,k) < 2$ if $\beta < \frac{1}{k-1+2\nu_{k+1}}$. Therefore it suffices to check the condition (1.6). (i) follows immediately from Theorem 2.5; however this special case follows already from Greenleaf's paper [11].

We may therefore assume that $(k+1+2\nu_{k+1})/(k+2\nu_{k+1}) < p \le (2\eta+1)/2\eta$. Let

$$\underline{\theta} = 1+\beta-\frac{\beta}{p-1}$$
$$\overline{\theta} = \frac{(p-1)(k+2\nu_{k+1})-1}{(p-1)(k+2\nu_{k+1}-2\eta)}.$$

Note that $\overline{\theta} > 0$ since $(k+1+2\nu_{k+1})/(k+2\nu_{k+1}) < p$ and $\overline{\theta} \le 1$ since $p \le (2\eta+1)/2\eta$. Moreover a computation shows that the inequality $\underline{\theta} < \overline{\theta}$ is equivalent with

$$1+2\eta-2\beta(k+2\nu_{k+1}-2\eta) < p\big(2\eta-\beta(k+2\nu_{k+1}-2\eta)\big). \tag{2.22}$$

If $\beta \ge \frac{1}{k-1+2\nu_{k+1}}$ then (2.22) holds for all $p \in (\frac{k+1+2\nu_{k+1}}{k+2\nu_{k+1}}, 2)$ and if $\beta < \frac{1}{k-1+2\nu_{k+1}}$ then (2.22) is satisfied precisely for $p > p_0(\beta,\eta,k)$. In either case it is therefore possible to choose $0<\theta<1$ such that $\underline{\theta} \le \theta < \overline{\theta}$. We now estimate using Proposition 2.1

$$\begin{aligned}
\Big(\int_{|w|\le T} |\mathcal{B}(w,\delta)|^{\frac{p}{2-p}-1} dw\Big)^{\frac{2-p}{p}} &= \Big(\int_{|w|\le T} |\mathcal{B}(w,\delta)|^{\frac{2p-2}{2-p}(1-\theta)+\frac{2p-2}{2-p}\theta} dw\Big)^{\frac{2-p}{p}} \\
&\le \Big((A_1\delta^{\eta})^{\frac{2p-2}{2-p}\theta}(A_2\delta^{\frac{k}{2}+\nu_{k+1}})^{\frac{2p-2}{2-p}(1-\theta)}\int_{|w|\le T}[K_k(w)]^{-\frac12\frac{2p-2}{2-p}(1-\theta)}dw\Big)^{\frac{2-p}{p}} \\
&\le C\delta^{(2\eta\theta+(k+2\nu_{k+1})(1-\theta))\frac{p-1}{p}}\Big(\int_{|w|\le T}[K_k(w)]^{-\frac{p-1}{2-p}(1-\theta)}dw\Big)^{\frac{2-p}{p}}.
\end{aligned}$$

The integral is finite if $\frac{p-1}{2-p}(1-\theta) \le \beta$; a short computation shows that this is equivalent to the condition $\theta \ge \underline{\theta}$ hence satisfied in view of our choice of θ. Now according to Theorem 2.5 the L^p boundedness holds if $(2\eta\theta+(k+2\nu_{k+1})(1-\theta))\frac{p-1}{p} > \frac{1}{p}$ and another computation shows that this is precisely the restriction $\theta < \overline{\theta}$. □

As an easy consequence we obtain

Theorem 2.7. *Let $\Sigma \subset \mathbb{R}^d$, $d \ge 3$, be a convex hypersurface of finite line type and let $K(x)$ the Gaussian curvature. Suppose that*

$$\int_{\Sigma}[K(x)]^{-\beta}d\sigma(x) < \infty \quad \textit{for all } \beta < \frac{1}{d-2}.$$

Then the maximal operator in (1.2) is bounded on $L^p(\mathbb{R}^d)$, for $p > d/(d-1)$.

Proof. After localization we may assume that the averaging operator is of the form (2.14). Note that $|B(x,\delta)| \approx |B(y,\delta)|$ if $y \in B(x,\delta)$. Therefore by Proposition 2.1

$$|B(x,\delta)|^{1+2\beta} \lesssim \int |B(y,\delta)|^{2\beta}d\sigma(y) \lesssim \delta^{(d-1)\beta}\int |K(y)|^{-\beta}d\sigma(y)$$

Therefore $|B(x,\delta)| \lesssim \delta^{\eta_\beta}$ with $\eta_\beta = \frac{(d-1)\beta}{1+2\beta}$ and $\eta_\beta > 1/2$ if $\beta > (2d-4)^{-1}$. The assertion follows from an application of Proposition 2.6 with $k=d-1$, $\eta=\eta_\beta$, the critical exponent in case (ii) is then $p=\eta_\beta^{-1}$ and for $\beta = 1-\varepsilon$ we see that $\eta_\beta^{-1} = d/(d-1)+O(\varepsilon)$. □

3. Auxiliary Results

According to a result of Schulz [20] one can decompose a convex function at a given point into a main term, which after an affine change of variable exhibits some homogeneity, and a remainder term. We first need the following

Definition. Define the dilations A_s by

$$A_s x = (s^{\frac{1}{a_1}} x_1, \dots, s^{\frac{1}{a_n}} x_n). \tag{3.1}$$

We say that a smooth function $Q : \mathbb{R}^n \to \mathbb{R}$ is mixed homogeneous of degree $(a_1, a_2, \dots, a_n)$, $a_j > 0$, if

$$Q(A_s x) = sQ(x), s > 0. \tag{3.2}$$

The following Proposition summarizes and extends a result of [20]; the fact (3.5) below was already applied in the proof of Theorem 10 in [14].

Proposition 3.1. *Let $\Phi \in \mathcal{S}^n_T(b, M, m, 3N+2)$, where $N > m$. Suppose that $a_1 \le \cdots \le a_n \le m$ and $\mathfrak{a} = (a_1, a_2, \dots, a_n)$ is the multitype of Φ at 0. Then the following statements hold.*

There is a rotation L on $\mathbb{R}^n$ so that

$$\Phi(Lx) = Q(x) + R(x), \ |x| \le T \tag{3.3}$$

where Q is a convex mixed homogeneous polynomial of degree $(a_1, \dots, a_n)$, the a_i are even positive integers with $a_1 \le \cdots \le a_n$, the graph of Q is of finite line type $\le a_n \le m$ and $(a_1, \dots, a_n)$ is the multitype at 0 of the graph of Φ (considered as a subset of $\mathbb{R}^{n+1}$.) If $a_j < a_{j+1}$ then the linear subspace S^{a_j} consisting of all v such that $(\langle v, \nabla\rangle)^j[\Phi \circ L](0) = 0$ for $j < a_{j+1}$ is the image of span$\{e_{j+1}, \dots, e_n\}$ under L^{-1}. Moreover

$$Q(x) > 0 \quad \text{if } x \ne 0 \tag{3.4}$$

and

$$|Q(x)| \le C_1 |x| |\nabla Q(x)| \le C_2 |x|^2 \sum_{i,j} \Big| \frac{\partial^2 Q}{\partial x_i \partial x_j}(x) \Big|. \tag{3.5}$$

The remainder term R satisfies

$$\Big| s^{-1} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \big(R(A_s x)\big) \Big| \le C_{M,N} s^{1/m} \tag{3.6}$$

for $|x| \le T$ and all multiindices $\alpha = (\alpha_1, \dots, \alpha_{d-1})$ with $|\alpha| \le N$; A_s is as in (3.1).

If $a_1 = \cdots = a_k = 2$ for some k, then the rotation L can be chosen so that

$$Q(x) = c_1 x_1^2 + \cdots + c_k x_k^2 + \widetilde{Q}(x_{k+1}, \dots, x_n) \tag{3.7}$$

where $\widetilde{Q}$ is mixed homogeneous of degree $(a_{k+1}, \dots, a_n)$; i.e. $\widetilde{Q}(s^{\frac{1}{a_{k+1}}} x_{k+1}, \dots, s^{\frac{1}{a_n}} x_n) = s\widetilde{Q}(x_{k+1}, \dots x_n)$ for all $x \in \mathbb{R}^n$.

Remark. We note that if Φ belongs to $\mathcal{S}^n_T(b, M, \mathfrak{a}, 3N+1)$ then Q belongs to a family $\mathcal{S}^n_T(\widetilde{b}, CM, m, 3N+1)$, with $\widetilde{b} > 0$, but unfortunately there is no good lower bound for $\widetilde{b}$ in terms of b.

Proof of Proposition 3.1. The decomposition (3.3) was obtained by Schulz [20] and the construction involved the subspaces S^{m_i} mentioned in the introduction. The polynomial Q was obtained as a Taylor-polynomial $\sum c_\gamma x^\gamma$ of $\Phi \circ L$ where each multiindex γ satisfies $\sum_{i=1}^n \gamma_i/a_i = 1$; the convexity and (3.4) is verified in [20]. As observed in [14], (3.5) is a consequence of Euler's homogeneity relation $Q(x) = \sum x_i a_i^{-1} Q_{x_i}(x)$. To see (3.6), fix α, and use Taylor's formula to write

$$R(x) = P_{2N}(x) + R_{2N}(x)$$

where $P_{2N}(x)$ is a linear combination of monomials $G_\beta(x) := x^\beta$ with $|\beta| \leq 2N$ and $\sum_{k=1}^n \frac{\beta_k}{\alpha_k} > 1$. If $\alpha_i \leq \beta_i$, $i = 1, \ldots, n$ it follows immediately that

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}\Big[s^{-1} G_\beta(A_s x)\Big] = c_{\alpha,\beta} x^{\beta-\alpha} s^{-1+\sum_{k=1}^n \frac{\beta_k}{\alpha_k}}$$

which is $\leq C s^{1/m}$ since β_k assume only integer values and $m^{-1} \leq a_n^{-1}$. Thus

$$\Big|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\Big[s^{-1} P_{2N}(A_s x)\Big]\Big| \leq C_{M,N} s^{1/m}.$$

Finally, the remainder $R_{2N}(x)$ satisfies $|\partial_\alpha R_{2N}(x)| \leq C_N |x|^{2N+1-|\alpha|}$, for $|\alpha| \leq N$. Therefore

$$\Big|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\Big[s^{-1} R(A_s x)\Big]\Big| \leq C|x|^{N+1} s^{-1} \max s^{(N+1)/a_i} \leq C'|s|^{1/m}$$

by the definition of N. This finishes the proof of (3.6).

We now turn to proving (3.7) and discuss first the case $k = 1$. Split $x = (x_1, x')$ with $x' = (x_2, \ldots, x_{d-1})$. Then Q can be decomposed as

$$Q(x) = c_1 x_1^2 + x_1 A(x') + B(x'),$$

where B is mixed homogeneous of degree $(a_2, \ldots, a_{d-1})$, and A is mixed homogeneous of degree $(a_2/2, \ldots, a_n/2)$. In order to prove that $A = 0$ it suffices to show that the partial derivatives $A_{x_i x_j}$ vanish for all $i, j \geq 2$. To see this we use homogeneity. Define

$$\delta_s x' = (s^{1/a_2} x_2, \ldots, s^{1/a_n} x_n)$$

and observe that

$$\begin{aligned} B_{x_i x_j}(\delta_s x') &= s^{1-1/a_i-1/a_j} B_{x_i x_j}(x') \\ A_{x_i x_j}(\delta_s x') &= s^{1/2-1/a_i-1/a_j} A_{x_j x_j}(x') \end{aligned} \tag{3.8}$$

for $s > 0$.

By the convexity of Q we have

$$\langle \eta, \nabla^2 Q(x) \eta \rangle \geq 0 \tag{3.9}$$

for all x near 0 and all η. With $\eta = e_j$, $j = 2, \ldots, n$ this yields

$$0 \leq B_{x_j x_j}(x') + x_1 A_{x_j x_j}(x'). \tag{3.10}$$

Suppose now that $A_{x_j x_j}(\tilde{x}') \neq 0$; then $G_j = B_{x_j x_j}/A_{x_j x_j}$ satisfies $G_j(\delta_s x') = s^{1/2} G_j(x')$ for x' near $\tilde{x}'$. Using this homogeneity property we see from (3.10) that if $A_{x_j x_j}$ is not identically zero, then

$e_j^t \nabla^2 Q(x) e_j$ changes sign arbitrarily close to the origin, a contradiction. Therefore $A_{x_j x_j}$ vanishes identically, for $j = 2, \ldots, n$.

Next we show that $A_{x_i x_j} = 0$ for $i \neq j$. We apply (3.9) with $\eta = \xi_i e_i + \xi_j e_j$. Since $A_{x_j x_j} = 0$, (3.10) becomes

$$0 \leq B_{x_i x_i}(x')\xi_2^2 + 2B_{x_i x_j}(x')\xi_i\xi_j + B_{x_j x_j}(x')\xi_j^2 + 2x_1 A_{x_i x_j}(x')\xi_i\xi_j. \tag{3.11}$$

Assume that $A_{x_i x_j}(\tilde{x}') \neq 0$; by homogeneity we have then $A_{x_i x_j}(\delta_s x') \neq 0$ for x' near $\tilde{x}'$. By (3.8) and (3.11) it follows that

$$\begin{aligned} 0 &\leq x_1 + \frac{\langle \eta, \nabla^2 B(\delta_s x')\eta\rangle}{\langle \eta, \nabla^2 A(\delta_s x')\eta\rangle} \\ &= x_1 + \frac{B_{x_i x_i}(x')\xi_2^2 s^{1/2-1/a_i+1/a_j} + 2B_{x_i x_j}(x')\xi_i\xi_j s^{1/2} + B_{x_j x_j}(x')\xi_j^2 s^{1/2+1/a_i-1/a_j}}{2A_{x_i x_j}(x')\xi_i\xi_j} \end{aligned}$$

and this expression tends to x_1 as $s \to 0$ since $|a_i^{-1} - a_j^{-1}| \leq 1/2$. Thus for each s sufficiently small, we can find a value of x_1, such that the right side of (3.11) vanishes. We see that the expression changes sign arbitrarily close to the origin, a contradiction. Hence $A_{x_i x_j}$ also vanishes.

We now turn to the case $k > 1$. Split $x = (x', x'')$ with $x' = (x_1, \ldots, x_k)$; then

$$Q(x) = Q_0(x') + \sum_{i=1}^{k} x_i A_i(x'') + B(x'').$$

where $Q_0(x')$ is a positive definite quadratic form on $\mathbb{R}^k$, the functions A_i are mixed homogeneous of degree $(a_{k+1}/2, \ldots, a_n/2)$ and B is mixed homogeneous of degree $(a_{k+1}, \ldots, a_n)$. By performing a rotation in the x' variables we can assume that $Q_0(x') = \sum_{i=1}^k c_i x_i^2$. Then we can apply the case $k = 1$ already proved to the functions $(x_i, x'') \mapsto Q(x_i e_i, x'')$ and deduce that $A_i = 0$. $\square$

Lemma 3.2. *Suppose that $\Phi \in \mathcal{S}^n_{2T}(b, M, m, N)$, $N > 4m$, $a_2 > 2$, and suppose that*

$$\frac{\partial^2 \Phi}{\partial x_1^2}(0) \neq 0, \qquad \frac{\partial^{a_2} \Phi}{\partial x_2^{a_2}}(0) \neq 0, \qquad \frac{\partial^j \Phi}{\partial x_2^j}(0) = 0 \text{ if } j < a_2.$$

Let $K_2[\Phi] = \Phi_{x_1 x_1}\Phi_{x_2 x_2} - (\Phi_{x_1 x_2})^2$. Then

$$\frac{\partial^{a_2-2} K_2[\Phi]}{\partial x_2^{a_2-2}}(0) \neq 0. \tag{3.12}$$

Moreover there is $\epsilon > 0$, $\delta > 0$ and C_γ (all depending on Φ) so that

$$\sup_{\substack{x_1, x_3, \ldots, x_n \\ \in[-\delta,\delta]}} \int_{-\delta}^{\delta} \big(K_2[\Psi](x)\big)^{-\gamma} dx_2 < C_\gamma, \qquad \text{if } \gamma < (m-2)^{-1}, \tag{3.13}$$

for all $\Psi \in \mathcal{S}^n_r(b/2, 2M, m, N)$ with $\|\Phi - \Psi\|_{C^N(|x| \leq r)} \leq \epsilon$.

Proof. We define $\phi(y_1, y_2) = \Phi(y_1, y_2, 0)$. Then $(1, 0, \ldots)$ is an eigenvector of the Hessian of ϕ and we can apply Proposition 2.1 to ϕ, without performing a rotation. Thus

$$\phi(y) = \frac{c_1}{2} y_1^2 + c_2 y_2^{a_2} + R(y)$$

where $c_1 > 0$, $c_2 > 0$ and R satisfies (3.6). Now

$$K(y) = c_1 c_2 a_2 (a_2 - 1) y_2^{a_2 - 2} + E(y)$$

where the error $E(y)$ is given by

$$E = (c_1 + R_{y_1 y_1}) R_{y_2 y_2} + c_2 a_2 (a_2 - 1) R_{y_1 y_1} y_2^{a_2 - 1} - R_{y_1 y_2}^2$$

Expanding R we see that

$$R(y) = \sum_{\beta} c_\beta y^\beta + R_{a_2+1}(y); \tag{3.14}$$

here we sum over multiindices β so that $|\beta| \le m$ and $\beta_1/2 + \beta_2/a_2 > 1$. All derivatives of order $\le a_2$ of R_{a_2+1} vanish for $y = 0$.

In order to show (3.12) we shall show that $\partial_{y_2}^{a_2-2} E(0) = 0$. To see this let $G_\beta(y) = y^\beta$. We have to verify that

$$\begin{aligned}
&\frac{\partial^{a_2} G_\beta}{\partial y_2^{a_2}} = O(y) \\
&\frac{\partial^{2+\ell} G_\beta}{\partial y_1^2 \partial y_2^\ell} \frac{\partial^{a_2-\ell} G_{\beta'}}{\partial y_2^{a_2-\ell}} = O(y), \qquad \ell \le a_2 - 2 \\
&\frac{\partial^2 G_\beta}{\partial y_1^2} = O(y) \\
&\frac{\partial^{\ell+1} G_\beta}{\partial y_1 \partial y_2^\ell} = O(y), \qquad 1 \le \ell \le \frac{a_2}{2}
\end{aligned}$$

whenever β or β' occur in the sum (3.14). Considering the term $\frac{\partial^{2+\ell} G_\beta}{\partial y_1^2 \partial y_2^\ell} \frac{\partial^{a_2-\ell} G_{\beta'}}{\partial y_2^{a_2-\ell}}$ it is clearly $O(y)$ unless $\beta_1 = 2$, $\beta_2 = \ell$, $\beta_1' = 0$, $\beta_2' = a_2 - \ell$ and $\beta_j = \beta_j' = 0$ for $j \ge 3$. But this implies that $a_2 - \ell = a_2$, hence $G_\beta(y) = y_1^2$, but y_1^2 is not an admissible term in (3.14). We argue similarly for each of the other terms and (3.12) is proved.

To see the second assertion we use a result related to van der Corput's lemma which is due to M. Christ [4] (alternatively one may use the Malgrange preparation theorem). It states that for any $k \in \mathbb{Z}_+$ there is a constant A_k such that for any interval $I \subset \mathbb{R}$, any $f \in C^k(I)$ and any $\gamma > 0$

$$\Big|\{t \in I : |f(t)| \le \gamma\}\Big| \le A_k \gamma^{1/k} \inf_{s \in I} |D^k f(s)|^{-1/k}. \tag{3.15}$$

By continuity we know that $\frac{\partial^{a_2} K_2}{\partial x_2^{a_2}}(x) \neq 0$ for small x and we can apply (3.15) with $k = a_2$ to obtain (3.13) □

Proposition 3.3. *Let $n = 2$, $\Phi \in \mathcal{S}_T^2(b, M, m, N)$ for large N and suppose that (a_1, a_2) is the multitype at 0; moreover assume*

$$\begin{aligned}
&\frac{\partial^j \Phi}{\partial x_i^j}(0) = 0 \quad \text{for } j < a_i, \\
&\frac{\partial^{a_i} \Phi}{\partial x_i^{a_i}}(0) \neq 0,
\end{aligned} \tag{3.16}$$

for $i = 1, 2$. *Let* $\rho(x) = x_1^{a_1} + x_2^{a_2}$ *and let*

$$\Omega_\ell = \{x : 2^{-\ell-1} \le \rho(x) \le 2^\ell\}.$$

Let $\nu = 1/a_1 + 1/a_2$. *There is* $\ell_0 > 0$ *so that for* $\ell > \ell_0$

$$\int_{\Omega_\ell} |\det \Phi''(x)|^{-\gamma} d\sigma(x) \le C_\gamma 2^{\ell(2\gamma - 2\gamma\nu - \nu)}, \quad \text{for} \quad \gamma < \frac{1}{a_2 - 2}. \tag{3.17}$$

Moreover $[\det \Phi'']^{-\gamma}$ *is integrable over a neighborhood of the origin.*

Proof. In view of assumption (3.16) we may decompose $\Phi = Q + R$ where Q is mixed homogeneous of degree (a_1, a_2), in fact $Q(x) \le c_1\rho(x) \le c_2 Q(x)$ for small x, by the homogeneity and positivity of Q and ρ. The function Q is of type $\le a_2$ near 0 and by homogeneity considerations it is easy to see that Q is of type $\le a_2$ everywhere. Moreover, by (3.5) the rank of Q in Ω_1 is at least 1.

Let $\Gamma = \{x \in \Omega : \det \Phi''(x) = 0\}$ and fix $x^0 \in \Gamma$. Then there is a rotation L_{x^0} so that $\psi(y) = Q(x^0 + L_{x^0} y)$ satisfies the assumption of Lemma 3.2 and therefore we can integrate $[\det Q'']^{-\gamma}$ over a small neighborhood of x^0; moreover the bound persists for small C^N perturbations of Q. Using compactness arguments we see that there is $\epsilon > 0$ so that

$$\int_{\Omega_1} [\det \Psi'']^{-\gamma} dx \le C_\gamma \tag{3.18}$$

if $\|\Psi - Q\|_{C^N(\Omega_1)} \le \varepsilon$ and $\gamma < 1/(a_2 - 2)$.

Let, for large ℓ

$$\Phi_\ell(y) = 2^\ell \Phi(2^{-\ell/a_1} y_1, 2^{-\ell/a_2} y_2).$$

Then

$$\Phi_\ell = Q + R_\ell \tag{3.19}$$

and all derivatives of R_ℓ tend to 0 uniformly in $\{y : \rho(y) \le 1\}$. Therefore there is ℓ_0, $2^{-\ell_0} \ll 1$ so that (3.18) applies for $\Psi = \Phi_\ell$, $\ell > \ell_0$, with a bound independent of ℓ. Since

$$\det \Phi_\ell''(y) = 2^{2\ell(1-\nu)} \det \Phi''(2^{-\ell/a_1} y_1, 2^{-\ell/a_2} y_2) \tag{3.20}$$

we obtain for $\ell > \ell_0$

$$\begin{aligned} \int_{\Omega_\ell} |\det \Phi''(y)|^{-\gamma} dy &= \int_{\Omega_1} 2^{-\ell\nu} |\det \Phi''(2^{-\ell a_1} y_1, 2^{-\ell a_2} y_2)|^{-\gamma} dy \\ &= 2^{-\ell\nu} 2^{2\gamma\ell(1-\nu)} \int_{\Omega_1} |\det \Phi_\ell''(y)|^{-\gamma} dy. \end{aligned} \tag{3.21}$$

If $\gamma < (a_2 - 2)^{-1}$ we can dominate the integrals by a constant independent of ℓ and the estimate (3.17) is proved.

Since $a_1 \le a_2$ we see that $-\nu + 2\gamma(1 - \nu) \le \frac{2}{a_2}((a_2 - 2)\gamma - 1) < 0$ and therefore we can sum the estimates (3.21) to obtain the integrability of $[\det \Phi'']^{-\gamma}$ near the origin. □

Proof of Theorem 1.2. Immediate from Proposition 3.3 □

We now examine the size of the balls in (2.6) near a point of given multitype.

Proposition 3.4. *Let $\Phi \in \mathcal{S}_T^n(b, M, m, N)$, where N is large, and let $\mathfrak{a} = (a_1, a_2, \dots, a_n)$ be the multitype of Φ at 0. We assume that (3.16) holds for $i = 1, \dots, n$.*

Let $\nu = \sum_{j=1}^n \frac{1}{a_j}$, let $\rho(y) = \sum_{i=1}^n y_i^{a_i}$ and $\Omega_\ell = \{x : 2^{-\ell-1} \le \rho(x) \le 2^\ell\}$. Then there are constants C_1, C_2 so that for $C_1\delta \le 2^{-\ell} \le C_2$, $y \in \Omega_\ell$

$$|\mathcal{B}(y,\delta)| \le C_\alpha \delta^\alpha 2^{\ell(\alpha-\nu)} \quad \text{if} \quad \alpha \le \frac{1}{2} + \frac{1}{a_n}. \tag{3.22}$$

Proof. Decompose $\Phi = Q + R$ as in (3.3). By our assumptions this holds with the rotation L being the identity. By the metric properties of the balls $\mathcal{B}(y,\delta)$ (in particular the triangle inequality for the pseudo-distance associated to these balls [2]) it follows that there are constants $C_1 > 1$, $C_2 > 1$ so that

$$\mathcal{B}(y,\delta) \subset \{x : C_1^{-1}Q(x) \le Q(y) \le C_1 Q(x)\} \quad \text{if} \quad Q(y) \ge C_2\delta.$$

Now let $Q(y) \ge C_2\delta$ and set $\Phi_\ell(w) = 2^\ell \Phi(A_{2^{-\ell}} w)$; note that $\Phi_\ell = Q + R_\ell$ where R_ℓ tends to zero in the C^∞ topology. Let ℓ be large so that $2^{-\ell-1} \le Q(y) \le 2^{-\ell}$. Then one computes that with $W = \{y' : C_1^{-1}/2 \le Q(y') \le C_1\}$ and $Y_\ell = A_{2^\ell} y \in W$

$$\{A_{2^\ell} z : z \in \mathcal{B}(y,\delta)\} = \{w : \Phi_\ell(w) - \Phi_\ell(Y_\ell) - \langle w - Y_\ell, \nabla\Phi_\ell(Y_\ell)\rangle \le 2^\ell\delta\} := W_{\ell,y,\delta}$$

and $W_{\ell,y,\delta}$ is contained in W. By Proposition 3.1 there is $C_2 > 0$ and $\ell_0 > 0$ such that for any $y \in W$ there is a unit vector θ with $\langle\theta,\nabla\rangle^2 \Phi_\ell(y) \ge C$, for all $\ell > \ell_0$. Moreover Φ_ℓ is of line type $\le a_n$, with uniform bounds for $\ell > \ell_0$, since this is the case for Q. This implies that

$$|W_{\ell,y,\delta}| \le C(2^\ell\delta)^\alpha$$

for $0 \le \alpha \le 1/2 + 1/a_n$. Since the Jacobian of the change of variable $z \to A_{2^\ell z}$ is $2^{\ell\nu}$ we obtain that

$$|\mathcal{B}(y,\delta)| \le C\delta^\alpha 2^{-\nu\ell+\alpha\ell}$$

and since $Q(y) \approx \rho(y)$ the assertion follows. □

Remark. Let $\alpha \le 1/2 + 1/a_n$. The estimate $|\mathcal{B}(y,\delta)| \le C\delta^\alpha[\Phi(y)]^{\nu-\alpha}$, for small y, is an easy consequence of Proposition 3.4.

4. Estimates involving the multitype

We shall first give a different proof of the following Theorem proved by the first two authors in [14].

Theorem 4.1. *Let $\mathcal{M}$ be as in (1.2). Suppose that $(a_1, a_2, \dots, a_d)$ is the multitype at P and that $\nu = \sum_{j=1}^{d-1} \frac{1}{a_j} \le \frac{1}{2}$. Then there is a neighborhood U of x_0 so that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^d)$ if $p > \nu^{-1}$, provided that* supp $\chi \subset U$.

Proof. We may assume that our averages are of the form (2.14) and $P = (0, c_d)$. Since $\nu^{-1} \ge 2$ we just have to verify (1.4). We now use Proposition 3.4, with $\alpha = \nu$ in the first term below and $\alpha < \nu$ in the second, and obtain

$$\begin{aligned}\Gamma_{\frac{p}{p-2}}(\delta) &\le \Big(\int_{\rho(w)\le C_1\delta} |\mathcal{B}(w,\delta)|^{\frac{2}{p-2}} dw\Big)^{\frac{p-2}{p}} + \sum_{C_1\delta\le 2^{-\ell}\ll 1} \Big(\int_{\rho(w)\approx 2^{-\ell}} |\mathcal{B}(w,\delta)|^{\frac{2}{p-2}} dw\Big)^{\frac{p-2}{p}} \\ &\le C\big(\delta^\nu + \sum_{C_1\delta\le 2^{-\ell}\ll 1} (\delta^\alpha 2^{\ell(\alpha-\nu)})^{\frac{2}{p}} \delta^{\nu\frac{p-2}{p}}\big) \le C'\delta^\nu.\end{aligned}$$

This implies (1.4) since $\nu > 1/p$. $\square$

Proof of Theorem 1.1. If $a_1 > 2$ then $\nu \le 1/4 + 1/a_2 \le 1/2$ and the assertion (i) follows from Theorem 4.1 (the necessity of the condition had also been shown in [13]). Now let $a_1 = 2$. Assertion (ii) follows from Proposition 2.6 (with $k = 2$, $\nu_3 = 0$, $\eta = 1/2 + 1/a_2$), and by Proposition 3.3 the hypothesis of (ii) is satisfied with $\beta < (a_2 - 2)^{-1}$; this shows assertion (iii). $\square$

Proof of Theorem 1.4. It is sufficient to assume that $\mathcal{A}$ is of the form (2.14) so that the multitype at 0 is $\mathfrak{a} = (a_1, a_2)$ and χ is supported near the origin; moreover we may assume that (3.16) holds for $i = 1, 2$.

We have boundedness for the cases $p = 1 = q$ trivially. Since $|\mathcal{B}(y,\delta)| \le C\delta^\nu$ for small y and δ it follows from Theorem 1.3 that $\mathcal{A}$ maps $B^p_{\beta,r}$ to $B^{p'}_{\beta+\alpha}$ if $1 \le p \le 2$, $\alpha \ge -\nu$ and $\frac{1}{p} - \frac{1}{2} \le \frac{\alpha+\nu}{2(\nu+1)}$. This is the asserted estimate for $1/p + 1/q = 1$. We remark that this result is well known and follows just from the assumption that $\widehat{d\sigma}(\xi) = O(|\xi|^{-\nu})$, see *e.g* [23, p. 371] and also the original references [25], [17].

We shall now consider the case $1/p + 1/q < 1$ and prove boundedness under the conditions (1.15.1-3); boundedness for $1/p + 1/q > 1$ under the conditions (1.16.1-3) follows then by duality. We shall verify the condition (1.13) by estimating the volume of the balls $\mathcal{B}(w,\delta)$ using Proposition 2.1 and then apply either Proposition 3.3 or Proposition 3.4 or both.

In what follows define r and σ by

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$$
$$\sigma = \frac{2q(p-1)}{p+q-pq}$$

so that $\sigma/r = 2/p'$. First observe that by Proposition 2.1

$$\delta^{-\alpha-\frac{1}{p}+\frac{1}{q}}\Big(\int_{\rho(w)\le C_2\delta} |\mathcal{B}(w,\delta)|^\sigma dw\Big)^{\frac{1}{r}} \le C\delta^{\nu(\frac{\sigma+1}{r})-\alpha-\frac{1}{p}+\frac{1}{q}} = C\delta^{-\alpha+\nu-\frac{\nu+1}{p}+\frac{\nu+1}{q}} \tag{4.1}$$

which is bounded uniformly in δ, by 1.15.1. Here we assume that C_2 is as in the statement of Proposition 3.4.

We use Proposition 2.1 to estimate $\mathcal{B}(w,\delta)$ and our conclusion follows if we can verify the estimate

$$\Big(\int_{C_2\delta\le\rho(w)\le c}\Big(\frac{\delta}{\sqrt{\det\Phi''(w)}}\Big)^{\sigma(1-\theta)}|\mathcal{B}(w,\delta)|^{\sigma\theta}dw\Big)^{1/r} \le C\delta^{\alpha+\frac{1}{p}-\frac{1}{q}} \tag{4.2}$$

for suitable $\theta \in [0,1]$ and small c.

In the present relevant case $1/p + 1/q > 1$ we distinguish three subcases

$$(a_2-1)(1-\frac{1}{p}) - \frac{1}{q} \ge 0 \quad \text{and} \quad \frac{a_1-1}{p} + \frac{1}{q} < a_1 - 1, \tag{4.3.1}$$

$$(a_2-1)(1-\frac{1}{p}) - \frac{1}{q} \ge 0 \quad \text{and} \quad \frac{a_1-1}{p} + \frac{1}{q} \ge a_1 - 1, \tag{4.3.2}$$

$$(a_2-1)(1-\frac{1}{p}) - \frac{1}{q} < 0. \tag{4.3.3}$$

We begin by assuming that the third estimate (4.3.3) holds. Here we check (4.2) with $\theta = 0$; by Proposition 3.3 the desired estimate holds if

$$\sigma > \frac{1}{a_2 - 2} \tag{4.4}$$

$$\frac{\sigma}{r} \le \alpha + \frac{1}{p} - \frac{1}{q}. \tag{4.5}$$

It is easily checked that (4.4) is equivalent to (4.3.3) which is presently assumed and (4.5) is equivalent to the assumption (1.15.3).

Next we assume that the inequalities (4.3.2) hold. In order to carry out the integration in (4.2) we have to assume that $\sigma(1-\theta) < (a_2-1)^{-1}$ which is equivalent to saying that θ is larger than the critical value

$$\theta_{\rm cr} = \frac{1}{a_2-2}\Big(a_2 - 1 - \frac{p}{(p-1)q}\Big). \tag{4.6}$$

Under the conditions $(a_2-1)(1-\frac{1}{p}) - \frac{1}{q} \geq 0$ (i.e. in (4.3.1) and (4.3.2)) we have that $\theta_{\rm cr} \geq 0$; moreover one can check that the assumption $1/p + 1/q > 1$ is equivalent with $\theta_{\rm cr} < 1$. We may therefore choose $\theta = \theta_{\rm cr} + \varepsilon < 1$ where ϵ is small.

Let $\Omega_\ell = \{w : 2^{-\ell-1} \leq \rho(y) \leq 2^{-\ell}\}$. By Propositions 3.3 and 3.4 we estimate

$$\begin{aligned} &\delta^{-\alpha-\frac1p+\frac1q}\Big(\int_{\Omega_\ell}\Big(\frac{\delta}{\sqrt{\det\Phi''(w)}}\Big)^{\sigma(1-\theta)}|\mathcal{B}(w,\delta)|^{\sigma\theta}dw\Big)^{1/r} \\ &\leq C\delta^{-\alpha-\frac1p+\frac1q+\frac{\sigma(1-\theta)}{r}+(\frac12+\frac1{a_2})\theta\frac{\sigma}{r}}2^{\frac{\ell}{r}(-\nu+\sigma(1-\theta)(1-\nu)+(\frac12+\frac1{a_2}-\nu)\sigma\theta)} \end{aligned} \tag{4.7}$$

Now one computes

$$\frac1r\Big(-\nu+\sigma(1-\theta)(1-\nu)+(\frac12+\frac1{a_2}-\nu)\sigma\theta\Big) = \frac{a_1-1}{a_1} - \frac1p(\frac{a_1-1}{a_1}) - \frac{1}{a_1 q} - \frac{\varepsilon}{p'}\frac{a_2-2}{a_2}$$

so that (4.3.2) implies the sum $\sum_{\ell>0}2^{\ell(\cdots)}$ in (4.7) converges. Moreover

$$-\alpha - \frac1p + \frac1q + \frac{\sigma(1-\theta)}{r} + (\frac12+\frac1{a_2})\theta\frac{\sigma}{r} = \widetilde{\alpha} - \varepsilon(\frac{a_2-2}{a_2}p') - \alpha$$

where

$$\begin{aligned} \widetilde{\alpha} &= 2 - \frac3p + \frac1q - \theta_{\rm cr}\frac{a_2-2}{a_2p'} \\ &= \frac{a_2+1}{a_2} - (2+\frac1{a_2})\frac1p + (1+\frac1{a_2})\frac1q. \end{aligned}$$

Therefore if (4.3.2) is satisfied we can choose $\varepsilon = \theta - \theta_{\rm cr}$ so small that the exponent of δ in (4.7) becomes nonnegative. This settles the estimate in case (4.3.2).

Finally assume that (4.3.1) holds, and again choose $\theta = \theta_{\rm cr} + \varepsilon$. The assumption $\frac{a_1-1}{p} + \frac1q < a_1 - 1$ implies that the terms $2^{\ell(\cdots)}$ in (4.7) form an increasing geometric progression if $\varepsilon > 0$ is chosen small enough. We compute

$$\begin{aligned} &\delta^{-\alpha-\frac1p+\frac1q}\sum_{2^{-\ell}\geq C_2\delta}\Big(\int_{\Omega_\ell}\Big(\frac{\delta}{\sqrt{\det\Phi''(w)}}\Big)^{\sigma(1-\theta)}|\mathcal{B}(w,\delta)|^{\sigma\theta}dw\Big)^{1/r} \\ &\leq C\delta^{-\alpha-\frac1p+\frac1q+\frac{\sigma(1-\theta)}{r}+(\frac12+\frac1{a_2})\theta\frac{\sigma}{r}}\delta^{-(-\frac{\nu}{r}+\frac{\sigma}{r}(1-\theta)(1-\nu)+(\frac12+\frac1{a_2}-\nu)\frac{\sigma}{r}\theta)} \\ &= C\delta^{-\alpha-\frac1p+\frac1q+\nu\frac{\sigma+1}{r}} = C\delta^{-\alpha+\nu-\frac{\nu+1}{p}+\frac{\nu+1}{q}}. \end{aligned}$$

We have proved the asserted estimate in the remaining case (4.3.1). □

5. Some Examples

As pointed out before Theorems 1.1 and 1.4 are sharp for the surfaces given as a graph $x_3 = x_1^{a_1} + x_2^{a_2}$. We now discuss a class examples for which the multitype does not suffice to get the best possible results. In order to prove improved $L^p \to L^q_\alpha$ results we shall directly apply Theorem 2.5.

Maximal operators. Let $\Sigma \subset \mathbb{R}^3$ be the graph of

$$\Phi(x) = x_1^2 + x_2^4 + x_1^2 x_2^2 - c_2 \tag{5.1}$$

over the set $|x_1|+|x_2| \le 1/4$ and consider the averages (2.14), with χ supported where $|x_1|+|x_2| \le 1/8$. The Hessian

$$\det \Phi'' = 4x_1^2 + 24x_2^2(1+x_2^2) - 16x_1^2x_2^2$$

is nonnegative in the support of χ and since trace$(\Phi'') \ge 1$ we see that Φ'' has two positive eigenvalues away from 0. Therefore Φ is convex, of multi-type $(2,4)$ at 0 and of type 2 at $(x_1,x_2) \neq 0$ near 0. The sufficient condition for L^p boundedness which only depends on the multitype yields boundedness for $p > 8/5$, by Theorem 1.1 (iii). However $|\det \Phi''|^{-1+\varepsilon}$ is integrable near 0, for all $\varepsilon > 0$, and therefore we obtain L^p boundedness for $p > 3/2$, which the best possible result.

More generally we consider

$$\Phi(x) = x_1^2 + x_2^M + x_1^a x_2^b - c_2 \tag{5.2}$$

where a and b are positive even integers with $a/2 + b/M > 1$. The graph of Φ is convex near the origin and the multitype at $(0,0)$ is $(2,M)$. Therefore, if the cutoff function χ has small support one ceratinly obtains boundedness for $p > 2(M+1)/(M+2)$. One computes

$$\det \Phi''(x) = cx_2^{M-2} + dx_1^a x_2^{b-2} + o(x_2^{M-2} + x_1^a x_2^{b-2})$$

with $c, d > 0$. Then for small ε

$$\int_{|x|\le\varepsilon} [\det \Phi'']^{-\gamma} dx < \infty$$

if

$$\gamma < \gamma_{\text{cr}} = \min\Big\{\frac{1}{b-2}, \frac{1}{M-2} + \frac{M-b}{a(M-2)}\Big\}$$

Note that $\gamma_{\text{cr}} > (M-2)^{-1}$ if $b < M$. In this case part (ii) of Theorem 1.1 gives us L^p boundedness for $p > p_0$ where the critical exponent p_0 is less than $2(M+1)/(M+2)$.

$L^p \to L^q$-estimates. Consider again the example (5.1). Let $Q_0 = (6/5, 1/2)$, $Q_0^* = (1/2, 1/6)$, and $R = (5/7, 2/7)$. Then the result of Theorem 1.4 implies $L^p \to L^q$ boundedness in the interior of the convex hull of the points $(0,0)$, $(1,1)$, Q_0, Q_0^* and R.

Let ℓ be the line $2 - 3/p + 1/q = 0$ and let σ be the lower edge of the boundary of the boundedness region which contains the point $(1,1)$. All points on ℓ with abscissae $1/p \in [5/6, 1]$ belong to σ. We shall show that this segment is in fact longer and thereby obtain a larger boundedness region. We use the estimate (2.10) with $k = 2$ and $\nu_{k+1} = 0$. L^p to L^q boundedness ($p \le q$, $p \le 2$) holds by Theorem 1.3 if

$$\delta^{\frac{1}{q} - \frac{1}{p} + \frac{2q(p-1)}{p+q-pq}(\frac{1}{p}+\frac{1}{q}-1)} \Big(\int_{\widetilde{\Sigma}} |\det \Phi''|^{-\frac{q(p-1)}{p+q-pq}} dx\Big)^{\frac{1}{p}+\frac{1}{q}-1} < \infty$$

and the exponent of δ is positive. The last requirement is equivalent to the restriction $2 - 3/p + 1/q > 0$. Since $|\det \Phi''|^{-\gamma}$ is integrable for $\gamma < 1$ we obtain boundedness if the restriction $\frac{q(p-1)}{p+q-pq} < 1$ is satisfied. A computation shows that all points on ℓ with abscissae $1/p \in [4/5, 1]$ belong to

σ. Therefore if $(1/p, 1/q)$ belongs to the interior of the pentagon with vertices $(1,1)$, $(4/5, 2/5)$, $(5/7, 2/7)$, $(3/5, 1/5)$ and $(0,0)$ then the averaging operator maps L^p to L^q. Similar considerations yield improved $L^p \to L^q_\alpha$ estimates.

We remark that the preceding $L^p \to L^q$ estimates for the example in (5.1) could also be obtained by a scaling argument in the spirit of [15]; one uses isotropic dilations since the curvature vanishes at an isolated point. The rescaled operators can be embedded in analytic families and the estimations are variants of those in [17].

References

1. J. Bourgain, *Averages in the plane over convex curves and maximal operators*, J. Analyse Math. **47** (1986), 69–85.
2. J. Bruna, A. Nagel and S. Wainger, *Convex hypersurfaces and Fourier transform*, Ann. Math. **127** (1988), 333–365.
3. D. Catlin, *Boundary invariants of pseudoconvex domains*, Ann. Math. **120** (1984), 529–586.
4. M. Christ, *Hilbert transforms along curves, I. Nilpotent groups*, Ann. Math. **122** (1985), 575–596.
5. ———, *Endpoint bounds for singular fractional integral operators*, preprint 1988.
6. ———, *Failure of an endpoint estimate for integrals along curves*, Fourier analysis and partial differential equations, ed. by J. Garcia-Cuerva, E. Hernandez, F. Soria and J. L. Torrea, CRC Press, 1995.
7. M. Cowling and G. Mauceri, *Inequalities for some maximal functions II*, Trans. Amer. Math. Soc. **296** (1986), 341–365.
8. ———, *Oscillatory integrals and Fourier transforms of surface carried measures*, Trans. Amer. Math. Soc. **304** (1987), 53–68.
9. M. Cowling, S. Disney, G. Mauceri and D. Müller, *Damping oscillatory integrals*, Invent. Math **101** (1990), 237–260.
10. E. Ferreyra, T. Godoy and M. Urciuolo, *Endpoint bounds for convolution operators with singular measures*, Coll. Math. **76** (1998), 35–47.
11. A. Greenleaf, *Principal curvature in harmonic analysis*, Indiana Math. J. **30**, 519–537.
12. A. Iosevich, *Maximal operators associated to families of flat curves in the plane*, Duke Math. J. **76**, 633-644.
13. A. Iosevich and E. Sawyer, *Oscillatory integrals and maximal averages over homogeneous surfaces*, Duke Math. J. **82**, 103–141.
14. ———, *Maximal averages over surfaces*, Adv. Math. **132** (1997), 46–119.
15. ———, *Sharp $L^p \to L^q$ estimates for a class of averaging operators*, Ann. Inst. Fourier **46** (1993), 903–927.
16. S. H. Lee, *Convolution operators with singular measures*, preprint.
17. W. Littman, *$L^p - L^q$-estimates for singular integral operators*, Proc. Symp. Pure and Appl. Math. Amer. Math. Soc. **23** (1973), 479–481.
18. A. Nagel, A. Seeger and S. Wainger, *Averages over convex hypersurfaces*, Amer. J. Math. **115** (1993), 903–927.
19. F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals I. Oscillatory integrals*, J. Funct. Anal. **73** (1987), 179–194.
20. H. Schulz, *Convex hypersurfaces of finite type and the asymptotics of their Fourier transforms*, Indiana U. Math. J. **40** (1991), 1267–1275.
21. C. D. Sogge and E. M. Stein, *Averages of functions over hypersurfaces in $\mathbb{R}^n$*, Invent. Math. **82** (1985), 543–556.
22. E. M. Stein, *Maximal functions: Spherical Means*, Proc. Nat. Acad. Sci. USA **73** (1976), 2174–2175.
23. ———, *Harmonic Analysis*, Princeton University Press, 1993.
24. E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295.
25. R. Strichartz, *Convolutions with kernels having singularities on a sphere*, Trans. Amer. Math. Soc. **148** (1970), 461–471.
26. J. Yu, *Multitypes of convex domains*, Indiana Math. J. **41** (1992), 837–849.

Georgetown University, Department of Mathematics, Washington, DC 20057

E-mail address: iosevich@gumath1.math.georgetown.edu

Indiana University - Purdue University Indianapolis, Department of Mathematical Sciences, Indianapolis, IN 46202

E-mail address: esawyer@math.iupui.edu

University of Wisconsin-Madison, Department of Mathematics, Madison, WI 53706

E-mail address: seeger@math.wisc.edu