ORTHOGONAL EXPONENTIALS, DIFFERENCE SETS, AND ARITHMETIC COMBINATORICS

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ABSTRACT. We prove that if A is a set of exponentials mutually orthogonal with respect to any symmetric convex set K in the plane with a smooth boundary and everywhere non-vanishing curvature, then $\#(A_q \equiv A \cap [0,q]^d) \lesssim q$. This extends and clarifies in the plane the result of Iosevich and Rudnev. As a corrollary, we obtain the result from [IKP01] and [IKT01] that if K is a centrally symmetric convex body with a smooth boundary and non-vanishing curvature, then $L^2(K)$ does not possess an orthogonal basis of exponentials. We also give ground to the conjecture that the disk has not more than 3 orthogonal exponentials. This is done by proving that if a set A has a difference set $\Delta(A) \subset \mathbb{Z}^+ + s$ for suitable s than A has at most 3 elements.

Contents

1. Introduction

Let K be a convex set in $\mathbb{R}^d$, symmetric with respect to the origin, such that the boundary of K is smooth and has everywhere non-vanishing curvature. In [IR03] the authors proved that if the collection $\{e^{2\pi i x\cdot a}\}_{a\in A}$ is orthogonal with respect to K in the sense that

$$\int_K e^{2\pi i x\cdot(a-a')}dx = 0, \tag{1}$$

whenever $a \neq a' \in A$, and A is infinite, then A is contained in a line. Moreover, they proved that if $d \neq 1 \mod (4)$, then A is necessarily finite. When $d = 1 \mod (4)$ the authors produced examples of convex bodies K for which there exists infinite A's satisfying (1). Unfortunately, the proof in [IR03] does not provide a finite upper bound for the size of A. The reason for this is reliance on an asymptotic generalization of the following combinatorial principle due to Anning and Erdös ([Er45a] *see also* [Er45b]).

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Theorem 1.1 (Anning and Erdös, [Er45a, Er45b])**.** *Let $d \geq 2$, and let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^d$. For $A \subset \mathbb{R}^d$, let*

$$\Delta(A) = \{|a - a'| : a, a' \in A\}. \tag{2}$$

If $\#A = \infty$ and $\Delta(A) \subset \mathbb{Z}^+$, then A is a subset of a line.

The reason explicit bounds are difficult to extract from any application of this principle is that for any N there exists $A_N \subset \mathbb{R}^d$ not contained in a line such that $\#A_N = N$ and $\Delta(A_N) \subset \mathbb{Z}^+$. This suggests that a different geometric point of view is needed to extract an explicit numerical bound if one exists. Moreover, such a point of view is likely to be dimension specific because, as we mention, above, when $d = 1 \mod (4)$, the set of orthogonal exponentials may be infinite.

An explicit numerical upper bound still eludes us, but we can prove the following.

Theorem 1.2. *Let K be a convex planar set, symmetric with respect to the origin. Suppose that the boundary of K is smooth and has everywhere non-vanishing curvature. Let $A \subset \mathbb{R}^2$ be such that (1) holds. Then*

$$\#(A \cap [-q, q]^2) \lesssim q. \tag{3}$$

In order to put these results into perspective, we contrast Theorem 1.2 with the following result due to Solymosi ([So04a], *see also* [So04b]).

Theorem 1.3 (Solymosi, [So04a])**.** *Let $A \subset \mathbb{R}^2$ such that $\Delta(A) \subset \mathbb{Z}^+$. Then*

$$\#(A \cap [-q, q]^2) \lesssim q. \tag{4}$$

This result is essentially sharp as can be seen in the following way. Let A_N be the subset of the plane consisting of $(n, 0)$, where n is a large positive integer, and pairs of the form $(0, m)$ such that m is a positive integer and $n^2 + m^2 = l^2$ for some positive integer l. By elementary number theory, one can find approximately $\frac{N}{\sqrt{\log(N)}}$ such integers m that are less than N.

A second aim of this paper is to give some ground on the conjecture that the disk $\mathbb{D}$ has no orthonormal set of exponentials of cardinality 4. This is done by proving a perturbation of Erdös-Solymosi's theorems that we may state as follows:

Theorem 1.4. *Let $A \subset \mathbb{R}^2$ be a set such that $\Delta(A) \subset \mathbb{Z}^+ + s$ where s is either a trancendental number or a rational that is not a half-integer nor an integer. Then A has at most* 3 *elements .*

This is related to the previous question in the following way. If $\{e^{2i\pi ax}\}_{a\in A}$ is an orthonormal set in $L^2(\mathbb{D})$, then $\Delta(A) \subset \mathcal{Z}_J$ where $\mathcal{Z}_J$ is the zero set of an apropriate Bessel function. This zero set is a perturbation of $\mathbb{Z}^+ + 1/4$. If we omit the perturbation, then the previous theorem would show that the disk has no orthonormal set of more than 3 exponentials. Moreover, we show that the same result is true if we slightly perturb the result by truncating the asymptotic formula of the zeros of the Bessel function.

This paper is organized as follows. In Section 2 we prove Theorem 1.2 and then explain why, if K is a symmetric convex body in $\mathbb{R}^d$, $d \geq 2$, with a smooth boundary and everywhere non-vanishing curvature, then $L^2(K)$ does not possess an orthogonal basis of exponentials. The last section is devoted to the proof of Theorem 1.4.

2. Orthogonal exponentials for planar convex sets

2.1. **Preliminaries.** We will need the following well known facts about convex sets.
Notation: For a convex set K, we call ρ_K its Minkowski function of K, so that $K = \{x : \rho(x) \leq 1\}$, and ρ_K^* its support function given by

$$\rho_K^*(\xi) = \sup_{x\in K} \langle x, \xi\rangle. \tag{5}$$

By the method of stationary phase ([He62], *see e.g.* [St93, Chapter 3]),

$$\widehat{\chi}_K(\xi) = C_1|\xi|^{-\frac{3}{2}} \sin\left(2\pi\left(\rho_K^*(\xi) - \frac{1}{8}\right)\right) + E(\xi), \tag{6}$$

with

$$|E(\xi)| \leq C_2 |\xi|^{-\frac{5}{2}}, \tag{7}$$

where C_1 and C_2 are universal constants provided, say, K contains the ball of radius 1 and is contained in the ball of radius 2.

It should also be noted that if $\{e^{2i\pi ax}\}_{a\in A}$ is orthogonal with respect to $L^2(K)$ then $\widehat{\chi_K}(a-a')=0$ for $a,a'\in A$. But $\widehat{\chi_K}$ is continuous and $\widehat{\chi_K}\neq 0$ so there exists η_0 such that $|a-a'|\geq \eta_0$, that is, the set A is separated with separation depending only on K.

2.2. **Proof of Theorem 1.2.** An immediate consequence of (6) and (7) is that, if A is as in the statement of Theorem 1.2, there exists a constant C_3 such that, whenever $a,a'\in A$, then

$$\left|\rho_K^*(a-a') - \frac{k}{2} - \frac{1}{8}\right| \leq \frac{C_3}{k+1} \tag{8}$$

for some integer k. We may now cut A into a finite number of pieces, such that in each piece, any two elements a,a' are separated enough to have $k\geq 100C_3$ in (8).

Now, if $a,a',a''\in A$ are in a q by α rectangle $\mathcal{R}_\alpha$ then $a-a', a-a'', a'-a''$ are all in an angular sector with direction some vector e that depends only on $\mathcal{R}_\alpha$. More precisely, the angle $\theta=\theta(e,a-a')$ between e and $(a-a')$ is at most θ with $\sin\theta = \alpha/|a-a'|$. In particular, $\theta\leq\theta_m$ where $\sin\theta_m=\alpha/L$ and L is the minimal distance between two elements of A. Further, from the curvature assumption on K, for u in such a sector,

$$\left|\rho_K^*(u) - c|u|\right| \leq c'\theta(u,e)^2|u|,$$

where $c=c(e)$ and $c'=c'(e)$ are two consants that depend on e, provided $\theta(u,e)$ is small enough (that is α is taken to be small enought). In particular, for $u=a-a'$,

$$\left|\rho_K^*(a-a') - c|a-a'|\right| \leq \frac{C_4}{|a-a'|}.$$

It follows from this and (8), that there exists an integer k such that

$$\left|c|a-a'| - \frac{k}{2} - \frac{1}{8}\right| \leq \frac{C_5}{k+1}. \tag{9}$$

There is no loss of generality to assume that elements in A are sufficiently separated to have $C_5/(k+1)<1/100$. Similarely, there also exist integers l,m such that

$$\left|c|a-a''| - \frac{l}{2} - \frac{1}{8}\right| \leq \frac{1}{100} \quad\text{and}\quad \left|c|a'-a''| - \frac{m}{2} - \frac{1}{8}\right| \leq \frac{1}{100}. \tag{10}$$

Now, since a,a',a'' are in a box of size q by α, if $|a-a''|\geq |a-a'|,|a'-a''|$ then

$$\begin{aligned}
|a-a''| &\geq (|a-a'|^2-\alpha^2)^{1/2} + (|a'-a''|^2-\alpha^2)^{1/2}\\
&= |a-a'|+|a'-a''| - \frac{\alpha^2}{(|a-a'|^2-\alpha^2)^{1/2}+|a-a'|} - \frac{\alpha^2}{(|a'-a''|^2-\alpha^2)^{1/2}+|a'-a''|}\\
&\geq |a-a'|+|a'-a''| - \frac{1}{100c}
\end{aligned}$$

where c is the constant in (9), provided we have taken α small enough. It follows that

$$\left|c|a-a''| - c|a-a'| - c|a'-a''|\right| \leq \frac{1}{100}.$$

But the, from (9) and (10),

$$\left|\frac{l-m-k}{2} - \frac{1}{8}\right| \leq \frac{1}{25}$$

a clear contradiction. Thus every q by α rectangle contains at most 2 elements of a. It follows that A has at most $2q/\alpha$ elements in a $q\times q$ square.

Remark: The same proof in dimension $d\neq 1 \mod (4)$ works provided we use $q\times\alpha\times\cdots\times\alpha$ tubes. We would then obtain that A has at most $\lesssim q^{d-1}$ elements in any cube of side d.

2.3. **Orthogonal exponential bases.** The following result is proved in [IKP01] in the case of the ball, and in [IKT01] in the general case. We shall give a completely self-contained and transparent proof below.

Theorem 2.1 (Iosevich, Katz, Pedersen, Tao, [IKP01, IKT01])**.** *Let K be a symmetric convex set in $\mathbb{R}^d$ with a smooth boundary and everywhere non-vanishing curvature. Then $L^2(K)$ does not possess an orthogonal basis of exponentials.*

Proof. To prove Theorem 2.1, assume that $L^2(K)$ does possess an orthogonal basis of exponentials $\{e^{2\pi ix\cdot a}\}_{a\in A}$. From Theorem 1.2, $\#(A\cap[-q,q]^2)\lesssim q$.

But, as is well known [Be66, La67, GR96, IKP01], if $\{e^{2\pi ix\cdot a}\}_{a\in A}$ is an orthonormal basis of exponentials of $L^2(K)$, then $\limsup \dfrac{\#(A\cap[-q,q]^2)}{q^2}>0$, a contradiction. □

3. A shifted Erdös-Solymosi theorem

In the remaining of the paper, we will identify $\mathbb{R}^2$ and $\mathbb{C}$.

3.1. **Proof of Theorem 1.4.** In this section, we will prove the following slightly stronger version of Theorem 1.4

Theorem 3.1. *Let $s\in\mathbb{R}$ be such that $8s^8\notin\mathbb{Z}+4s\mathbb{Z}+2s^2\mathbb{Z}+4s^3\mathbb{Z}+s^4\mathbb{Z}+2s^5\mathbb{Z}+2s^6\mathbb{Z}+4s^7\mathbb{Z}$. If $A\subset\mathbb{R}^2$ is such that $\Delta(A)\subset\mathbb{Z}^++s$, then $\#A\leq 3$.*

The result is best possible as if $a_0=0$, $a_1=r$, $a_2=re^{2i\pi/3}$, then all distances are equal to r.

Proof. Assume that a_0,a_1,a_2,a_3 are four different elements from A. For $j=1,\ldots,3$ let $\alpha_j=a_j-a_0$ and write $\alpha_j=(k_j+s)e^{i\theta_j}$. There is no loss of generality in assuming that α_1 is real, that is $\theta_1=0$.

For $j=2,3$ let $\beta_j=a_j-a_1=\alpha_j-\alpha_1$. Write $|\beta_j|=l_j+s$ with $l_j\in\mathbb{Z}^+$.

From $|\beta_j|^2=|\alpha_j-\alpha_1|^2=|\alpha_1|^2+|\alpha_j|^2-2|\alpha_1||\alpha_j|\cos\theta_j$, we deduce that

$$2|\alpha_1||\alpha_j|\cos\theta_j=|\alpha_1|^2+|\alpha_j|^2-|\beta_j|^2. \tag{11}$$

From this, we get that

$$(2|\alpha_1||\alpha_j|\sin\theta_j)^2=4|\alpha_1|^2|\alpha_j|^2-(|\alpha_1|^2+|\alpha_j|^2-|\beta_j|^2)^2 \tag{12}$$

On the other hand $\alpha_3-\alpha_2=a_3-a_2$ thus $|\alpha_3-\alpha_2|=m+s$ for some $m\in\mathbb{Z}^+$ and

$$2|\alpha_2||\alpha_3|\cos(\theta_3-\theta_2)=|\alpha_2|^2+|\alpha_3|^2-|\alpha_3-\alpha_2|^2, \tag{13}$$

from which we get that

$$\begin{aligned}
4|\alpha_1|^2|\alpha_2||\alpha_3|\cos(\theta_3-\theta_2) &= 2|\alpha_1|^2(|\alpha_2|^2+|\alpha_3|^2-|\alpha_3-\alpha_2|^2)\\
&= 2(k_1+s)^2\big((k_2+s)^2+(k_3+s)^2-(m+s)^2\big)\\
&= 2k_1^2(k_2^2+k_3^2-m^2)+4s\big(k_1^2(k_2+k_3-m)+k_1(k_2^2+k_3^2-m^2)\big)\\
&\qquad +s^2\big(k_1^2+k_2^2+k_3^2-m^2+4k_1(k_2+k_3-m)\big)\\
&\qquad\quad +4s^3(k_1+k_2+k_3-m)+2s^4\\
&\in 2\mathbb{Z}+s4\mathbb{Z}+s^22\mathbb{Z}+s^34\mathbb{Z}+2s^4.
\end{aligned} \tag{14}$$

On the other hand,

$$4|\alpha_1|^2|\alpha_2||\alpha_3|\cos(\theta_3-\theta_2)=4|\alpha_1|^2|\alpha_2||\alpha_3|(\cos\theta_2\cos\theta_3+\sin\theta_2\sin\theta_3).$$

But, with (11),

$$\begin{aligned}
4|\alpha_1|^2|\alpha_2||\alpha_3|\cos\theta_2\cos\theta_3 &= \big((k_1+s)^2+(k_2+s)^2-(l_2+s)^2\big)\\
&\qquad\times\big((k_1+s)^2+(k_3+s)^2-(l_3+s)^2\big)\\
&\in \mathbb{Z}+s2\mathbb{Z}+s^2\mathbb{Z}+s^32\mathbb{Z}+s^4,
\end{aligned} \tag{15}$$

as can be seen by expanding the expression in the first line. It follows that

$$4|\alpha_1|^2|\alpha_2||\alpha_3|\sin\theta_2\sin\theta_3=4|\alpha_1|^2|\alpha_2||\alpha_3|\cos(\theta_3-\theta_2)-4|\alpha_1|^2|\alpha_2||\alpha_3|\cos\theta_2\cos\theta_3$$

has to be in $\mathbb{Z}+s2\mathbb{Z}+s^2\mathbb{Z}+s^32\mathbb{Z}+s^4$, thus

$$(4|\alpha_1|^2|\alpha_2||\alpha_3|\sin\theta_2\sin\theta_3)^2\in\mathbb{Z}+s4\mathbb{Z}+s^22\mathbb{Z}+s^34\mathbb{Z}+s^4\mathbb{Z}+s^54\mathbb{Z}+s^62\mathbb{Z}+s^74\mathbb{Z}+s^8.$$

Now, from (12), we get that

$$(16)\qquad \begin{array}{rcl}(2|\alpha_1||\alpha_j|\sin\theta_j)^2&=&4(k_1+s)^2(k_j+s)^2-\big((k_1+s)^2+(k_j+s)^2-(l_j+s)^2\big)^2\\ &\in&\mathbb{Z}+s4\mathbb{Z}+s^22\mathbb{Z}+s^34\mathbb{Z}+3s^4\end{array}$$

as previously, thus

$$(4|\alpha_1|^2|\alpha_2||\alpha_3|\sin\theta_2\sin\theta_3)^2\in\mathbb{Z}+4s\mathbb{Z}+s^22\mathbb{Z}+s^34\mathbb{Z}+s^4\mathbb{Z}+s^54\mathbb{Z}+s^62\mathbb{Z}+s^712\mathbb{Z}+9s^8.$$

We thus want that

$$8s^8\in\mathbb{Z}+4s\mathbb{Z}+2s^2\mathbb{Z}+4s^3\mathbb{Z}+s^4\mathbb{Z}+2s^5\mathbb{Z}+2s^6\mathbb{Z}+4s^7\mathbb{Z}$$

which contradicts our assumption on s. □

Remark: The assumption on s is quite mild as it is satisfied by all transcendental numbers, all algebraic numbers of order at least 9 and also by all rational numbers that are not integers nor half-integers. For this last fact, if $s=\frac{p}{q}$ with p,q mutually prime and $q\neq 1$, then the assumption reads

$$8p^8\notin q^8\mathbb{Z}+4pq^7\mathbb{Z}+2p^2q^6\mathbb{Z}+4p^3q^5\mathbb{Z}+p^4q^4\mathbb{Z}+2p^5q^3\mathbb{Z}+2p^6q^2\mathbb{Z}+4p^7q\mathbb{Z}\subset q\mathbb{Z}$$

so that q divides 8. But, then writing $q=2r$ with $r=1,2$ or 3, the assumption reads

$$8p^8\notin 2^8r^8\mathbb{Z}+2^9pr^7\mathbb{Z}+2^7p^2r^6\mathbb{Z}+2^6p^3r^5\mathbb{Z}+2^4p^4r^4\mathbb{Z}+2^4p^5r^3\mathbb{Z}+2^3p^6r^2\mathbb{Z}+2^3p^7r\mathbb{Z}\subset 8r\mathbb{Z}$$

so that $r=1$ and $s=\frac{p}{2}$.

Also, note that one may scale the assumption. For example, if we assume that $\Delta(A)\subset\frac{1}{2}\mathbb{Z}^++\frac{1}{8}$ then $\#A\leq 3$, since $B=2A=\{2a,\ a\in A\}$ satisifes $\Delta(B)\subset\mathbb{Z}+\frac{1}{4}$, which establishes the link with zeroes of the Bessel function J_1 (*see* next section).

Of course, we may scale both ways, and we then for instance get that if $\Delta(A)\subset 4\mathbb{Z}^++1$ or if $\Delta(A)\subset 4\mathbb{Z}^++3$ (that is $s=1/4$ and $s=3/4$ respectively) then $\#A\leq 3$. This result is false when $s=1/2$ since it is not hard to construct sets for which $\Delta(A)\subset 2\mathbb{Z}^++1$. Indeed, let $k,l\in\mathbb{Z}^+$ and assume that $2l+1\leq 2(2k+1)$ and let $\theta=\arccos\frac{2l+1}{2(2k+1)}$. Finally, let $a_0=0$, $a_1=1$, $a_2=(2k+1)e^{i\theta}$ and $a_3=-(2k+1)e^{-i\theta}$. It is then clear that a_i-a_0, $i=1,2,3$ all have odd integer modulus. Further $a_3-a_2=-2(2k+1)\cos\theta=-2l-1$ is an odd integer. Finally

$$\begin{array}{rcl}|a_2-a_1|&=&\big((2k+1)\cos\theta-1\big)^2+(2k+1)^2\sin^2\theta\\ &=&(2k+1)^2+1-2(2k+1)\cos\theta=4k^2+2(2k-l)+1\end{array}$$

while

$$\begin{array}{rcl}|a_3-a_1|&=&\big(-(2k+1)\cos\theta-1\big)^2+(2k+1)^2\sin^2\theta\\ &=&(2k+1)^2+1+2(2k+1)\cos\theta=4k^2+2(2k+l)+3\end{array}$$

are also an odd integers. It seems nevertheless clear from the previous proof that some sparcity should happen in this case.

3.2. **Asymptotic of zeroes of Bessel functions.** Recall that $\widehat{\chi_B}(\xi)=\frac{1}{|\xi|}J_1(2\pi|\xi|)$ where J_1 is the Bessel function of order 1. Further, J_1 has the following asymptotic expansion when $r\to+\infty$ (*see e.g.* [St93, VIII 5.2, page 356-357] :

$$J_1(r)\sim-\left(\frac{2}{\pi r}\right)^{1/2}\left[\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma\left(\frac{3}{2}+2j\right)}{2^{2j}(2j)!\Gamma\left(\frac{3}{2}-2j\right)}\frac{\sin\left(r-\frac{\pi}{4}\right)}{r^{2j}}-\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma\left(\frac{5}{2}+2j\right)}{2^{2j+1}(2j+1)!\Gamma\left(\frac{1}{2}-2j\right)}\frac{\cos\left(r-\frac{\pi}{4}\right)}{r^{2j+1}}\right].$$

In particular, we get

$$\widehat{\chi_B}(\xi)=\frac{-1}{\pi|\xi|^{3/2}}\left[\sin 2\pi\left(|\xi|-\frac{1}{8}\right)-\frac{3}{16\pi|\xi|}\cos 2\pi\left(|\xi|-\frac{1}{8}\right)+O\left(\frac{1}{|\xi|^2}\right)\right]\tag{17}$$

$$=\frac{-1}{\pi|\xi|^{3/2}}\left[\left(1+\frac{5\times3}{2^9\pi^2|\xi|^2}\right)\sin 2\pi\left(|\xi|-\frac{1}{8}\right)-\left(\frac{3}{16\pi|\xi|}-\frac{7\times5\times3}{2^{13}\pi^3|\xi|^3}\right)\cos 2\pi\left(|\xi|-\frac{1}{8}\right)+O\left(\frac{1}{|\xi|^4}\right)\right].\tag{18}$$

$$=\frac{-1}{\pi|\xi|^{3/2}}\left[\left(1+\frac{5\times3}{2^9\pi^2|\xi|^2}-\frac{7\times5^2\times3^3}{2^{19}\pi^4|\xi|^4}\right)\sin 2\pi\left(|\xi|-\frac{1}{8}\right)-\left(\frac{3}{16\pi|\xi|}-\frac{7\times5\times3}{2^{13}\pi^3|\xi|^3}\right)\cos 2\pi\left(|\xi|-\frac{1}{8}\right)+O\left(\frac{1}{|\xi|^5}\right)\right].\tag{19}$$

From the first asymptotic formula, we get that if $\chi_B(\xi)=0$ and if $|\xi|$ is big enough, then there exists $k\in\mathbb{Z}$ such that $|\xi|=\dfrac{4k+1}{8}+e_k$ with $|e_k|\leq\frac{C}{k}$. (17) further gives that

$$\sin 2\pi e_k-\frac{3}{2\pi(4k+1+8e_k)}\cos 2\pi e_k=O\left(\frac{1}{k^2}\right).$$

As $e_k=O\left(\frac{1}{k}\right)$ we get

$$2\pi e_k-\frac{3}{2\pi(4k+1)}=O\left(\frac{1}{k^2}\right),$$

that is $e_k=\dfrac{3}{4\pi^2(4k+1)}+\varepsilon_k$ with $\varepsilon_k=O\left(\dfrac{1}{k^2}\right)$. But then, using (18), we obtain

$$I:=\left(1+\frac{5\times3}{2^9\pi^2|\xi|^2}\right)\sin 2\pi\left(|\xi|-\frac{1}{8}\right)=\frac{3}{2\pi(4k+1)}+2\pi\varepsilon_k+O\left(\frac{1}{k^4}\right)$$

and

$$II:=\left(\frac{3}{16\pi|\xi|}-\frac{7\times5\times3}{2^{13}\pi^3|\xi|^3}\right)\cos 2\pi\left(|\xi|-\frac{1}{8}\right)=\frac{3}{2\pi(4k+1)}+O\left(\frac{1}{k^3}\right)$$

so that $\varepsilon_k=O\left(\frac{1}{k^3}\right)$. Bootstaping this, we obtain that

$$I=\frac{3}{2\pi(4k+1)}+2\pi\varepsilon_k-\frac{9}{4\pi^3(4k+1)^3}+O\left(\frac{1}{k^4}\right)$$

while

$$II=\frac{3}{2\pi(4k+1)}-\frac{69}{4\pi^3(4k+1)^3}+O\left(\frac{1}{k^4}\right)$$

thus $\varepsilon_k=-\frac{15}{2\pi^4(4k+1)^3}+O\left(\frac{1}{k^4}\right)$.

So far, we proved that

$$|\xi|=\frac{1}{8}(4k+1)\left(1+\frac{6}{\pi^2(4k+1)^2}-\frac{120}{\pi^4(4k+1)^4}+O\left(\frac{1}{k^5}\right)\right).$$

Using (19), it is not hard to see that the $O(1/k^5)$ is actually $O(1/k^6)$ thus

$$\begin{aligned}|\xi| &= \frac{1}{2}(k+1/4)\left(1+\frac{3}{8\pi^2(k+1/4)^2}-\frac{15}{2^5\pi^4(k+1/4)^4}+O\left(\frac{1}{k^6}\right)\right)\\ |\xi|^2 &= \frac{1}{4}(k+1/4)^2\left(1+\frac{3}{4\pi^2(k+1/4)^2}-\frac{51}{2^6\pi^4(k+1/4)^4}+O\left(\frac{1}{k^6}\right)\right).\end{aligned}$$

It is not hard to see that if one keeps going on like this, then

$$|\xi|^2=\frac{1}{4}(k+1/4)^2\left(1+\frac{3}{4\pi^2(k+1/4)^2}+\sum_{j=2}^{N}\frac{c_j}{\pi^{2j}(k+1/4)^{2j}}+O\left(\frac{1}{k^{2N+2}}\right)\right). \tag{20}$$

where the c_j's are rational constants.

3.3. **A perturbation of Theorem 1.4.** We will now show that we can still perturbate Theorem 1.4 so as that the difference set $\Delta(A)$ be almost zeroes of the Bessel function. Up to a harmless scale, this reads as follows:

Proposition 3.2. *Let $s\in\mathbb{Q}\setminus\frac{1}{2}\mathbb{Z}$ and let η be either algebraic of order at least 5 or trancendental. Let $A\subset\mathbb{R}^2$ be such that every element $\alpha\in\Delta(A)$ has the property that $|\alpha|^2=(k+s)^2+\eta$, then A has at most 3 elements.*

The example we have in mind is of course $s=1/4$ and $\eta=\frac{3}{4\pi^2}$. The assumption $|\alpha|^2=(k+1/4)^2+\frac{3}{4\pi^2}$ then ammounts, up to scaling everything by 2, to approximate the zeroes of the Bessel function at first order in the asymptotic expension (20).

Proof. We use the same notation and asumptions as in the proof of Theorem 1.4. Identity (14) then becomes

$$\begin{aligned}4|\alpha_1|^2|\alpha_2||\alpha_3|\cos(\theta_3-\theta_2) &= 2\big((k_1+s)^2+\eta\big)\big((k_2+s)^2+(k_3+s)^2-(m+s)^2+\eta\big)\\ &= 2k_1^2(k_2^2+k_3^2-m^2)+4s\big(k_1^2(k_2+k_3-m)+k_1(k_2^2+k_3^2-m^2)\big)\\ &\quad+s^2\big(k_1^2+k_2^2+k_3^2-m^2+4k_1(k_2+k_3-m)\big)\\ &\quad+4s^3(k_1+k_2+k_3-m)+2s^4\\ &\quad+2\eta\big((k_1+s)^2+(k_2+s)^2+(k_3+s)^2-(m+s)^2\big)+2\eta^2\\ &\in \mathbb{Z}+4s\mathbb{Z}+s^2\mathbb{Z}+4s^3\mathbb{Z}+2s^4+\eta\mathbb{Q}+2\eta^2.\end{aligned}$$

Identity (15) becomes

$$\begin{aligned}4|\alpha_1|^2|\alpha_2||\alpha_3|\cos\theta_2\cos\theta_3 &= \big((k_1+s)^2+(k_2+s)^2-(l_2+s)^2+\eta\big)\\ &\quad\times\big((k_1+s)^2+(k_3+s)^2-(l_3+s)^2+\eta\big)\\ &\in \mathbb{Z}+2s\mathbb{Z}+s^2\mathbb{Z}+2s^3\mathbb{Z}+s^4+\eta\mathbb{Q}+\eta^2.\end{aligned}$$

It follows that $4|\alpha_1|^2|\alpha_2||\alpha_3|\big(\cos(\theta_3-\theta_2)-\cos\theta_2\cos\theta_3\big)$ belongs to

$$\mathbb{Z}+2s\mathbb{Z}+s^2\mathbb{Z}+2s^3\mathbb{Z}+s^4+\eta\mathbb{Q}+\eta^2.$$

Squaring, we get that $\big(4|\alpha_1|^2|\alpha_2||\alpha_3|\sin\theta_2\sin\theta_3\big)^2$ is in

$$\mathbb{Z}+s4\mathbb{Z}+s^22\mathbb{Z}+s^34\mathbb{Z}+s^4\mathbb{Z}+s^54\mathbb{Z}+s^62\mathbb{Z}+s^74\mathbb{Z}+s^8+\eta\mathbb{Q}+\eta^2\mathbb{Q}+\eta^3\mathbb{Q}+\eta^4.$$

On the other hand, Identity (16) becomes

$$\begin{aligned}(2|\alpha_1||\alpha_j|\sin\theta_j)^2 &= 4\big((k_1+s)^2+\eta\big)\big((k_j+s)^2+\eta\big)\\ &\quad-\big((k_1+s)^2+(k_j+s)^2-(l_j+s)^2+\eta\big)^2\\ &\in \mathbb{Z}+s4\mathbb{Z}+s^22\mathbb{Z}+s^34\mathbb{Z}+3s^4+\eta\mathbb{Q}+3\eta^2.\end{aligned}$$

It follows that $\big(4|\alpha_1|^2|\alpha_2||\alpha_3|\sin\theta_2\sin\theta_3\big)^2$ is in

$$\mathbb{Z}+4s\mathbb{Z}+s^22\mathbb{Z}+s^34\mathbb{Z}+s^4\mathbb{Z}+s^54\mathbb{Z}+s^62\mathbb{Z}+s^712\mathbb{Z}+9s^8+\eta\mathbb{Q}+\eta^2\mathbb{Q}+\eta^3\mathbb{Q}+9\eta^4$$

We still have a contradiction:

$$8s^8 \in \mathbb{Z} + 4s\mathbb{Z} + 2s^2\mathbb{Z} + 4s^3\mathbb{Z} + s^4\mathbb{Z} + 2s^5\mathbb{Z} + 2s^6\mathbb{Z} + 4s^7\mathbb{Z} + \mathbb{R}\setminus\mathbb{Q}$$

since, as noticed at the end of the proof of Theorem 3.1, $8s^8 \in \mathbb{Q}\setminus(\mathbb{Z} + 4s\mathbb{Z} + 2s^2\mathbb{Z} + 4s^3\mathbb{Z} + s^4\mathbb{Z} + 2s^5\mathbb{Z} + 2s^6\mathbb{Z} + 4s^7\mathbb{Z})$ when $s \in \mathbb{Q}\setminus\frac{1}{2}\mathbb{Z}$. □

Remark: We have used that the perturbation by η is fixed only in a mild way in order to simplify computations. Actually, it is not hard to see that if we assume that if each $\alpha \in \Delta(A)$ has the property that $|\alpha|^2 = (k+s)^2 + P_k(\eta)$ where k is an integer, η is a fixed *trancendental* number and P_k is a polynomial with rational coefficients, then the above proof still gives the result. If moreover the degrees of the $P_k's$ are bounded by M, then the proof works provided η is algebraic of order at least $4M+1$.

In particular, the result is valid for any truncation of the assymptotic formula of the zeroes of the Bessel function J_1, that is, if we assume that if each $\alpha \in \Delta(A)$ has the property that $|\alpha|^2$ be of the form

$$\frac{1}{4}(k+1/4)^2\left(1 + \sum_{j=1}^{N} \frac{c_j}{\pi^{2j}(k+1/4)^{2j}}\right).$$

An even more carefull examination of the proof shows that, if each $\alpha \in \Delta(A)$ is of the form $|\alpha|^2 = (k+s)^2 + \eta_k$ then only 6 η_k's intervene in the proof (corresponding to $k_1, \ldots, k_3$, l_2, l_3 and m). Moreover they are raised to the power at most 4, so A has at most 3 elements as soon as there exists no rational polynomial relation of degree at most 4 between any 6 η_k's *i.e* if, for any $j_1, \ldots, j_6$, the only polynomial of degree ≤ 4 of 6 variables with coefficients in $\mathbb{Q}$ such that $P(\eta_{j_1}, \ldots, \eta_{j_6}) = 0$ is $P = 0$. Such relations are highly unlikely beetween zeroes of the Bessel function J_1, it is thus natural to conjecture, following Fluglede [Fu72] that the disk has no more than 3 orthogonal exponentials.

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