Combinatorial methods or integer tiling

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Tiling the integers: an introduction
Tiling the integers with a finite set

Let $A \subset \mathbb{Z}$ be a finite set. We say that $A$ tiles $\mathbb{Z}$ by translations if $\mathbb{Z}$ can be covered by a union of disjoint translates of $A$. (There is an infinite set $T \subset \mathbb{Z}$ such that every $x \in \mathbb{Z}$ can be uniquely represented as $x = a + t$, with $a \in A$, $t \in T$.)
Let $A \subset \mathbb{Z}$ be a finite set. We say that $A$ tiles $\mathbb{Z}$ by translations if $\mathbb{Z}$ can be covered by a union of disjoint translates of $A$. (There is an infinite set $T \subset \mathbb{Z}$ such that every $x \in \mathbb{Z}$ can be uniquely represented as $x = a + t$, with $a \in A$, $t \in T$.)

$A = \{0, 2\}$ and $A = \{0, 4, 8\}$ tile $\mathbb{Z}$; $A = \{0, 1, 3\}$ does not.

How to determine whether a given $A$ tiles the integers?
Newman (1977): all tilings of \( \mathbb{Z} \) by a finite set \( A \) are periodic, with period \( M \).

Tijdeman (1993) + Coven-Meyerowitz (1998): if \( A \) tiles the integers, then it also tiles a finite cyclic group \( \mathbb{Z}_M \), where \( M \) has the same prime factors as \( |A| \).

This reduces the problem to the study of tilings of finite cyclic groups \( \mathbb{Z}_M = \{0, 1, \ldots, M - 1\} \), with addition mod \( M \).

Notation:

\[
A \oplus B = \mathbb{Z}_M.
\]
Further reductions

We measure distances between elements in $\mathbb{Z}_M$ in terms of the GCD with $M$. In particular

$$\text{Div}(A) := \{(a - a', M) : a, a' \in A\}$$
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$$\text{Div}(A) := \{(a - a', M) : a, a' \in A\}$$

Sands (1979): let $A, B \subset \mathbb{Z}_M$. Then $A \oplus B = \mathbb{Z}_M$ if and only if $|A||B| = M$ and

$$\text{Div}(A) \cap \text{Div}(B) = \{M\}.$$
Suppose that $A \oplus B = \mathbb{Z}_M$, with $M = \prod_{i=1}^{K} p_i^{n_i}$, $p_i$ distinct primes, $n_i \geq 1$. By the Chinese Remainder Theorem, we have

$$\mathbb{Z}_M = \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_K^{n_K}},$$

which we can represent geometrically as a $K$-dimensional lattice. Then $A \oplus B$ can be interpreted as a (modular) tiling of that lattice.
\[ \mathbb{Z}_M = \mathbb{Z}_{p_i^2} \oplus \mathbb{Z}_{p_j^2}, \quad M = p_i^2 p_j^2 \]
Geometric representation of sets

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\[ \mathbb{Z}_M = \mathbb{Z}_{p_i^{n_i}} \oplus \mathbb{Z}_{p_j^{n_j}} \oplus \mathbb{Z}_{p_k^{n_k}}, M = p_i^{n_i} p_j^{n_j} p_k^{n_k} \]

\[ \{x \in \mathbb{Z}_M : p_j^{n_j} p_k^{n_k} |x - a\} = a + M/p_i^{n_i} \mathbb{Z}_M \]

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Geometric representation of sets

\[ \{0, M/p_i, 2M/p_i, \ldots, (p_i - 1)M/p_i\} \]

\[ \{x \in \mathbb{Z}_M : M/p_ip_j|x\} \]
Examples of tilings
Examples of tilings
The Coven-Meyerowitz tiling conditions
By translational invariance, we may assume that $A, B \subset \{0, 1, \ldots \}$ and that $0 \in A \cap B$. The characteristic polynomials (aka mask polynomials) of $A$ and $B$ are

$$A(X) = \sum_{a \in A} X^a, \quad B(X) = \sum_{b \in B} X^b.$$ 

Then $A \oplus B = \mathbb{Z}_M$ is equivalent to

$$A(X)B(X) = 1 + X + \cdots + X^{M-1} \mod (X^M - 1).$$
Recall the $s$-th cyclotomic polynomial is the unique monic, irreducible polynomial $\Phi_s(X)$ whose roots are the primitive $s$-th roots of unity.

Then the tiling condition $A(X)B(X) = 1 + X + \cdots + X^{M-1} \mod (X^M - 1)$ is equivalent to

$$|A||B| = M \text{ and } \Phi_s(X) \mid A(X)B(X) \text{ for all } s \mid M, \ s \neq 1.$$ 

Since $\Phi_s$ are irreducible, each $\Phi_s(X)$ with $s \mid M, \ s \neq 1$, must divide at least one of $A(X)$ and $B(X)$. 

Cyclotomic polynomials
Let \( S_A = \{ p^\alpha : \Phi_{p^\alpha}(X)|A(X) \} \). Consider the following conditions.

\((T1)\) \( A(1) = \prod_{s \in S_A} \Phi_s(1) \),

\((T2)\) if \( s_1, \ldots, s_k \in S_A \) are powers of different primes, then \( \Phi_{s_1 \ldots s_k}(X) \) divides \( A(X) \).

Then:

- if \( A \) satisfies \((T1)\), \((T2)\), then \( A \) tiles \( \mathbb{Z} \);
- if \( A \) tiles \( \mathbb{Z} \) then \((T1)\) holds;
- if \( A \) tiles \( \mathbb{Z} \) and \( |A| \) has at most two prime factors, then \((T2)\) holds.
Cyclotomic polynomials and distribution

Divisibility by prime power cyclotomic polynomials $\Phi_{p_i^\alpha}$ can be interpreted in terms of distribution of the elements of $A$:

- $\Phi_{p_i} | A \iff A$ is equidistributed mod $p_i$,
- $\Phi_{p_i^{n_i}} | A \iff A$ is equidistributed mod $p_i^{n_i}$ within residue classes mod $p_i^{n_i-1}$.

$M/p_i \\{ a_0 \in A \}$

$M/p_i \\{ a_1, a_2, \ldots, a_{p_i-1} \in A \}$

$p_i^{n_i-1} \parallel a_\nu - a_{\nu'}$
Assume $M = \prod_i p_i^{n_i}$ and let $M_i = M/p_i^{n_i}$. Given $1 \leq \alpha \leq n_i$, we define

$$F_{i,\alpha} = \{0, M_ip_i^{\alpha-1}, 2M_ip_i^{\alpha-1}, \ldots, (p_i - 1)M_ip_i^{\alpha-1}\}$$
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$$F_{i,\alpha}(X) = \prod_{d|M_i} \Phi_{dp_i^\alpha}(X)$$
Reformulation of T2

Assume $A \oplus B = \mathbb{Z}_M$ and define

$$B^b(X) := \prod_i \prod_{\alpha : p_i^\alpha \in S_B} F_{i,\alpha}(X)$$
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Note that
- for $m \mid M \Phi_m \nmid B^b$ if and only if
  \[ m = \prod_{i \in I \subseteq \{1, \ldots, K\}} p_i^{\beta_i} \text{ and } \Phi_{p_i^{\beta_i}} \mid A \text{ for all } i (p_i^{\beta_i} \in S_A) \]
- since $S_A \cup S_B = \{ p^\alpha : p^\alpha \mid M \}$ and disjoint, $B^b$ is uniquely determined by $A$. 
Reformulation of T2

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  and $\Phi_{p_i^{\{i}}} | A$ for all $i$ ($p_i^{\beta_i} \in S_A$)

- since $S_A \cup S_B = \{p^\alpha : p^\alpha | M\}$ and disjoint, $B^b$ is uniquely determined by $A$.

Then

$A$ satisfies T2 if and only if $A$ is a maximal set satisfying

$$\#(A' \cap B^b) = 1$$

for all $A'$-translation of $A$. 
Reformulation of T2

Example:

Suppose that $M = p_i^2 p_j^2 p_k^2$ and $A \subset \mathbb{Z}_M, |A| = p_i p_j p_k$ is uniformly distributed modulo $p_i, p_j$ and $p_k$. Then

The following are equivalent

- $A$ satisfies T2
- any translation of $A$ intersects $p_i p_j p_k \mathbb{Z}_M$ at exactly one point
- $A$ is uniformly distributed modulo $p_i p_j p_k$. 
Standard T2 sets

\[ A = A^b, \ B = B^b \]
Main result
**Theorem.** Suppose that $A \oplus B = \mathbb{Z}_M$, with $M = p_i^2 p_j^2 p_k^2$. Then $A$ and $B$ satisfy T2.
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Additionally:

- The proof essentially provides a classification of all tilings of period $M = p_i^2p_j^2p_k^2$. (It does not get much more complicated than Szabó-type examples.)
- Methods and some intermediate results extend to more general $M$. 
Cuboids and cyclotomic divisibility

Particular case: Let $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$. An $M$-cuboid is a weighed set with the mask polynomial

$$\Delta(X) = X^a \prod_i (1 - X^{d_i M/p_i}) \mod (X^M - 1)$$

where $a \in \mathbb{Z}_M$ and $(d_i, p_i) = 1$. 

$$M/p_i \{ \bullet \ -1 \text{ weighted} \ \bullet \ +1 \text{ weighted} \}$$
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- For $A \subset \mathbb{Z}_M$ we have
  $$\Phi_M|A \iff \text{for every } \Delta, A|\Delta \text{ sums up to } 0$$
Let $M = \prod_i p_i^{n_i}$. An $M$-fiber in the $p_i$ direction is a set

$$\{a, a + M/p_i, a + 2M/p_i, \ldots, a + (p_i - 1)M/p_i\} \subset \mathbb{Z}_M.$$ 

- A set $A \subset \mathbb{Z}_M$ is $M$ fibered in the $p_i$ direction if it is a union of disjoint $M$ fibers in that direction.
Application: structure on grids

The cyclotomic $\Phi_M$ determines the structures of sets on $\Lambda(x, D(M))$ grids, where $D(M) = M/\prod_i p_i$ and

$$\Lambda(x, D(M)) := \{y \in \mathbb{Z}_M : D(M)|x - y\}.$$
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We prove the following basic structure lemma:
Let $A \subset \mathbb{Z}_M$, $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$
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Lemma

If $\Phi_M|A$ and $M/p_i \notin \text{Div}(A)$ for some $p_i \neq 2$. Then on every fixed grid $\Lambda(x, D(M))$, the set $A$ is $M$ fibered either in the $p_j$ or the $p_k$ direction.
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**Lemma**

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- Note that this lemma does not require the tiling assumption.
- The lemma is not true when $p_i = 2$. 
Application: structure on grids

Proof: Fix $a \in A$. We prove first that $a$ has to belong to an $M$ fiber in either the $p_j$ or the $p_k$ directions.
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Suppose this is not true and find $x_j, x_k \in \mathbb{Z}_M \setminus A$ with
\[(a - x_\nu, M) = M/p_\nu \text{ for } \nu = j, k.\]
Proof: Fix $a \in A$. We prove first that $a$ has to belong to an $M$ fiber in either the $p_j$ or the $p_k$ directions. Suppose this is not true and find $x_j, x_k \in \mathbb{Z}_M \setminus A$ with
\[(a - x_\nu, M) = M/p_\nu \text{ for } \nu = j, k.\]
Also, since $p_i \neq 2$ and $M/p_i$ is not a difference in $A$, we may find $x_i, x'_i \in \mathbb{Z}_M \setminus A$ with
\[(a - x_i, M) = (a - x'_i, M) = (x_i - x'_i, M) = M/p_i.\]
Application: structure on grids
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\[ M/p_i \]

\[ M/p_j \]

\[ M/p_k \]

\( x_k \)

\( x_i \)

\( x_j \)

\( x'_i \)

\( a \)
Application: structure on grids
We, therefore, conclude that every element in $A \cap \Lambda(a, D(M))$ is contained in an $M$ fiber in either the $p_j$ or $p_k$ direction. If the choice of direction is uniform across all elements of $A \cap \Lambda(a, D(M))$ - we’re done. Otherwise
We, therefore, conclude that every element in $A \cap \Lambda(a, D(M))$ is contained in an $M$ fiber in either the $p_j$ or $p_k$ direction. If the choice of direction is uniform across all elements of $A \cap \Lambda(a, D(M))$ - we’re done. Otherwise
Let $M = \prod_i p_i^{n_i}$ and define the slab

$$A_{p_i} = \{a \in A : 0 \leq a \mod p_i^{n_i} \leq p_i^{n_i-1} - 1\}$$
Let $M = \prod_i p_i^{n_i}$ and define the slab
\[ A_{p_i} = \{ a \in A : 0 \leq a \mod p_i^{n_i} \leq p_i^{n_i-1} - 1 \} \]
Theorem (Łaba-L, 2021)

Let \( A \oplus B = \mathbb{Z}_M \), and \( \Phi_{p_i^{n_i}}|A \). The following are equivalent:

(i) For any translate of \( A' \) of \( A \) we have \( A'_p \oplus B = \mathbb{Z}_{M/p_i} \).

(ii) For every \( p_i^{n_i}|m|M \) we have

\[
m \in \text{Div}(A) \Rightarrow m/p_i \notin \text{Div}(B).
\]

Remarks:

• The case \( M = p_2^{2}p_3^{2}p_2^{2} \).

• Open problem: prove the conclusion of the theorem hold for arbitrary \( M \).

If true then tiling would imply T2, by induction.
Theorem (Łaba-L, 2021)

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Remarks:

- The case $M = p_i^2p_j^2p_k^2$
- Open problem: prove the conclusion of the theorem hold for arbitrary $M$.
  If true then tiling would imply T2, by induction.
Slab reduction - application

Suppose that every element of $A$ belongs to an $M$-fiber in the $p_i$ direction, i.e.

$$a + M/p_i, \ldots, a + (p_i - 1)M/p_i \in A$$

for all $a \in A$.

We claim that $A$ satisfies the conditions of the slab reduction.
Suppose that every element of $A$ belongs to an $M$-fiber in the $p_i$
direction, i.e.

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We claim that $A$ satisfies the conditions of the slab reduction.

- $(1 + X^{M/p_i} + \ldots + X^{(p_i - 1)M/p_i})|A$, in particular $\Phi_{p_i^{n_i}}|A$. 

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We claim that $A$ satisfies the conditions of the slab reduction.

- $(1 + X^{M/p_i} + \ldots + X^{(p_i-1)M/p_i})|A$, in particular $\Phi_{p_i^{n_i}}|A$.
- $A$ satisfies the condition: for every $p_i^{n_i}|m|M$

$$m \in Div(A) \iff m/p_i \in Div(A).$$
Slab reduction - application

\[(a - a'', M) = m/p_i\]

\[(a - a', M) = m, p_i^{n_i}|m\]
Let \( A \oplus B = \mathbb{Z}_M \) with \( M = p_i^{n_i} p_j^{n_j} p_k^{n_k} \).

**Lemma**

Assume that \( \Phi_M \) divides both \( A \) and \( B \). Then either \( A \) or \( B \) has to be \( M \)-fibered in some direction. Moreover, if \( n_i = n_j = n_k = 2 \) then \( A \) and \( B \) satisfy T2.
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**Proof.**

- By PH principle, WLOG, $M/p_i, M/p_j \notin Div(A)$, and $\max\{p_i, p_j\} > 2$
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**Proof.**

- By PH principle, WLOG, $M/p_i, M/p_j \notin \text{Div}(A)$, and $\max\{p_i, p_j\} > 2$
- Since $\Phi_M|A$, by structure lemma $A$ must be $M$ fibered in the $p_k$ direction
Let $M = (p_ip_jp_k)^2$, $|A| = |B| = p_ip_jp_k$. 

\[
\begin{array}{c}
\Phi_M|A \\
\Phi_M|B \\
\Phi_M \not| B
\end{array}
\]
Let $M = (p_ip_jp_k)^2$, $|A| = |B| = p_ip_jp_k$. 

**Structure of proof**

$\Phi_M|A$

$\Phi_M|B$

$\Phi_M \not| B$

A, B satisfy T2
Let $M = (p_ip_jp_k)^2$, $|A| = |B| = p_ip_jp_k$.

A, B satisfy T2

A is fibered on every $M/p_ip_jp_k$-grid possibly in different directions

Split $A = \cup_i A(i)$ and go down one scale
Let $M = (p_ip_jp_k)^2$, $|A| = |B| = p_ip_jp_k$. 

A, B satisfy $T2$ 

$A$ is fibered on every $M/p_ip_jp_k$-grid possibly in different directions 

There exists an $M/p_ip_jp_k$-grid $\Lambda$ and $A \cap \Lambda$ has one of 4 possible structures 

Split $A = \bigcup_i A(i)$ and go down one scale
Structure of proof

“Corner”

“Full Plane”

“Almost Corner”
Let $M = (p_ip_jp_k)^2$, $|A| = |B| = p_ip_jp_k$. 

**Structure of proof**

- $\Phi_M|A$
- $\Phi_M|B$
- $\Phi_M \parallel B$

A, B satisfy T2

- A is fibered on every $M/p_ip_jp_k$-grid
  - possibly in different directions
  - Split $A = \cup_i A(i)$
  - and go down one scale
- There exists an $M/p_ip_jp_k$-grid $\Lambda$
  - and $A \cap \Lambda$ has one of 4 possible structures
  - Solve each of the structures
Thank you!