MEAN LATTICE POINT DISCREPANCY BOUNDS, II: CONVEX DOMAINS IN THE PLANE

ALEXANDER IOSEVICH   ERIC T. SAWYER   ANDREAS SEEGER

Abstract. We consider planar curved strictly convex domains with no or very weak smoothness assumptions and prove sharp bounds for square-functions associated to the lattice point discrepancy.

1. Introduction

This paper is a sequel to [13] in which the authors proved bounds for the mean square lattice point discrepancy for convex bodies with smooth boundary in $\mathbb{R}^d$. Here we reconsider the case $d = 2$ but admit now domains with rough boundary.

Let Ω be a convex domain in $\mathbb{R}^2$ containing the origin in its interior. Let

$$\mathcal{N}_\Omega(t) = \operatorname{card}(t\Omega \cap \mathbb{Z}^2),$$

the number of integer lattice point inside the dilate $t\Omega$. It is well known that $\mathcal{N}_\Omega(t)$ is asymptotic to $t^2\operatorname{area}(\Omega)$ as $t \to \infty$ and we denote by

$$E_\Omega(t) = \mathcal{N}_\Omega(t) - t^2\operatorname{area}(\Omega) \tag{1.1}$$

the error, or lattice rest. A trivial estimate for the lattice rest is $E_\Omega(t) \le Ct$ which holds for any convex set . For the case that the boundary is smooth and has positive curvature everywhere this estimate has been significantly improved. It is conjectured that in this case $E_\Omega(t) = O(t^{1/2+\varepsilon})$ for any $\varepsilon > 0$ but by the best result published at this time, due to Huxley [12], one only knows that $E_\Omega(t) = O(t^{46/73}(\log t)^A)$ for suitable A (according to [21] Huxley has improved the exponent 46/73 to 131/208).

On average however better estimates hold. We consider the mean-square discrepancy of the lattice rest over the interval $[R, R+h]$ where $h \le R$ and R is large; it is given by

$$\mathcal{G}_\Omega(R,h) = \Big(\frac{1}{h}\int_R^{R+h} |E_\Omega(t)|^2 dt\Big)^{1/2}. \tag{1.2}$$

Provided that the boundary is smooth (say C^4) and the Gaussian curvature never vanishes it has been shown by Nowak [20] that $\mathcal{G}_\Omega(R,R) = O(R^{1/2})$; later Huxley [11] showed that $\mathcal{G}_\Omega(R,1) = O(R^{1/2}\log^{1/2} R)$. A result which unifies both estimates is in the authors' paper [13], namely

Date: September 26, 2004.

Research supported in part by NSF grants.

$\mathcal{G}_\Omega(R,h) \le CR^{1/2}$ if $\log R \le h \le R$. We note that Nowak [21] has independently proved the same bound. Moreover he obtained asymptotics for $\mathcal{G}_\Omega(R,H(R))$ as $R \to \infty$, provided that $H(R)/\log R \to \infty$; see also earlier asymptotics by Bleher [2] where essentially $H(R) \approx R$.

The purpose of this paper is to prove versions of these estimates under minimal (or no) smoothness assumptions on the boundary of the domain. The main difficulty is that the oscillation of the Fourier transforms of densities on the boundary cannot be used in a straightforward way as in [19], [13], or [21], because of the lack of asymptotic expansions.

Our first result deals with domains for which the curvature is bounded below with very weak regularity assumptions on the curvature. Here we assume that Ω has C^1 boundary, that the components of the tangent vectors are absolutely continuous functions of the arclength parameter so that the second derivatives of a regular parametrization are well defined as L^1 functions on the boundary. The following theorem yields an analogue of the above result with a slightly more restrictive assumption on these second derivatives.

Theorem 1.1. *Let Ω be a convex domain in $\mathbb{R}^2$ containing the origin in its interior, and assume that Ω has C^1 boundary and that the components of the tangent vector are absolutely continuous functions. Suppose also that curvature κ is bounded below, i.e. $|\kappa(x)| \ge a > 0$ for almost every $x \in \partial\Omega$ and that $\kappa \in L\log^{2+\epsilon} L(\partial\Omega)$, for some $\epsilon > 0$. Then there is a constant C_Ω so that for all $R \ge 2$*

$$\mathcal{G}_\Omega(R,h) \le C_\Omega R^{1/2} \quad \text{if} \quad \log R \le h \le R. \tag{1.3}$$

Of course this result applies to all convex domains with C^2 boundary and nonvanishing curvature; but it also applies to rougher domains, the simplest examples are $\{x : |x_1|^{a_1} + |x_2|^{a_2} \le 1\}$ when $1 < a_1, a_2 \le 2$. Moreover if $\mathcal{D}$ is a convex domain with smooth finite type boundary, containing the origin, then the polar set $\Omega = \mathcal{D}^* = \{x : \sup_{\xi\in\mathcal{D}}\langle x,\xi\rangle \le 1\}$ satisfies the assumptions of Theorem 1.1. For these examples the second derivatives belong to $L^p(\partial\Omega)$ for some $p > 1$ (*cf.* the calculations in the proof of Lemma 5.1 in [13]).

An immediate consequence of (1.3) is Huxley's bound ([11]) who proved that $\mathcal{G}_\Omega(R,1) = O(\sqrt{R\log R})$ under the assumption that Ω has C^4 boundary and the curvature is bounded below. We shall see (*cf.* Theorem 1.3 below) that it is possible to prove this estimate for convex domains in which even the weak regularity assumption of Theorem 1.1 is removed. Moreover in this case we shall prove (*cf.* Theorem 1.2 below) that the optimal bound $\mathcal{G}_\Omega(R,h) = O(R^{1/2})$ holds in the more restricted range of h's $R^{1/2} \le h \le R$.

In this rough case the assumption of the curvature bounded below has to be reformulated (as now we are not actually assuming that the curvature is a well defined function). Let ρ^* be the Minkowski functional of the polar set Ω^*, *i.e.*

$$\rho^*(\xi) = \sup\{\langle x,\xi\rangle : x \in \Omega\} \tag{1.4}$$

so that $\Omega^* = \{\xi : \rho^*(\xi) \le 1\}$. For $\theta \in S^1$ and $\delta > 0$ consider the arc (or "cap")

$$\mathcal{C}(\theta,\delta) \equiv \mathcal{C}_\Omega(\theta,\delta) = \{x \in \partial\Omega : \langle x,\theta\rangle = \rho^*(\theta) - \delta\}. \tag{1.5}$$

Let

$$\mu(\theta,\delta) = \operatorname{diam}(\mathcal{C}(\theta,\delta)). \tag{1.6}$$

We note that if $d\sigma$ is the arclength measure on $\partial\Omega$ then $\sum_\pm \mu(\pm\theta,\delta)$ controls the size of the Fourier transform $\widehat{d\sigma}(\pm\theta/\delta)$, see [5] and also [4]. If the curvature is absolutely continuous and bounded below then it is easy to see that $\mu(\theta,\delta) = O(\sqrt{\delta})$ uniformly in $\theta \in S^1$, and in the general case we shall simply assume the validity of this inequality; see §10 for the equivalence with other natural definitions of bounded below curvature for rough domains.

Theorem 1.2. *Let Ω be a convex domain in $\mathbb{R}^2$ containing the origin in its interior. Suppose that*

$$\sup_{\theta\in S^1}\sup_{\delta>0} \delta^{-1/2}\mu(\theta,\delta) < \infty. \tag{1.7}$$

Then for $R \ge 2$,

$$\mathcal{G}_\Omega(R,h) \le C_{\Omega,\varepsilon} R^{1/2} \quad \textit{if} \quad R^{1/2} \le h \le R. \tag{1.8}$$

If we admit an additional factor of $\sqrt{\log R}$ the range of h can be vastly improved to obtain a version of Huxley's theorem ([11]) for rough domains with nonzero curvature (which is much more elementary than Theorem 1.2).

Theorem 1.3. *Let Ω be as in Theorem 1.2 (satisfying (1.7)). Then for $R \ge 2$*

$$\mathcal{G}_\Omega(R,h) \le C_\Omega (R\log R)^{1/2} \quad \textit{if} \quad 1 \le h \le R. \tag{1.9}$$

Remark. An examination of the proof of Theorem 1.3 shows that the constants depend only on the bound in (1.7) and the radii of inscribed and circumscribed circles centered at the origin. This uniform version of inequality (1.9) as well as the statement of Theorem 1.1 is close to sharp as one can show that they fail for $h \le (\log R)^{-1}$. To see this one uses Jarník's curve ([14]) to produce a sequence $R_j \to \infty$ and domains Ω_j, so that the maximal inscribed and minimal circumscribed radii of Ω_j are bounded above and below, the curvature on the boundary is bounded above and below and $E_{\Omega(R_j)} \ge R_j^{2/3}$ ([14], [16]). By Huxley's mean-max inequality ([12], p. 136)

$$\Big(\frac{1}{\delta}\int_{R_j-\delta}^{R_j+\delta} E_{\Omega_j}(s)^2 ds\Big)^{1/2} \ge E_{\Omega_j}(R_j)/2$$

which holds under the assumptions that $|E_{\Omega_j}(R_j)| \ge 5(\operatorname{area}(\Omega_j))\delta R_j$ and $0 < \delta \le R_j/2$. We apply this for $\delta \approx R_j^{-1/3} \le h$ to see that under the assumption of $\mathcal{G}_{\Omega_j}(R_j,h) \lesssim (R_j\log R_j)^{1/2}$ we have

$$R_j^{2/3} \lesssim E_{\Omega_j}(R_j) \lesssim R_j^{1/6} h^{1/2} \mathcal{G}_{\Omega_j}(R_j,h) \lesssim R_j^{2/3}(h\log R_j)^{1/2}.$$

thus $h \gtrsim (\log R_j)^{-1}$. *Cf.* also Plagne [23] for the construction of a single strictly convex curve C and a sequence R_j so that R_jC contains $R_j^{2/3}w(R_j)$ lattice points, with $w(R)$ converging to zero at a slow rate.

It is certainly conceivable that the result of Theorem 1.2 may hold for some $h \ll R^{1/2}$. However this could not be established by simple extensions of our method, see the discussion below and in §9.

Finally, if we consider arbitrary convex domains (dropping the curvature assumptions on the boundary) then the estimate (1.9) may fail as does the classical estimate $E_\Omega(t) = O(t^{2/3})$ (*cf.* [25]). However for almost all rotations $A \in SO(2)$ it is still true that $E_{A\Omega}(t) = O(t^{2/3}\log^{1/2+\varepsilon} t)$, see [3]. In fact for domains with smooth finite type boundary one has the better estimate $E_{A\Omega}(t) = O(t^{2/3-\delta})$, for almost all rotations, for some $\delta > 0$, see Nowak's article [20]. Likewise for such domains it is proved in [13] that for almost all rotations A we have $\mathcal{G}_{A\Omega}(R,h) \lesssim R^{1/2}$ for all $R \geq 2$, $\log R \leq h \leq R$. For arbitrary convex domains we can prove an analogous result but lose an additional power of a logarithm.

Theorem 1.4. *Let Ω be a convex domain in $\mathbb{R}^2$ containing the origin in its interior. For $\vartheta \in [-\pi,\pi]$ denote by A_ϑ the rotation by the angle ϑ and by $A_\vartheta\Omega$ the rotated domain $\{A_\vartheta x : x \in \Omega\}$. Then for $\epsilon > 0$, $R \geq 2$,*

$$\mathcal{G}_{A_\vartheta\Omega}(R,h) \leq \mathcal{C}_{\varepsilon,\Omega}(\vartheta)R^{1/2}(\log R)^{1+\epsilon} \quad \text{if} \quad 1 \leq h \leq R \tag{1.10}$$

where $\mathcal{C}_{\epsilon,\Omega}(\vartheta) < \infty$ for almost all $\vartheta \in [-\pi,\pi]$; in fact the function $\mathcal{C}_{\epsilon,\Omega}$ belongs to the weak type space $L^{2,\infty}$.

Structure of the paper: The first part of the proofs is identical for Theorems 1.1–1.4. One uses essentially a "T^*T-argument" to reduce to a weighted estimate for lattice points in thin annuli formed by dilations of the polar domain. This argument is straightforward in the smooth case ([13], [21]) but there are considerable technical complications in the rough case. The relevant estimates are given in §2. One is led to the estimation of quantities such as

$$\mathcal{K}(R,h) = \sum_{\substack{(k,\ell)\in\mathbb{Z}^2\times\mathbb{Z}^2 \\ |k|,|\ell|\leq R \\ |\rho^*(k)-\rho^*(\ell)|\leq h^{-1}}} |k|^{-2}\mu(\tfrac{k}{|k|},\tfrac{1}{|k|R})\,\mu(\tfrac{\ell}{|\ell|},\tfrac{1}{|\ell|R}) \tag{1.11}$$

when $h \gg 1$, and some variants with tails. §3 contains further discussion of these quantities in the case of curvature bounded below and the main technical propositions needed for the proofs of Theorems 1.1, 1.2, 1.3.

The estimation of (1.11) needed for Theorems 1.3 and 1.4 is rather straightforward. For Theorem 1.3 one uses the bound $\mu(\theta,\delta) = O(\delta^{1/2})$ in conjunction with the trivial bound $E_{\Omega^*}(t) = O(t)$. For Theorem 1.4 one argues similarly but uses an averaged estimate for the size of caps for the rotated domains. The proofs are contained in §3 and §4. The mild regularity assumption in Theorem 1.1 can be used to improve on the trivial bound for

$E^*_\Omega(t)$. This leads to boundedness of $\mathcal{K}(R,h)$ for $h \geq \log R$ and then in this range to the optimal bound (1.3); the argument is carried out in §5.

The main technical estimate needed for the proof of Theorem 1.2 is stated as Proposition 3.3. Here we need to efficiently estimate a *weighted* version of the lattice point discrepancy for Ω^*, and we shall use a more geometrical approach for which we need the assumption $h \geq R^{1/2}$. The proof is carried out in §6-§8.

We do not know whether in the generality of Theorem 1.2 the assumption $h \geq R^{1/2}$ is really necessary. In §9 we construct some examples of sets with rough boundary (and curvature bounded below) which show that at least for the estimation of $\mathcal{K}(R,h)$ the condition $h \geq R^{1/2}$ is necessary (which only shows the sharpness of the method). In §10 we shall discuss several notions of "curvature bounded below" for rough convex domains in the plane.

Finally we shall discuss in an appendix §11 a connection between mean discrepancy results and generalized distance sets for integer point lattices. Here our previous results in [13] yield a new result in three and higher dimensions.

Notation: Given two quantities A, B we write $A \lesssim B$ if there is an absolute positive constant, depending only on the specific domain Ω, so that $A \leq CB$. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2. The first step

Our purpose here is to show an estimate involving the quantities $\mu(\theta, \delta)$, which holds without any regularity or curvature assumptions on the boundary of the convex domain.

However we shall first make the *a priori*

Assumption: The boundary of Ω is a C^1 curve and the components of the outer unit normal vectors are absolutely continous.

This means we assume that the curvature is integrable. Below we shall remove this *a priori* assumption by a limiting argument.

We now begin with a standard procedure using mollifiers to regularize the characteristic function of Ω. Suppose that $r_1 < 1 < r_2$ and r_1, r_2 are the radii of inscribed and circumscribed circles centered at the origin. Let ζ be a smooth nonnegative radial cutoff function supported in the ball $B_{r_1/2}(0)$ so that $\int \zeta(x)dx = 1$ and let $\zeta_\varepsilon(x) = \varepsilon^{-2}\zeta(x/\varepsilon)$. Let

$$N_\varepsilon(t) = \sum_{k \in \mathbb{Z}^2} \chi_{t\Omega} * \zeta_\varepsilon(k) \tag{2.1}$$

and

$$E_\varepsilon(t) = \sum_{k \in \mathbb{Z}^2} \chi_{t\Omega} * \zeta_\varepsilon(k) - t^2 \text{area}(\Omega). \tag{2.2}$$

It suffices to estimate the modified square-function

$$G(R,h)=\Big(\frac{1}{h}\int_R^{R+h}|E_{1/R}(t)|^2dt\Big)^{1/2} \tag{2.3}$$

since there is the elementary estimate (with $E\equiv E_\Omega$)

$$\Big(\frac{1}{h}\int_R^{R+h}|E(t)|^2dt\Big)^{1/2}\le G(R,h)+C(R/h)^{1/2}, \tag{2.4}$$

valid for $R^{-1}\le h\le R$; see Lemma (2.2) of [13].

Basic decompositions. Fix a nonnegative $\eta_0\in C^\infty(\mathbb{R})$ so that $\eta_0(t)=1$ for $t\in[0,1]$ and η_0 is supported in $(-1/2,3/2)$ and let

$$\eta_{R,h}(t)=\frac{1}{\sqrt{h}}\eta_0(\frac{t-R}{h}). \tag{2.5}$$

Then

$$\frac{1}{h}\int_R^{R+h}|E_{1/R}(t)|^2dt\lesssim\int|E_{1/R}(t)\eta_{R,h}(t)|^2dt. \tag{2.6}$$

By the Poisson summation formula

$$\begin{aligned} E_{1/R}(t) &= \sum_{k\neq0}(2\pi t)^2\widehat{\chi_\Omega}(2\pi tk)\widehat{\zeta}(2\pi k/R)\\ &= \sum_{0<|k|\le R^2}(2\pi t)^2\widehat{\chi_\Omega}(2\pi tk)\widehat{\zeta}(2\pi k/R)+O(R^{-10}) \end{aligned} \tag{2.7}$$

since always $|\widehat{\chi_\Omega}(2\pi tk)|\lesssim|tk|^{-1}$ and $|\widehat{\zeta}(2\pi k/R)|\le C_N(1+|k|/R)^{-N}$.

As in [9] and elsewhere we have by the divergence theorem $\widehat{\chi_\Omega}(\xi)=-i\sum_{i=1}^2(\xi_i/|\xi|^2)\widehat{\mathfrak{n}_i d\sigma}(\xi)$ where $\mathfrak{n}$ denotes the unit outer normal vector. We may assume that the boundary of Ω is parametrized by $\alpha\mapsto x(\alpha)$ where $x'(\alpha)$ is a unit vector and $x(\alpha)=x(\alpha+L)$ if L is the length of $\partial\Omega$.

Then $\mathfrak{n}(\alpha)=-x'_\perp(\alpha)$ where $x_\perp(\alpha)=(x_2(\alpha),-x_1(\alpha))$ and

$$\widehat{\chi_\Omega}(\xi)=-i\int_0^L\frac{\langle\xi,x'_\perp(\alpha)\rangle}{|\xi|^2}e^{-i\langle x(\alpha),\xi\rangle}d\alpha. \tag{2.8}$$

Assuming that $R/2\le t\le 2R$ we shall now introduce a finer microlocal decomposition of $\widehat{\chi_\Omega}(2\pi tk)$, depending on k and R and based on (2.8). This is somewhat inspired by [5], [18] and in particular by [27] where a related construction is used.

Suppose that β_0 is an even function which is supported in $(-3/4,3/4)$ and which is equal to one in $[-1/2,1/2]$. Let $\beta(s)=\beta_0(s/2)-\beta_0(s)$ and let, for $n\ge1$, $\beta_n(s)=\beta(2^{-n}(s))$. Let

$$\Psi(k,\alpha) = \frac{\langle k, x'_\perp(\alpha)\rangle}{|k|^2}\Big(1 - \beta_0\big(2r_1^{-1}(\langle \tfrac{k}{|k|}, x(\alpha)\rangle - \rho^*(\tfrac{k}{|k|}))\big) - \beta_0\big(2r_1^{-1}(\langle \tfrac{-k}{|k|}, x(\alpha)\rangle - \rho^*(\tfrac{-k}{|k|}))\big)\Big)$$

and

$$\Phi_n^\pm(k,\alpha) = \frac{\langle k, x'_\perp(\alpha)\rangle}{|k|^2} \times \beta_n\big(R(\langle \pm k, x(\alpha)\rangle - \rho^*(\pm k)\big)\beta_0\big(2r_1^{-1}(\langle \tfrac{\pm k}{|k|}, x(\alpha)\rangle - \rho^*(\tfrac{\pm k}{|k|}))\big).$$

The cutoff function $\Phi_n^+(k,\cdot)$ localizes to those points P on the boundary for which the distance of P to the supporting line $\{x : \langle k, x\rangle = \rho^*(k)\}$ is small and $\approx 2^n(R|k|)^{-1}$ (or $\lesssim (R|k|)^{-1}$ if $n = 0$). Also $\Phi_n^-(k,\cdot)$ gives a localization in terms of the distance to the supporting line $\{x : \langle -k, x\rangle = \rho^*(-k)\}$. The factors $\beta_0(2r_1^{-1}(\langle \frac{\pm k}{|k|}, x(\alpha)\rangle - \rho^*(\frac{\pm k}{|k|})))$ are included in this definition to make sure that the supports of Φ_n^+ and Φ_n^- are disjoint.

Note that

$$\Psi(k,\alpha) + \sum_{n=0}^\infty \Phi_n^+(k,\alpha) + \sum_{n=0}^\infty \Phi_n^-(k,\alpha) = \frac{\langle k, x'_\perp(\alpha)\rangle}{|k|^2}$$

and also that $\Phi_n^\pm(k,\alpha) = 0$ if $2^n \geq |k|R$.

Define (for fixed R and h)

$$I_n^\pm(k,t) = 2\pi t\, \eta_{R,h}(t) \int \Phi_n^\pm(k,\alpha) e^{-2\pi i\langle x(\alpha), tk\rangle} d\alpha \tag{2.9}$$

$$II(k,t) = 2\pi t\, \eta_{R,h}(t) \int \Psi(k,\alpha) e^{-2\pi i\langle x(\alpha), tk\rangle} d\alpha \tag{2.10}$$

and

$$I_n^\pm(t) = \sum_{\frac{2^n}{R} < |k| \leq R^2} \widehat{\zeta}(2\pi k/R) I_n^\pm(k,t) \tag{2.11}$$

$$II(t) = \sum_{k\neq 0} \widehat{\zeta}(2\pi k/R) II(k,t), \tag{2.12}$$

and set

$$G_n^\pm(R,h) = \Big(\int |I_n^\pm(t)|^2 dt\Big)^{1/2}. \tag{2.13}$$

Using the decay of $\widehat{\zeta}$ we see that

$$G(R,h) \leq \sum_\pm \sum_{n=1}^\infty G_n^\pm(R,h) + \Big(\int |II(t)|^2 dt\Big)^{1/2} + CR^{-10}. \tag{2.14}$$

Pointwise bounds via van der Corput's lemma. We start with a simple pointwise estimate for the pieces $I^n_\pm$ and II which just relies on van der Corput's lemma for oscillatory integrals (see [28], p. 334). We set

$$\omega_R(\xi) = (1 + |\xi|/R)^{-N} \tag{2.15}$$

Lemma 2.1. *For $n \geq 0$ we have*

$$I^\pm_n(t) \leq C 2^{-n} R |\eta_{R,h}(t)| \sum_{|k|>2^n/R} \omega_R(k)|k|^{-1} \mu(\tfrac{k}{|k|}, \tfrac{2^n}{|k|R})$$

and

$$II(t) \leq C|\eta_{R,h}(t)| \log R.$$

Let

$$\Gamma^\pm_n(R,h) := R \sum_{|k|>2^n/R} \omega_R(k)|k|^{-1} \mu(\tfrac{k}{|k|}, \tfrac{2^n}{|k|R}). \tag{2.16}$$

Then

$$G^\pm_n(R,h) \leq C \Gamma^\pm_n(R,h); \tag{2.17}$$

moreover

$$\Big(\int |II(t)|^2 dt \Big)^{1/2} \leq C \log R. \tag{2.18}$$

Proof. We write down the argument for I^+_n as the estimate for I^-_n is analogous. The estimate for $n = 0$ is immediate if we observe that length of the support of $\Phi^+_0(k, \cdot)$ is $\leq \mu(k/|k|, (|k|R)^{-1})$.

Fix $\theta \in S^1$ and choose α_θ so that $\langle \theta, x(\alpha_\theta) \rangle = \rho^*(\theta)$ and thus also $\mathfrak{n}(\alpha_\theta) = \theta$. We first observe that if $\langle \theta, x(\alpha) \rangle - \rho^*(\theta) > \delta$ then $\tan \measuredangle\big(\mathfrak{n}(\alpha), \mathfrak{n}(\alpha_\theta)\big) \geq \delta/\mu(\theta, \delta)$.

We use this with $\delta = 2^n(|k|R)^{-1}$ to get a lower bound for the derivative of the phase function in the support of $\Phi^*_n(k, \cdot)$. This implies that for $t \in$ supp $\eta_{R,h}$

$$|\langle x'(\alpha), tk \rangle| \geq 2^n \big(\mu(\tfrac{k}{|k|}, \tfrac{2^n}{|k|R})\big)^{-1} \quad \text{if } \alpha \in \text{supp } \Phi^*_n(k, \cdot). \tag{2.19}$$

and this derivative is monotone in α. Moreover

$$\|\Phi^+_n(k, \cdot)\|_\infty + \|\partial_\alpha \Phi^+_n(k, \cdot)\|_1 \lesssim |k|^{-1};$$

here we use our *a priori* assumption on the integrability of the second derivatives of γ.

Consequently, by van der Corput's lemma, we obtain

$$|I^+_n(k,t)| \lesssim t|\eta_{R,h}(t)|\omega_R(k)|k|^{-1}|2^{-n}\mu(\tfrac{k}{|k|}, \tfrac{2^n}{|k|R})$$

which yields the asserted bound for $I^+_n(t)$.

Similarly, $|\partial_\alpha \langle tk, x(\alpha) \rangle| \geq c|k|R$ if $\alpha \in$ supp $\Psi(k, \cdot)$ and $\|\Psi(k, \cdot)\|_\infty$ and $\|\partial_\alpha \Psi(k, \cdot)\|_1$ are $O(|k|^{-1})$. Thus

$$|II(k,t)| \lesssim |\eta_{R,h}(t)||k|^{-2}(1 + |k|/R)^{-N} \omega_R(k)$$

and summing in k yields the asserted bound for $II(t)$. The bounds for the square functions are immediate from the pointwise estimates. $\square$

Square function estimates. We shall need to improve on the pointwise estimates for $G_n^{\pm}(R,h)$ in Lemma 2.1 which will only be useful for $2^n \geq R$.

We apply Plancherel's theorem with respect to the t-variable and obtain

$$\begin{aligned} G_n^{\pm}(R,h)^2 &= 2\pi \int |\widehat{I_n^{\pm}}(\lambda)|^2 d\lambda \\ &= 2\pi \sum_{\frac{2^n}{R}<|k|\leq R^2} \sum_{\frac{2^n}{R}<|\ell|\leq R^2} \widehat{\zeta}(2\pi k/R)\widehat{\zeta}(2\pi \ell/R) \int \widehat{I_n^{\pm}}(k,\lambda)\overline{\widehat{I_n^{\pm}}(\ell,\lambda)} d\lambda \end{aligned}$$

where

$$\widehat{I_n^{\pm}}(k,\lambda) = 2\pi \iint t\, \eta_{R,h}(t) \Phi_n^{\pm}(k,\alpha) e^{-it(\lambda + \langle x(\alpha), 2\pi k\rangle)} d\alpha dt. \tag{2.20}$$

The crucial estimate is

Lemma 2.2. *Suppose that $k \in \mathbb{Z}^2$ and $|k| > 2^n/R$. Then the following inequalities hold.*

If $2^n \leq R/h$ then

$$\begin{aligned} \big|\widehat{I_n^{+}}(k,\lambda)\big| &\lesssim Rh^{1/2}|k|^{-1}2^{-n}\mu(\tfrac{k}{|k|}, \tfrac{2^n}{|k|R})(1+h|\lambda+\rho^*(2\pi k)|)^{-2N} \\ \big|\widehat{I_n^{-}}(k,\lambda)\big| &\lesssim Rh^{1/2}|k|^{-1}2^{-n}\mu(-\tfrac{k}{|k|}, \tfrac{2^n}{|k|R})(1+h|\lambda-\rho^*(-2\pi k)|)^{-2N} \end{aligned} \tag{2.21}$$

and if $2^n \geq R/h$, then

$$\begin{aligned} \big|\widehat{I_n^{+}}(k,\lambda)\big| &\lesssim Rh^{1/2}|k|^{-1}2^{-n}\mu(\tfrac{k}{|k|}, \tfrac{2^n}{|k|R})(1+R2^{-n}|\lambda+\rho^*(2\pi k)|)^{-2N} \\ \big|\widehat{I_n^{-}}(k,\lambda)\big| &\lesssim Rh^{1/2}|k|^{-1}2^{-n}\mu(-\tfrac{k}{|k|}, \tfrac{2^n}{|k|R})(1+R2^{-n}|\lambda-\rho^*(-2\pi k)|)^{-2N} \end{aligned} \tag{2.22}$$

.

Proof. We prove the estimate for $I_n^+(k,\lambda)$; the estimate for $I_n^-(k,\lambda)$ is analogous.

We first consider the case $n = 0$. Interchange the order of integration in (2.20) and perform $2N$ integrations by parts with respect to t. This, together with the estimates $\Phi_0^+(k,\alpha) = O(|k|^{-1})$, $\eta_{R,h}(t) = O(h^{-1/2})$ yields

$$|\widehat{I_n^{+}}(k,\lambda)| \lesssim R|h|^{-1/2}|k|^{-1} \iint_{\substack{t\in \mathrm{supp}\, \eta_{R,h} \\ \alpha \in \mathrm{supp}\, \Phi_0^+(k,\cdot)}} (1+h|\lambda+\langle x(\alpha), 2\pi k\rangle|)^{-2N} dt d\alpha.$$

By definition we have $|\langle x(\alpha), k\rangle - \rho^*(k)| \lesssim R^{-1}$ for $\alpha \in \mathrm{supp}\ \Phi_0^+(k,\cdot)$. Since $R \geq h$ this implies

$$(1+h|\lambda+\langle x(\alpha), 2\pi k\rangle|) \approx (1+h|\lambda+\rho^*(2\pi k)|). \tag{2.23}$$

Also observe that the length of the support of $\eta_{R,h}$ is $O(h)$ and that the length of the support of $\Phi_0^+(k,\cdot)$ is $\le \mu(k/|k|,(|k|R)^{-1})$. Thus

$$\begin{aligned}|\widehat{I_0^+}(k,\lambda)| &\lesssim R|h|^{-1/2}|k|^{-1}\iint\limits_{\substack{t\in\text{supp }\eta_{R,h}\\ \alpha\in\text{supp }\Phi_0^+(k,\cdot)}} (1+h|\lambda+\rho^*(2\pi k)|)^{-2N}dtd\alpha\\ &\lesssim R|h|^{1/2}|k|^{-1}\mu(\tfrac{k}{|k|},\tfrac{1}{|k|R})(1+h|\lambda+\rho^*(2\pi k)|)^{-2N}\end{aligned}$$

which is the asserted estimate for $n=0$.

We now suppose that $n\ge 1$, and begin by performing an integration by parts with respect to α in (2.20). Observe that $\langle x'(\alpha),k\rangle\ne 0$ if $\alpha\in$ supp $\Phi_n^+(k,\cdot)$. We obtain

$$I_n^+(k,\lambda)=F_{n,1}(k,\lambda)+F_{n,2}(k,\lambda)$$

where

$$F_{n,1}(k,\lambda)= 2\pi\iint \eta_{R,h}(t)\Phi_n^+(k,\alpha)\frac{\partial}{\partial\alpha}\Big(\frac{1}{i\langle x'(\alpha),2\pi k\rangle}\Big)e^{-it(\lambda+\langle x(\alpha),2\pi k\rangle)}d\alpha dt, \tag{2.24}$$

and

$$F_{n,2}(k,\lambda)=2\pi\iint \eta_{R,h}(t)\frac{\partial\Phi_n^+}{\partial\alpha}(k,\alpha)\frac{e^{-it(\lambda+\langle x(\alpha),2\pi k\rangle)}}{i\langle x'(\alpha),2\pi k\rangle}d\alpha dt \tag{2.25}$$

As above we interchange the order of integration and integrate by parts in t. This yields the estimate

$$\begin{aligned}&|F_{n,1}(k,\lambda)|\lesssim R|h|^{-1/2}|k|^{-1}\\ &\quad\times\iint\limits_{\substack{t\in\text{supp }\eta_{R,h}\\ \alpha\in\text{supp }\Phi_n^+(k,\cdot)}}(1+h|\lambda+\langle x(\alpha),2\pi k\rangle|)^{-2N}\Big|\frac{\partial}{\partial\alpha}\Big(\frac{1}{i\langle x'(\alpha),2\pi k\rangle}\Big)\Big|d\alpha dt.\end{aligned}$$

If $2^n\le R/h$ then (2.23) is still valid if $\alpha\in$ supp $\Phi_n^+(k,\cdot)$. Moreover we claim that

$$\int\limits_{\text{supp }\Phi_n^+(k,\cdot)}\Big|\frac{\partial}{\partial\alpha}\Big(\frac{1}{\langle x'(\alpha),2\pi k\rangle}\Big)\Big|d\alpha\lesssim R2^{-n}\mu(\tfrac{k}{|k|},\tfrac{2^n}{|k|R}). \tag{2.26}$$

To see this we choose α_k so that $\langle x(\alpha_k),k\rangle=\rho^*(k)$ (this choice may not be unique). The support of $\Phi_n^+(k,\cdot)$ consists of two connected intervals (on $\mathbb{R}/L\mathbb{Z}$) and on each of these the function $\alpha\to\langle x'(\alpha),k\rangle$ is monotone;

moreover this function vanishes for $\alpha = \alpha_k$. Thus

$$\begin{aligned} 2^{n-1}R^{-1} &\leq \big|\langle x(\alpha), k\rangle - \rho^*(k)\big| \\ &= \Big|(\alpha-\alpha_k)\int_0^1 \langle x'(\alpha_k+\tau(\alpha-\alpha_k)), k\rangle d\tau\Big| \\ &\leq |\alpha-\alpha_k||\langle x'(\alpha), k\rangle|. \end{aligned} \tag{2.27}$$

Note that $|\alpha-\alpha_k| \lesssim \mu(k/|k|, 2^n/(|k|R))$ for $\alpha \in \text{supp } \Phi_n^+(k,\cdot)$. Since $\partial_\alpha((\langle x'(\alpha), 2\pi k\rangle)^{-1})$ is single-signed on the two components of supp $\Phi_n^+(k,\cdot)$ we can apply the fundamental theorem of calculus on these intervals and we see that the left hand side of (2.26) is bounded by $4\sup|\langle x'(a), 2\pi k\rangle|^{-1})$ where the supremum is taken over all $\alpha \in \text{supp } (\Phi_n^+(k,\cdot))$. But by (2.27) this bound is $O(R2^{-n}\mu(k/|k|, 2^n/(|k|R)))$.

Combining (2.23) and (2.26) we obtain for $2^n \leq R/h$

$$\begin{aligned} &|F_{n,1}(k,\lambda)| \\ &\lesssim |h|^{1/2}|k|^{-1}(1+h|\lambda+\rho^*(2\pi k)|)^{-2N} \int\limits_{\text{supp } \Phi_n^+(k,\cdot)} \Big|\frac{\partial}{\partial\alpha}\Big(\frac{1}{\langle x'(\alpha), 2\pi k\rangle}\Big)\Big| d\alpha \\ &\lesssim 2^{-n}R|h|^{1/2}|k|^{-1}(1+h|\lambda+\rho^*(2\pi k)|)^{-2N}\mu(\tfrac{k}{|k|}, \tfrac{2^n}{|k|R}) \end{aligned} \tag{2.28}$$

which is the estimate we were aiming for. Next we consider the term $F_{n,2}(k,\lambda)$ and arguing as above we see that

$$|F_{n,2}(k,\lambda)| \lesssim h^{1/2}(1+h|\lambda+\rho^*(2\pi k)|)^{-2N} \int \Big|\frac{\partial\Phi_n^+}{\partial\alpha}(k,\alpha)\Big|\Big|\frac{1}{\langle x'(\alpha), 2\pi k\rangle}\Big| d\alpha \tag{2.29}$$

and

$$\begin{aligned} &\int \Big|\frac{\partial\Phi_n^+}{\partial\alpha}(k,\alpha)\Big|\Big|\frac{1}{\langle x'(\alpha), 2\pi k\rangle}\Big| d\alpha \\ &\quad\lesssim \int_{\text{supp } \Phi_n^+(k,\cdot)} \frac{|\langle k, x''_\perp(\alpha)\rangle|}{|k|^2}\Big|\frac{1}{\langle x'(\alpha), 2\pi k\rangle}\Big| d\alpha \\ &\quad+\frac{1}{|k|}\int_{\text{supp } \Phi_n^+(k,\cdot)} \big|\partial_\alpha\big(\beta(R2^{-n}(\langle x'(\alpha), k\rangle - \rho^*(2\pi k)))\big)\big| d\alpha \\ &\quad+\frac{1}{|k|}\int_{\text{supp } \Phi_n^+(k,\cdot)} \big|\partial_\alpha\big(\beta_0(2r_1^{-1}(\langle \tfrac{k}{|k|}, x(\alpha)\rangle - \rho^*(\tfrac{k}{|k|})))\big)\big| d\alpha \\ &:= A_1(k)+A_2(k)+A_3(k). \end{aligned}$$

We now use that by (2.26)

$$|\langle x'(a), 2\pi k\rangle|^{-1} \lesssim R2^{-n}\mu(k/|k|, 2^n/(|k|R))$$

on the support of $\Phi_n^+(k,\cdot)$. Since $\langle k, x''_\perp(\alpha)\rangle$ is single-signed on the components we see that

$$A_1(k) \lesssim R2^{-n}|k|^{-1}\mu(\tfrac{k}{|k|}, \tfrac{2^n}{|k|R}).$$

Next

$$A_2(k) \lesssim |k|^{-1}R2^{-n}\text{meas}(\text{supp } \Phi_n^+(k,\cdot)) \lesssim |k|^{-1}R2^{-n}\mu(\tfrac{k}{|k|}, \tfrac{2^n}{|k|R}).$$

Finally, on the support of the derivative of the term $\beta_0(\dots)$ we have the better bound $|\langle x'(\alpha), k\rangle|^{-1} = O((|k|)^{-1})$ so that

$$A_3(k) \lesssim |k|^{-2}\text{meas}(\text{supp } \Phi_n^+(k,\cdot)) \lesssim |k|^{-2}\mu(\tfrac{k}{|k|}, \tfrac{2^n}{|k|R}) \lesssim |k|^{-1}R2^{-n}\mu(\tfrac{k}{|k|}, \tfrac{2^n}{|k|R})$$

in view of our restriction $k \ge 2^n/R$. If we use these estimates in (2.29) then we obtain the desired estimate for $F_{n,2}(k,\lambda)$, at least for the case $2^n \le R/h$.

The estimates for $F_{n,1}(k,\lambda)$ and $F_{n,2}(k,\lambda)$ in the case $2^n > R/h$ are derived analogously. The only difference is that (2.23) does not hold in all of the support of $\Phi_n^\pm(k,\cdot)$. However we still have $|\rho^*(2\pi k) - \langle x'(\alpha), 2\pi k\rangle| \lesssim 2^n/R$ in this set so that (2.23) is now replaced by

$$\tag{2.30} \begin{aligned}&(1 + h|\lambda + \langle x(\alpha), 2\pi k\rangle|)\\ &\qquad \lesssim (1 + R2^{-n}|\lambda + \langle x(\alpha), 2\pi k\rangle|) \approx (1 + R2^{-n}|\lambda + \rho^*(2\pi k)|)\end{aligned}$$

and the remainder of the above arguments applies without change to yield the inequalities in (2.14) $\square$

Lemma 2.3. *Let*

$$\tag{2.31} \begin{aligned}&\mathfrak{B}_\pm^n(R,h) := 2^{-n}R \times\\ &\Big(\sum_{\substack{k\in\mathbb{Z}^2\\ \frac{2^n}{R}<|k|\le R^2}} \sum_{\substack{\ell\in\mathbb{Z}^2\\ \frac{2^n}{R}<|\ell|\le R^2}} \frac{\omega_R(k)\omega_R(\ell)}{|k||\ell|} \frac{\mu(\frac{k}{|k|}, \frac{2^n}{|k|R})\,\mu(\frac{\ell}{|\ell|}, \frac{2^n}{|\ell|R})}{(1 + h|\rho^*(\pm k) - \rho^*(\pm \ell)|)^N}\Big)^{1/2},\end{aligned}$$

and

$$\tag{2.32} \begin{aligned}&\widetilde{\mathfrak{B}}_\pm^n(R,h) := 2^{-n}R \times\\ &\Big(\sum_{\substack{k\in\mathbb{Z}^2\\ \frac{2^n}{R}<|k|\le R^2}} \sum_{\substack{\ell\in\mathbb{Z}^2\\ \frac{2^n}{R}<|\ell|\le R^2}} \frac{\omega_R(k)\omega_R(\ell)}{|k||\ell|} \frac{\mu(\frac{k}{|k|}, \frac{2^n}{|k|R})\,\mu(\frac{\ell}{|\ell|}, \frac{2^n}{|\ell|R})}{(1 + R2^{-n}|\rho^*(\pm k) - \rho^*(\pm \ell)|)^N}\Big)^{1/2}.\end{aligned}$$

Then for $1 \le h \le R$, $R \ge 2$ we have the estimates

$$\tag{2.33} G_n^\pm(R,h) \le C\mathfrak{B}_n^\pm(R,h) \qquad \text{if } 2^n \le R/h$$

$$\tag{2.34} G_n^\pm(R,h) \le C\widetilde{\mathfrak{B}}_n^\pm(R,h) \qquad \text{if } 2^n > R/h$$

Proof. We observe that $|\zeta(k/R)| \le \omega_R(k)$ and use the elementary convolution inequality

$$\tag{2.35} \int (1 + a|A+\lambda)|)^{-2N}(1 + a|B+\lambda)|)^{-2N} d\lambda \lesssim a^{-1}(1 + a|A-B|)^{-N}$$

We use (2.35) for $A = \rho^*(\pm 2\pi k)$, $B = \rho^*(\pm 2\pi \ell)$ and $a = h$ if $2^n \le R/h$ and $a = R2^{-n}$ if $2^n > R/h$. The estimates (2.33), (2.34) are now straightforward from (2.21), (2.22) and (2.35). □

We shall now combine the previous lemmata to state the definitive result of this section; here we abandon the *a priori* assumption that the outer unit normals are absolutely continous functions.

Proposition 2.4. *Let Ω be a convex domain containing the origin in its interior, and suppose that $r_1 < 1 < r_2$ where r_1, r_2 are the radii of inscribed and circumscribed circles centered at the origin. Let $\mathfrak{B}_n^{\pm}$, $\widetilde{\mathfrak{B}}_n^{\pm}$ and $\Gamma_n^{\pm}$ be as in (2.31), (2.32) and (2.16).*

There exists a constant C, depending only on r_1, r_2 and N, so that for $1 \le h \le R$, $R \ge 2$ we have the estimate

$$\mathcal{G}_\Omega(R,h) \le C \sum_{\pm} \Big[\sum_{2^n \le R/h} \mathfrak{B}_n^{\pm}(R,h) + \sum_{R/h < 2^n \le R} \widetilde{\mathfrak{B}}_n^{\pm}(R,h) + \sum_{2^n > R} \Gamma_n^{\pm}(R,h) \Big] + C[\log R + (R/h)^{1/2}]. \tag{2.36}$$

Proof. Under our previous *a priori* assumption on the boundary of Ω this statement follows by simply putting together the estimates (2.4), (2.14), (2.17), (2.18), (2.33), and (2.34). We note that all bounds just depend on $r_1 \le 1 \le r_2$ and N, and that the L^1 bound for the second derivatives does not enter in the result.

In the general case we note that there is a sequence of convex domains Ω_j which contain the origin, such that $\Omega_j \subset \Omega_{j+1} \subset \Omega$ and $\cup_j \Omega_j = \Omega$ and Ω_j has smooth boundary. Moreover, let $\mu_j(\theta, \delta)$ for fixed $\theta \in S^1$ and $\delta > 0$ be the quantity (1.6) but associated to the domain Ω_j. Then $\mu_j(\theta, \delta)$ converges to the corresponding quantity associated to Ω, $\mu(\theta, \delta)$. Moreover the square functions defined by the smoothed errors $E_{1/R}$ associated to Ω_j converge to the corresponding expression associated to Ω and the same statement applies to the expressions $\mathfrak{B}_n^{\pm}(R,h)$, $\widetilde{\mathfrak{B}_n}^{\pm}(R,h)$, $\Gamma_n^{\pm}(R,h)$. For one explicit construction of the approximation see the proof of Lemma 2.2 in [27]. The constant C in the statement of Lemmata 2.1 and 2.3 can be chosen uniformly in j and the assertion follows. □

3. Estimates for the case of nonzero curvature

In this section we estimate the quantities $\mathfrak{B}_n^{\pm}$ etc. in the case of nonvanishing curvature and rough boundary; that is, we assume inequality (1.7). Proposition 2.4 reduces matters to estimates for lattice points in thin annuli

$$\mathcal{A}^{\pm}(r,h) := \{\xi \in \mathbb{R}^2 : |\rho^*(\pm\xi) - r| \le h^{-1}\}; \tag{3.1}$$

here $r \ge 2$ and $h \ge 1$. Let

$$S^{\pm}(r,h) := \mathrm{card}(\mathcal{A}^{\pm} \cap \mathbb{Z}^2), \tag{3.2}$$

the number of lattice points in the annulus $\mathcal{A}^{\pm}(r,h)$.

We also need to consider for $\delta \le r^{-1}$ the weighted sum

$$\mathfrak{S}^{\pm}(r,\delta,h) = \delta^{-1/2} \sum_{\substack{\ell\in\mathbb{Z}^2:\\ |\rho^*(\pm\ell)-r|\le h^{-1}}} \mu(\tfrac{\ell}{|\ell|},\delta). \tag{3.3}$$

Notice that assuming (1.7) we have

$$\mathfrak{S}^{\pm}(r,\delta,h) \lesssim \operatorname{card}(S_{\pm}(r,h)); \tag{3.4}$$

however $\mathfrak{S}(r,\delta,h)$ could be much smaller than $\operatorname{card}(S(r,h))$. Indeed consider the case that $S(r,h)$ contains a long line segment; then the boundary Ω^* becomes nearly flat and by duality the curvature of Ω at the corresponding points gets large causing $\mu(\frac{\ell}{|\ell|})$ to be smaller than $\delta^{1/2}$ for many ℓ on the line segment. This phenomenon will be exploited in the proof of Theorem 1.2.

Proposition 3.1. *Suppose that 1.7 is satisfied and that $R\ge 2$ and $1\le h\le R/\log^2 R$. Then*

$$\begin{aligned}\mathcal{G}_{\Omega}(R,h) &\le CR^{1/2} \\ &+ CR^{1/2}\sum_{0\le 2^n\le \log R} 2^{-n/2}\Big(\sum_{1\le l\le \log R} 2^{-l} \sup_{2^{l-1}\le r\le 2^l} \mathfrak{S}\big(r,\tfrac{2^n}{rR},h\big)\Big)^{1/2}.\end{aligned} \tag{3.5}$$

Proof. The proof relies on rather straightforward calculations which use $\mu(\frac{k}{|k|},\frac{2^n}{|k|R}) = O(2^{n/2}(R|k|)^{-1/2}$ (by assumption 1.7) the definition of $\mathfrak{S}$, (2.14) and bounds for $G_n^{\pm}(R,h)$ established in Lemma 2.1 and Lemma 2.3.

First, for the case $2^n\ge R$ we use (2.16)

$$\Gamma_n^{\pm}(R,h) \lesssim 2^{-n/2}R^{1/2}\sum_{|k|>0}|k|^{-3/2}\omega_R(k) \lesssim R2^{-n}$$

and thus by (2.17) we can bound

$$\sum_{2^n\ge R} G_n^{\pm}(R,h) \lesssim \sum_{2^n\ge R}\Gamma_n^{\pm}(R,h) \lesssim R^{1/2}.$$

To bound $\sum_{2^n<R} G_n^{\pm}(R,h)$ we use (2.33), (2.34) depending on whether $2^n\le R/h$ or $2^n > R/h$. It turns out that the trivial bound (3.4) will suffice in the range $2^n\ge \log R$, and in the complementary range we shall seek an estimate involving the terms $\mathfrak{S}$ explicitly.

Now assume that $R/h\le 2^n\le R$ (and in view of our assumption $h\le R(\log R)^{-2}$ this certainly implies $2^n\ge \log R$). In this case we compute and the assumption $R2^{-n}\ge 1$ from (2.32)

$$\begin{aligned}\widetilde{\mathfrak{B}}_n^{\pm}(R,h) &\lesssim 2^{-n/2}R^{1/2}\Big(\sum_{\substack{k\in\mathbb{Z}^2\\ \frac{2^n}{R}<|k|\le R^2}}\ \sum_{\substack{\ell\in\mathbb{Z}^2\\ \frac{2^n}{R}<|\ell|\le R^2}} \frac{\omega_R(k)\omega_R(\ell)(|k||\ell|)^{-3/2}}{(1+|\rho^*(\pm k)-\rho^*(\pm\ell)|)^N}\Big)^{1/2} \\ &\lesssim h^{1/2}\log^{1/2} R, \qquad R/h\le 2^n\le R\end{aligned}$$

and therefore

$$\sum_{R/h\le 2^n\le R} \widetilde{\mathfrak{B}}_n^{\pm}(R,h) \lesssim (h\log(2+h^{-1})\log R)^{1/2} \lesssim R^{1/2}$$

since we assume $h \le R/\log^2 R$.

Next we assume that $2^n \le R/h$ and we bound $\mathfrak{B}_n^{\pm}(R,h)$.

The argument above for $\widetilde{\mathfrak{B}}_n^{\pm}(R,h)$ also applies to $\mathfrak{B}_n^{\pm}(R,h)$ which is $O(2^{-n/2}R^{1/2}(\log R)^{1/2})$ when $R/h \le 2^n \le R$. Thus

$$\sum_{\log R\le 2^n\le R/h} \mathfrak{B}_n^{\pm}(R,h) \lesssim R^{1/2};$$

note that if we summed instead over $1 \le 2^n \le R/h$ we would only get the weaker bound $O(R^{1/2}(\log R)^{1/2})$.

Finally we have to bound $\sum_{2^n\le\log R}\mathfrak{B}_n^{\pm}(R,h)$. To this end we observe that the sum of the contributions of the terms in (2.31) which involve either $|k| \ge R$ or $|\ell| \ge R$ or $|\rho*(\pm k) - \rho^*(\pm\ell)| \ge \sqrt{\rho^*(\pm k)}$

Now let $E^{\pm}(k,n)$ denote the set of all $\ell \in \mathbb{Z}^2$ which also satisfy $0 < |\ell| \le R$ and $|\rho*(\pm k) - \rho^*(\pm\ell)| < \sqrt{\rho^*(\pm k)}$. We use the bound $\mu(\frac{k}{|k|}, \frac{2^n}{|k|R}) \lesssim 2^{n/2}|k|^{-1/2}R^{-1/2}$ and estimate

$$\mathfrak{B}_n^{\pm}(R,h) \le C_1R^{1/2} + C_2 2^{-n3/4}R^{3/4} \times \Big(\sum_{\substack{k\in\mathbb{Z}^2\\0<|k|\le R}} |k|^{-5/2} \sum_{\ell\in E^{\pm}(k,n)} \frac{\mu(\frac{\ell}{|\ell|}, \frac{2^n}{|\ell|R})}{(1+h|\rho^*(\pm k)-\rho^*(\pm\ell)|)^N}\Big)^{1/2}.$$

By the property $\mu(\theta, A\delta) \le C_A\mu(\theta,\delta)$ we obtain

$$\sum_{\ell\in E^{\pm}(k,n)} \frac{\mu(\frac{\ell}{|\ell|}, \frac{2^n}{|\ell|R})}{(1+h|\rho^*(\pm k)-\rho^*(\pm\ell)|)^N} \lesssim \sum_{|m|\le C|k|^{1/2}} (1+|m|)^{-N}2^{n/2}(R|k|)^{-1/2}\mathfrak{S}\big(\rho^*(\pm k)+m, \tfrac{2^n}{|k|R}, h\big)$$

and thus

$$\begin{aligned} &2^{-n3/4}R^{3/4} \Big(\sum_{\substack{k\in\mathbb{Z}^2\\0<|k|\le R}} |k|^{-5/2} \sum_{\ell\in E^{\pm}(k,n)} \frac{\mu(\frac{\ell}{|\ell|}, \frac{2^n}{|\ell|R})}{(1+h|\rho^*(\pm k)-\rho^*(\pm\ell)|)^N}\Big)^{1/2} \\ &\lesssim 2^{-n/2}R^{1/2} \Big(\sum_{\substack{k\in\mathbb{Z}^2\\0<|k|\le R}} |k|^{-3} \sup_{\frac{\rho^*(\pm k)}{2}\le r\le 2\rho^*(\pm k)} \mathfrak{S}(r, \tfrac{2^{n+1}}{rR}, h)\Big)^{1/2} \\ &\lesssim 2^{-n/2}R^{1/2} \Big(\sum_{1\le l\le 2+\log R} 2^{-l} \sup_{2^{l-1}\le r\le 2^l} \mathfrak{S}(r, \tfrac{2^n}{rR}, h)\Big)^{1/2}. \quad \square \end{aligned}$$

We now state the crucial propositions needed in the proof of Theorems 1.1 and 1.2. The mild regularity assumption in Theorem 1.1 gives us a favorable estimate for the cardinality of $S_\pm(r,h)$ which will be proved in §5.

Proposition 3.2. *Let Ω be as in the statement of Theorem 1.1, i.e. with $\kappa \in L\log^{2+\epsilon} L$ and κ bounded below. Assume that $2 \le r$ and $1 \le h$. Then*

$$\operatorname{card}(S_\pm(r,h)) \lesssim r\big[h^{-1} + \log^{-1-\varepsilon/2}(2+r)\big]. \tag{3.6}$$

If we only make the assumption that the curvature is bounded below (in the sense of (1.7)) then there is no nontrivial pointwise bound for $\operatorname{card}(S_\pm(r,h))$, but we still have a favorable bound for the weighted sums $\mathfrak{S}(r,(Rr)^{-1},h)$ provided that $h \gtrsim R^{1/2}$. This estimate is more difficult than Proposition 3.2 and will be proved in §6-8.

Proposition 3.3. *Let Ω be as in the statement of Theorem 1.2, i.e. with κ bounded below. Assume that $R \ge 10$, $10 \le r \le R$ and $R^{1/2} \le h \le R$.*

Then if $1 \le h \le R$ the estimate

$$\sqrt{Rr}\,\mathfrak{S}(r,\tfrac{1}{Rr},h) \lesssim r^{17/18} \tag{3.7}$$

holds.

We finish this section by showing the implication of the above propositions to the results stated in the introduction.

Proof of Theorem 1.3. For this result we just use the trivial estimate $\operatorname{card}(S_\pm(r,h)) = O(r)$ if $r \ge 1$, $|h| \ge 1$. Then the bound $\mathcal{G}_\Omega(R,h) = O((R\log R)^{1/2})$ follows easily from a combination of Proposition 3.1 and (3.4). □

Proposition 3.2 implies Theorem 1.1. Now we still use (3.4) and observe that by Proposition 3.2

$$\sum_{1\le l\le \log R} 2^{-l} \sup_{2^{l-1}\le r\le 2^l} \mathfrak{S}\big(r,\tfrac{2^n}{rR},h\big) \lesssim \sum_{1\le l\le \log R} (h^{-1} + (1+l)^{-1-\varepsilon/2}) \lesssim (1 + h^{-1}\log R)$$

and thus we obtain the bound $\mathcal{G}_\Omega(R,h) = O(R^{1/2})$ from Proposition 3.1 if $h \ge \log R$. □

Proposition 3.3 implies Theorem 1.2. We argue similarly but now use the inequality (3.7) in the application of Proposition 3.1. Here (3.7) is applied with R replaced by $R2^{-n}$, and since $2^n \le \log R$ this application is certainly valid for $R^{1/2} \le h \le R/(\log R)^2$. We obtain

$$\sum_{1\le l\le \log(R2^{-n})} 2^{-l} \sup_{2^{l-1}\le r\le 2^l} \mathfrak{S}\big(r,\tfrac{2^n}{rR},h\big) \lesssim \sum_{1\le l\le \log(R2^{-n})} \big(2^{-l/18} + (\log(R2^{-n}))^{-1}\big) \le C; \tag{3.8}$$

moreover for the terms with $\log(R2^{-n}) < l \le \log R$ we simply use the trivial bound $\mathfrak{S}\big(r, \frac{2^n}{rR}, h\big) = O(r)$ and get

$$\sum_{\log(R2^{-n})<l\le\log R} 2^{-l} \sup_{2^{l-1}\le r\le 2^l} \mathfrak{S}\big(r, \tfrac{2^n}{rR}, h\big) \lesssim n+1. \tag{3.9}$$

We use (3.8) and (3.9) in the application of Proposition 3.1, and in view of the exponential decay in n in (3.5) we obtain the bound $\mathcal{G}_\Omega(R,h) = O(R^{1/2})$. □

4. Proof of Theorem 1.4

We follow the same setup as in the proof of Theorem 1.3 and use a crucial fact from [3] according to which the maximal function defined by

$$\mu^*(\theta) = \sup\{\delta^{-1/2}\mu(\theta,\delta) : \delta > 0\}$$

belongs to $L^{2,\infty}(S^1)$; *i.e.*

$$\text{meas}(\{\theta \in S^1 : \mu^*(\theta)^2 > s\}) \le C^2/s$$

uniformly in s.

We now consider the sets $A_\vartheta\Omega$ and denote the quantities in (2.13) associated to $A_\vartheta\Omega$ by $G_n^\pm(R,h,\vartheta)$ etc. By averaging it suffices to assume $h=1$. We estimate $G_n^+(R,1,\vartheta)$.

From Lemma 2.3 we obtain for $2^n \le R$

$$G_n^+(R,1,\vartheta) \lesssim 2^{-n/2}R^{1/2} \times \Big(\sum_{\substack{0<|k|\le R^2\\ 0<|\ell|\le R^2}} \mu^*(A_\vartheta \tfrac{k}{|k|})\mu^*(A_\vartheta \tfrac{\ell}{|\ell|}) \frac{|k|^{-3/2}|\ell|^{-3/2}}{(1+|\rho^*(A_\vartheta k)-\rho^*(A_\vartheta \ell)|)^N}\Big)^{1/2}.$$

By symmetry we may restrict the summation to those pairs (k,ℓ) for which $\mu^*(\frac{k}{|k|}) \le \mu^*(\frac{\ell}{|\ell|})$ and we thus have the estimate

$$G_n^+(R,1,\vartheta) \lesssim 2^{-n/2}R^{1/2} \times \Big(\sum_{\substack{0<|k|\le R^2\\ 0<|\ell|\le R^2}} \mu^*(A_\vartheta \tfrac{k}{|k|})^2 \frac{|k|^{-3/2}|\ell|^{-3/2}}{(1+|\rho^*(A_\vartheta k)-\rho^*(A_\vartheta \ell)|)^N}\Big)^{1/2}.$$

Now as above it is easy to see that for fixed k

$$\sum_{\ell\ne 0} |\ell|^{-3/2}(1+|\rho^*(A_\vartheta k)-\rho^*(A_\vartheta \ell)|)^{-N} \le C|k|^{-1/2}$$

where C is independent from ϑ. Thus

$$G_n^+(R,1,\vartheta) \lesssim 2^{-n/2}R^{1/2} \sum_{0<|k|\le R^2} \mu^*(A_\vartheta \tfrac{k}{|k|})^2|k|^{-2}(1+\tfrac{|k|}{R})^{-N}; \tag{4.1}$$

moreover

$$
\begin{aligned}
&\sup_{j\ge n} 2^{-j} j^{-(2+\epsilon)} \sup_{2^j\le R\le 2^{j+1}} G_n^+(R,1,\vartheta)^2\\
&\le \sum_{j=n}^{\infty} 2^{-j} j^{-(2+\epsilon)} \sup_{2^j\le R\le 2^{j+1}} G_n^+(R,1,\vartheta)^2\\
&\lesssim 2^{-n}\sum_{j=n}^{\infty} j^{-(2+\epsilon)} \sum_{0<|k|\le 2^{2j}} \mu^*(A_\vartheta \tfrac{k}{|k|})^2 |k|^{-2}(1+2^{-j}|k|)^{-N}. \qquad (4.2)
\end{aligned}
$$

In order to complete the proof we have to show that the expression (4.2) defines a function in $L^{1,\infty}([-\pi,\pi])$. To do this we apply a well known lemma by Stein and N. Weiss [29] on adding functions in $L^{1,\infty}$ and the quasi-norm is bounded by a constant times the square-root of

$$
\begin{aligned}
&2^{-n}\sum_{j=n}^{\infty} j^{-(2+\epsilon)} \sum_{0<k\le 2^{2j}} |k|^{-2}(\log(1+|k|+j))\\
&\lesssim 2^{-n}\sum_{j=n}^{\infty} j^{-(1+\epsilon)} \le C_\varepsilon 2^{-n}.
\end{aligned}
$$

Thus the function

$$
\vartheta \mapsto \sup_{R\ge 2^n} R^{-1/2}(\log(2+R))^{-1-\varepsilon} G_{I,n}^+(R,1,\vartheta) \qquad (4.3)
$$

belongs to $L^{2,\infty}$ with norm $O(2^{-n/2})$.

For $R<2^n\le R^3$ we argue as in the proof of Theorem 1.2 and see that the estimate (4.1) is replaced by

$$
G_n^+(R,1,\vartheta) \lesssim \sum_{0<|k|\le R^2} \mu^*(A_\vartheta \tfrac{k}{|k|})^2|k|^{-2}
$$

and thus

$$
\begin{aligned}
&\sup_{j<n} 2^{-j\varepsilon} \sup_{2^j\le R\le 2^{j+1}} G_n^+(R,1,\vartheta)^2\\
&\quad\lesssim \sum_{n/3<j\le n} 2^{-j\varepsilon} \sum_{2^{n-j}<|k|\le 2^{2j}} \mu^*(A_\vartheta \tfrac{k}{|k|})^2|k|^{-2}
\end{aligned}
$$

and again this expression as a function of ϑ belongs to $L^{1,\infty}$ with quasi-norm $2^{-\varepsilon n/3}$. Thus the function

$$
\vartheta \mapsto \sup_{R<2^n} R^{-\varepsilon/2} G_n^+(R,1,\vartheta)
$$

belongs to $L^{2,\infty}$ with norm $O(2^{-\varepsilon n/3})$ (which is a better result than for the function (4.3), as expected). We may sum in n and get the required assertion for $\vartheta \mapsto \sum_{n=0}^{\infty} G_n^+(R,1,\vartheta)$ and the corresponding assertion involving $G_n^-(R,1,\vartheta)$ follows in the same way. □

5. Bounds for the lattice rest associated to the polar set – the proof of Proposition 3.2

We improve the the trivial estimate $E_{\Omega^*}(t) = O(t)$ under the given mild regularity assumption on $\partial\Omega$. Since

$$S(r,h) \lesssim rh^{-1} + \sup_{t \le r+h^{-1}} E_{\Omega^*}(t)$$

the Proposition 3.2 immediately follows from the following result.

Proposition 5.1. *Let Ω be a convex domain with C^1 boundary in $\mathbb{R}^2$ containing the origin in its interior, ands assume that the components of the tangent vector are absolutely continuous. Suppose also that curvature κ is uniformly bounded below, i.e. $|\kappa(x)| \ge a > 0$ for almost every $x \in \partial\Omega$ and that $\kappa \in L\log^\gamma L(\partial\Omega)$, for some $\gamma > 0$. Let $E_{\Omega^*}(t) = \mathcal{N}_{\Omega^*}(t) - t^d area(\Omega^*)$. Then for $t \ge 2$*

$$|E_{\Omega^*}(t)| \le Ct(\log t)^{-\gamma/2} \tag{5.1}$$

We need the following variant of van der Corput's Lemma.

Lemma 5.2. *Let f be a C^1 function on the interval $[a,b]$ and assume that f' is absolutely continuous and monotone. Let $\gamma > 0$ and suppose that the function $t \mapsto (\log(2 + \frac{1}{|f''(t)|}))^\gamma$ belongs to $L^{1,\infty}$, with operator (quasi-) norm bounded by A. Then*

$$\Big|\int_a^b e^{i\lambda f(t)}\chi(t)dt\Big| \le C(\gamma, A)\big(\|\chi\|_\infty + \|\chi'\|_1\big)(\log(2+\lambda))^{-\gamma}.$$

Proof. We may assume that $\lambda \ge 10$. In view of the monotonicity of f' the set $I = \{t \in [a,b] : |f'(t)| \le \lambda^{-1}(\log\lambda)^\gamma\}$ is an interval, $I = [c,d]$. The set $[a,b] \setminus I$ is a union of at most two intervals and on each of these we have $|\lambda f'(t)| \ge (\log\lambda)^\gamma$. By the standard van der Corput Lemma with first derivatives ([29]) it follows that

$$\Big|\int_{[a,b]\setminus I} e^{i\lambda f(t)}\chi(t)dt\Big| \le C\big(\|\chi\|_\infty + \|\chi'\|_1\big)(\log\lambda)^{-\gamma}. \tag{5.2}$$

To complete the proof we have to show that

$$|I| \lesssim (\log\lambda)^{-\gamma}. \tag{5.3}$$

Let $E_1 = \{t \in I : |f''(t)| \le (\log\lambda)^{2\gamma}\lambda^{-1}\}$ and $E_2 = I \setminus E_1$. On E_1 we have

$$\log^\gamma(2 + \tfrac{1}{|f''(t)|}) \ge \log^\gamma(2 + \tfrac{\lambda}{\log^{2\gamma}\lambda}) \ge c\log^\gamma(2+\lambda);$$

here c depends only on A and γ. Thus by our $L^{1,\infty}$ assumption $|E_1| \lesssim (\log(2+\lambda))^{-\gamma}$. By definition of I we also have

$$2\frac{(\log\lambda)^\gamma}{\lambda} \ge |f'(d) - f'(c)| = \Big|\int_I f''(s)ds\Big| \ge \Big|\int_{E_2} f''(s)ds\Big| \ge |E_2|\frac{(\log\lambda)^{2\gamma}}{\lambda}$$

thus $|E_2| \le \frac{1}{2}(\log(2+\lambda))^{-\gamma}$. Thus we have shown (5.3) and the proof is complete. □

As a consequence we obtain

Lemma 5.3. *Let Ω be as in Proposition 5.1. Then*

$$\big|\widehat{\chi_{\Omega^*}}(\xi)\big| \le C(1+|\xi|)^{-1}\log(2+|\xi|)^{-\gamma} \tag{5.4}$$

Proof. Let $\alpha \mapsto x(\alpha)$ be a parameterization of $\partial\Omega$ with $|x'(\alpha)| = 1$. A parametrization of $\partial\Omega^*$ is then given by $\alpha \mapsto \widetilde{x}(\alpha) = \langle x(\alpha), \mathfrak{n}(\alpha)\rangle^{-1}\mathfrak{n}(\alpha)$ but this parametrization is not sufficiently regular. We compute

$$\widetilde{x}' = \frac{n_1' n_2 - n_1 n_2'}{\langle x, \mathfrak{n}\rangle^2}(x_2, -x_1) \tag{5.5}$$

and we observe that $n_1'n_2 - n_1n_2' = \kappa$. Moreover, if $r_1 < r_2$ are the radii of inscribed and circumscribed circles centered at the origin then for $x \in \partial\Omega$

$$\langle x, \mathfrak{n}\rangle \ge \frac{r_1}{2r_2}|x| > \frac{r_1^2}{r_2}. \tag{5.6}$$

We introduce a new parameter $\tau = \tau(\alpha) = \int_{\alpha_0}^{\alpha} \frac{\kappa(\beta)}{\langle x(\beta), \mathfrak{n}(\beta)\rangle^2} d\beta$; then τ is invertible with inverse $\tau \mapsto \alpha(\tau)$, $\tau \in I$. We work with the parametrization

$$\tau \mapsto x_*(\tau) = \widetilde{x}(\alpha(\tau))$$

and then

$$x_*'(\tau) = (x_2(\alpha(\tau))), -x_1(\alpha(\tau)).$$

In view of an analogue of (2.8) it suffices to show that

$$\Big|\int_I e^{-i\langle x_*(\tau),\xi\rangle}\chi(\tau)d\tau\Big| \lesssim (\log(2+|\xi|))^{-\gamma} \tag{5.7}$$

Let $c_0 = r_1/2r_2$ and $g(\tau,\xi) = \langle \frac{\xi}{|\xi|}, \frac{x_*'}{|x_*'|}\rangle$. Fix $|\xi| \ge 2$. We split our interval I into no more than 16 subintervals, where on each interval J either $|g(\tau,\xi)| \ge c_0/4$ for all $\tau \in J$ or $|g(\tau,\xi)| \le c_0/4$ for all $\tau \in J$; and $\tau \mapsto g(\tau,\xi)$ is monotonic on J.

Suppose for all $\tau \in J$ we have $|g(\tau,\xi)| \ge c_0/4$. Then $|\langle x_*'(\tau), \xi\rangle| \ge c_1|\xi|$ in J and by van der Corput's Lemma we get

$$\int_J e^{-i\langle x_*(\tau),\xi\rangle}\chi(\tau)d\tau \lesssim |\xi|^{-1} \tag{5.8}$$

which of course is much better then the desired estimate.

Now fix J' with the property that $|g(\tau,\xi)| \le c_0/4$ for all $\alpha \in J'$. Now $x(\alpha(\tau))$ and $x_*'(\alpha(\tau))$ are orthogonal and thus $|\langle \frac{x}{|x|}, \frac{\xi}{|\xi|}\rangle| \ge (1 - c_0^2/16)^{1/2} \ge 1 - c_0/4$; moreover

$$\begin{aligned}
\Big|\langle \mathfrak{n}, \frac{\xi}{|\xi|}\rangle\Big| &\ge \Big|\langle \frac{x}{|x|}, \mathfrak{n}\rangle\langle \frac{x}{|x|}, \frac{\xi}{|\xi|}\rangle\Big| - \Big|\langle \frac{x_*'}{|x_*'|}, \frac{\xi}{|\xi|}\rangle\Big| \\
&\ge c_0(1 - c_0/4) - c_0/4 \ge c_0/4.
\end{aligned}$$

Now for $\alpha \in J'$ we have $\mathfrak{n}(\alpha) = (x_2'(\alpha), -x_1'(\alpha))$ and thus

$$|\langle x_*''(\tau), \xi\rangle| = |\langle \mathfrak{n}(\alpha(\tau)), \xi\rangle \alpha(\tau)| \ge \frac{c_0}{4}|\alpha'(\tau)|.$$

Therefore

$$\begin{aligned}\int_{J'} \log^\gamma(2+\frac{1}{|\langle x''_*(\tau),\xi/|\xi|\rangle|})^\gamma d\tau &\geq c_1 \int_{J'} \log^\gamma(2+\frac{1}{|\alpha'(\tau)|})d\tau \\ &= c_1 \int_{\alpha(J')} \log^\gamma(2+|\tau'(\alpha)|)|\tau'(\alpha)|d\alpha d\tau\end{aligned}$$

where c_1 is chosen independently of ξ. The latter expression is finite since $|\tau'(\alpha)| \approx |\kappa(\alpha)|$ which is assumed to be in $L\log^\gamma L$. Thus we may apply Lemma 5.2 with $\lambda = |\xi|$ and obtain

$$\Big|\int_{J'} e^{-i\langle x_*(\tau),\xi\rangle}\chi(\tau)d\tau\Big| \lesssim (\log(2+|\xi|))^{-\gamma} \tag{5.9}$$

and the assertion follows from combining the estimates (5.8) and (5.9) on the various subintervals. □

Proof of Proposition 5.1. Given Lemma 5.3 this is just an application of the standard argument. Let $N^*_\varepsilon(t)$, $E^*_\varepsilon(t)$ be defined as in (2.1) and (2.2), but for the set Ω^* in place of Ω. Then by Lemma 5.3 for $t \geq 2$

$$\begin{aligned}|E^*_\varepsilon(t)| &\leq (2\pi t^2)\sum_{k\neq 0} |\widehat{\chi_{\Omega^*}}(2\pi t k)||\widehat{\zeta}(2\pi\varepsilon k)| \\ &\lesssim t^2 \sum_{k\neq 0}(t|k|)^{-1}\log(2+|t|k|)^\gamma(1+\varepsilon|k|)^{-N} \lesssim t(\log t)^{-\gamma}\varepsilon^{-1}.\end{aligned}$$

Also $N^*_\varepsilon(t-C\varepsilon) \leq N^*(t) \leq N^*_\varepsilon(t+C\varepsilon)$ and applying the previous estimate with $t \pm C\varepsilon$ in place of t we obtain that

$$|E_{\Omega^*}(t)| \lesssim \big[t(\log t)^{-\gamma}\varepsilon^{-1} + t\varepsilon\big].$$

Thus for the choice $\varepsilon = (\log t)^{-\gamma/2}$ we obtain the asserted estimate. □

6. A weighted estimate for lattice points on lines in thin annuli

The purpose of this section is to prove a bound for sums $\sum_\ell \mu(\frac{\ell}{|\ell|}, \frac{1}{R|\ell|})$ where the sum runs over the lattice points contained on a given line segment in the ρ^*-annulus

$$\mathcal{A}(r,h) = \left\{x \in \mathbb{R}^2 : r \leq \rho^*(x) \leq r + h^{-1}\right\}. \tag{6.1}$$

It turns out that if $h \gtrsim R^{1/2}$ and the line segment is sufficiently long then the trivial bound $\mu(\frac{\ell}{|\ell|}, \frac{1}{R|\ell|}) \lesssim (Rr)^{-1/2}$ may be substantially improved for most of the lattice points on the line segment; *i.e.* the fact that the thin ρ^* annulus contains long line segments reflects a rather fast turning of the normals for the original domain which may only happen if $\partial\Omega$ lacks smoothness.

Throughout this and the next two chapters we shall adopt the following notations. Given certain subsets A, B, G, I, J etc. we shall use blackboard bold fonts to denote by $\mathbb{A}$, $\mathbb{B}$, $\mathbb{G}$, $\mathbb{I}$, $\mathbb{J}$ the intersections of these sets with the integer lattice $\mathbb{Z}^2$ (the standard notation for the plane $\mathbb{R}^2$ remains an exception to this convention). We shall adopt the convention that a *line segment* $I = \overrightarrow{PQ}$ is a nontrivial segment whose endpoints P, Q lie in the lattice $\mathbb{Z}^2$. The corresponding collection $\mathbb{I} = I \cap \mathbb{Z}^2$ of lattice points in I will be called a *lattice line segment*.

If n is an odd natural number and I is a line segment, we let nI denote the line segment concentric with and parallel to I but with n times the length. The distance between consecutive lattice points on the line segment I is constant. Let $d \equiv d(I)$ denote this distance; then $d^{-1} = (\operatorname{card} \mathbb{I} - 1)/|I|$ is the density of lattice points on I.

The following result is the key for the proof of Proposition 3.3. We are seeking to improve the bound from (1.7) (with $\delta = (R|\ell|)^{-1}$)

$$\sqrt{Rr} \sum_{\ell \in \mathbb{I}} \mu\big(\tfrac{\ell}{|\ell|}, \tfrac{1}{R|\ell|}\big) \lesssim \operatorname{card}(\mathbb{I}), \tag{6.2}$$

if $\operatorname{card}(\mathbb{I})$ is sufficiently large.

Lemma 6.1. *Let Ω be an open convex bounded set in the plane $\mathbb{R}^2$, with positive curvature, and containing the origin. Let $10 \le r \le R < \infty$, and let $h \ge R^{1/2}$. Let J be a closed line segment whose endpoints are contained in $\mathbb{Z}^2$ and let $\mathbb{J} = J \cap \mathbb{Z}^2$. Assume that the ninefold dilate $9J$ is contained in $\mathcal{A}(r, h)$ and that* $\operatorname{card}(\mathbb{J}) \ge 10$.

Let

$$\mathcal{T} \equiv \mathcal{T}(R, d, h, r) := R^{3/4} d^{1/2} h^{-1/2} r^{-1/4}. \tag{6.3}$$

Then

$$\sqrt{Rr} \sum_{\ell \in \mathbb{J}} \mu\big(\tfrac{\ell}{|\ell|}, \tfrac{1}{R|\ell|}\big) \lesssim \begin{cases} \left(\frac{Rr}{h^2 d^2}\right)^{1/4} & \text{if } \operatorname{card}(\mathbb{J}) \le \mathcal{T}, \\ \left(\frac{r \operatorname{card}(\mathbb{J})}{h d^2}\right)^{1/3} & \text{if } \operatorname{card}(\mathbb{J}) \ge \mathcal{T}. \end{cases} \tag{6.4}$$

Proof.

We begin by introducing some additional notation. For $P \in \mathbb{R}^2$, let $\mathfrak{n}_P^*$ denote the unit outward normal to Ω^* at the point $\frac{P}{\rho^*(P)} \in \partial\Omega^*$. Note that this unit normal is uniquely determined since it is parallel to the uniquely determined position vector $\overrightarrow{O\mathcal{P}_P}$ joining the origin O to the point $\mathcal{P}_P \in \partial\Omega$ having P as one of its outward normal vectors.

We set $I = 3J$ and let P, Q be the endpoints of I. Our first observation is that the angle $\Psi = \measuredangle(\mathfrak{n}_P^*, \mathfrak{n}_Q^*)$ between the vectors $\mathfrak{n}_P^*$ and $\mathfrak{n}_Q^*$ (the nonnegative angle less than π) satisfies

$$\Psi = \measuredangle\left(\mathfrak{n}_P^*, \mathfrak{n}_Q^*\right) \le C_1 \arctan\left(\frac{h^{-1}}{|I|}\right), \tag{6.5}$$

where $|I| = |\overline{QP}|$ is the length of the segment I and C_1 is a constant depending only on Ω. To see this one notes that there is a rectangle of width ch^{-1} which contains the line segment $3I$ and which is contained in the annulus$\mathcal{A}(r, h)$. If we had $\tan \measuredangle\left(\mathfrak{n}_P^*, \mathfrak{n}_Q^*\right) \ge \widetilde{C} h^{-1} |I|^{-1}$ for large $\widetilde{C}$ then it is easy to see that the triangle OPQ would contain points in the complement of $\{\rho^* \leq r+h\}$. Thus we have (6.5).

For $x \in I_m$, define the collection $\Gamma(x)$ of subsegments $J = \overrightarrow{UV}$ of I by

$$\Gamma(x) = \left\{J = \overrightarrow{UV} \subset I : x \in J, |J| \geq d\right\},$$

and the corresponding uncentered maximal function $\mathcal{M}$ on I by

$$\mathcal{M}(x) = \sup_{J=\overrightarrow{UV}\in\Gamma(x)} \frac{1}{|J|} \measuredangle\big(\mathfrak{n}_U^*, \mathfrak{n}_V^*\big).$$

Then $\mathcal{M}$ is in weak L^1 on I by F. Riesz's lemma ([26], ch. 1); it satisfies the inequality

$$|\{x \in I : \mathcal{M}(x) > \lambda\}| \leq \frac{2}{\lambda}\Psi. \tag{6.6}$$

We shall now consider a decomposition of the set $\mathbb{I}$ depending on a parameter q; we shall see that the choice

$$q = \begin{cases} \left(\frac{Rrd^2}{h^2}\right)^{1/4} & \text{if } \operatorname{card}(\mathbb{J}) \leq \mathcal{T} \\ \left(\frac{rd\,\operatorname{card}(\mathbb{J})}{h}\right)^{1/3} & \text{if } \operatorname{card}(\mathbb{J}) \geq \mathcal{T} \end{cases} \tag{6.7}$$

will be (essentially) optimal.

Let $\mathbb{B} \subset \mathbb{I} := I \cap \mathbb{Z}^2$ denote the set

$$\mathbb{B} = \left\{\ell \in \mathbb{I} : \mathcal{M}(\ell) > q^{-1}\Psi\right\}$$

to which, following the terminology in Calderón-Zygmund theory, we refer as the set of *bad* lattice points. Let $\mathbb{G} = \mathbb{I} \setminus \mathbb{B}$ be the set of *good* lattice points. Denote by u a unit vector parallel to I. Then the segments $\overrightarrow{(\ell - du/2)(\ell + du/2)}$ are pairwise disjoint and for each $\ell \in \mathbb{B}$, either $\overrightarrow{(\ell - du/2)\ell}$ or $\overrightarrow{\ell(\ell + du/2)}$ is contained in

$$\left\{x \in I : \mathcal{M}(x) > \Psi/q\right\}.$$

Thus by (6.6) we have

$$\begin{aligned} \operatorname{card}(\mathbb{B}) &= d^{-1}\Big|\bigcup_{\ell\in\mathbb{B}} \overrightarrow{(\ell - du/2)(\ell + du/2)}\Big| \\ &\leq 2d^{-1}\big|\left\{x \in I : \mathcal{M}(x) > \Psi/q\right\}\big| \lesssim q/d. \end{aligned}$$

By (1.7) we obtain

$$\sum_{\ell\in\mathbb{B}} \sqrt{Rr}\mu\big(\tfrac{\ell}{|\ell|}, \tfrac{1}{R|\ell|}\big) \lesssim q/d \tag{6.8}$$

We will obtain now obtain an estimate for the sum over $\ell \in \mathbb{G} \cap \mathbb{J}$ (rather than over *all* of $\mathbb{G}$). Note that if $\ell \in \mathbb{G} \cap \mathbb{J}$, then

$$\measuredangle\left(\mathfrak{n}_{\ell-\alpha du}^*, \mathfrak{n}_{\ell+\alpha du}^*\right) \leq 2\alpha d\Psi/q, \tag{6.9}$$

$$\measuredangle(\ell - \alpha du, \ell), \measuredangle(\ell, \ell + \alpha du) \geq c\alpha d/r \tag{6.10}$$

for $1/2 \le \alpha \le |J|/d$ upon using $|\ell| \approx r$.

Passing to the dual set $\Omega^{**} = \Omega$ with defining Minkowski functional ρ, we have that for $\alpha \ge 0$, the points $\mathcal{A}^{\pm}_{\ell,\alpha} = \frac{\mathfrak{n}^*_{\ell\pm\alpha du}}{\rho(\mathfrak{n}^*_{\ell\pm\alpha du})}$ in $\partial\Omega$ have unit normals $\nu^{\pm}_{\alpha} = \frac{\ell\pm\alpha du}{|\ell\pm\alpha du|}$, and that

$$(6.11)\qquad \begin{aligned} \big|\measuredangle\big(\nu^{\pm}_{\alpha}, \tfrac{\ell}{|\ell|}\big)\big| &\ge c\alpha d/r \\ |\measuredangle\left(\mathcal{A}^{-}_{\alpha}, \mathcal{A}^{+}_{\alpha}\right)| &\le 2\alpha d\Psi/q \end{aligned}$$

for $1/2 \le \alpha \le |J|/d$.

Using this estimate we derive a bound on the diameter of the cap $\mathcal{C}_\Omega(\frac{\ell}{|\ell|}, \frac{1}{R|\ell|})$. Let

$$D = 2C_1 d(h|I|q)^{-1},$$

where C_1 is as in (6.5).

Suppose that $\partial\Omega$ is parametrized in a neighborhood of $\frac{\mathfrak{n}^*_\ell}{\rho(\mathfrak{n}^*_\ell)}$ by $t \mapsto \gamma(t)$, $|t| \le c'$, with

$$\langle \gamma(t) - \mathcal{A}_{\ell,0}, \tfrac{\ell}{|\ell|}\rangle = \varphi_\ell(t);$$

here $\mathcal{A}_\ell \equiv \mathcal{A}^{\pm}_{\ell,0} = \frac{\mathfrak{n}^*_\ell}{\rho(\mathfrak{n}^*_\ell)}$, and $\varphi_\ell(t)$ is convex and nonnegative with $\varphi_\ell(0) = 0$, $\varphi'_\ell(0) = 0$.

Now by (6.5) we have $D \ge 2d\Psi/q$, and so (6.11) shows that

$$|\varphi'_\ell(\alpha D)| \ge |\varphi'_\ell(\alpha 2d\Psi/q)| \ge c\alpha d/r, \qquad 1/2 \le \alpha \le |J|/d,$$

and it follows that for $D/2 \le |t| \le |J|d^{-1}D$,

$$(6.12)\qquad \varphi_\ell\,(t) \ge \int_{D/2}^{t} \frac{cd}{Dr} s\,ds = \frac{1}{2}\frac{cd}{Dr}\left[t^2 - (D/2)^2\right].$$

Now assume $|J|/d \ge 1$ and observe that then for $t = D|J|/d$, the upper bound for $|t|$, we have by (6.12) and the definition of D

$$\begin{aligned} \varphi_\ell\big(\tfrac{|J|}{d}D\big) &\ge \frac{c}{2}\frac{dD}{r}\left[(|J|/d)^2 - 1/4\right] \\ &\ge \frac{3c|J|^2 D}{8dr} = \frac{3c|J|^2}{8h|I|q} =\ge \frac{c|J|}{8qhr}. \end{aligned}$$

Since $h \le R$ and in view of $h \ge R^{1/2}$, $1 \le r \le R$ we see that the choice of q in (6.7) certainly implies

$$\varphi_\ell(\tfrac{|J|}{d}D) \ge c'(Rr)^{-1} \ge c''(R|\ell|)^{-1}$$

This estimate and (6.12) imply that

$$\begin{aligned} \mu(\tfrac{\ell}{|\ell|}, \tfrac{1}{R|\ell|}) &\le C(c'')\mu(\tfrac{\ell}{|\ell|}, \tfrac{1}{c''R|\ell|}) \lesssim \varphi_\ell^{-1}(\tfrac{1}{c''R|\ell|}) \\ &\lesssim \varphi_\ell^{-1}(\tfrac{1}{R|\ell|}) \le D + 2\sqrt{\frac{2Dr}{cd}\frac{1}{R\,|\ell|}} \\ &\lesssim (D + (D/dR)^{1/2}), \end{aligned}$$

and since $D = 2C_1 d(h|I|q)^{-1}$ we then have

$$\sqrt{Rr}\mu(\tfrac{\ell}{|\ell|}, \tfrac{1}{R|\ell|}) \lesssim \frac{R^{1/2}}{h} \frac{dr^{1/2}}{|I|q} + \Big(\frac{r}{h|I|q}\Big)^{1/2},$$

for $\ell \in \mathbb{G} \cap \mathbb{J}$. Since $|I| \approx d\,\mathrm{card}(\mathbb{I}) \approx d\,\mathrm{card}(\mathbb{J})$ we obtain

$$\sum_{\ell \in \mathbb{G}\cap\mathbb{J}} \sqrt{Rr}\mu(\tfrac{\ell}{|\ell|}, \tfrac{1}{R|\ell|}) \lesssim \frac{(Rr)^{1/2}}{hq} + \Big(\frac{r\mathrm{card}(\mathbb{J})}{hdq}\Big)^{1/2}. \tag{6.13}$$

Altogether (6.13) and (6.8) yield

$$\sum_{\ell \in \mathbb{J}} \sqrt{Rr}\mu(\tfrac{\ell}{|\ell|}, \tfrac{1}{R|\ell|}) \lesssim \frac{q}{d} + \frac{(Rr)^{1/2}}{hq} + \Big(\frac{r\mathrm{card}(\mathbb{J})}{hdq}\Big)^{1/2}. \tag{6.14}$$

We essentially choose q to minimize this expression. Its square is comparable to $d^{-2}F(q)$ where

$$F(q) = q^2 + b_1 q^{-2} + b_2 q^{-1}, \tag{6.15}$$

where the positive coefficients are given by

$$b_1 = Rrd^2h^{-2}, \qquad b_2 = r\mathrm{card}(\mathbb{J})dh^{-1}.$$

In what follows we shall need the relation

$$b_2^{4/3}/b_1 = \big(\mathrm{card}(\mathbb{J})/\mathcal{T}\big)^{4/3}$$

where $\mathcal{T}$ is as in (6.3). To analyze (6.15) it is natural to distinguish two cases, where in the first case the second term b_1q^{-2} dominates the third term b_2q^{-1} and in the second case we have the opposite inequality.

In the first case, $q \le b_1/b_2$, we have that $F(q) \approx q^2 + b_1q^{-2}$ and the latter expression has a local minimum where $q = b_1^{1/4}$. This value $b_1^{1/4}$ belongs to the currently relevant interval $(0, b_1/b_2]$ when $b_2 \le b_1^{3/4}$ which is equivalent to $\mathrm{card}(\mathbb{J}) \le \mathcal{T}$. In this case we thus make the choice $q = b_1^{1/4}$ (which is the value $(Rrh^{-2}d^{-2})^{1/4}$ in (6.7)). Then the right hand side of (6.14) is bounded by a constant times

$$d^{-1}\sqrt{F(b_1^{1/4})} \lesssim d^{-1}b_1^{1/4} = (Rrd^{-2}h^{-2})^{1/4}. \tag{6.16}$$

In the second case, $q \ge b_1/b_2$ we have $F(q) \approx q^2 + 2b_2q^{-1}$ and the latter expression has a local minimum at $b_2^{1/3}$ which lies in the interval $[b_1/b_2, \infty)$ when $b_2^{4/3} \ge b_1$; this condition is equivalent with $\mathrm{card}(\mathbb{J}) \ge \mathcal{T}$. Thus we now make the choice $q = b_2^{1/3} = (r\mathrm{card}(\mathbb{J})dh^{-1})^{1/3}$. Now the right hand side of (6.14) is bounded by a constant times

$$d^{-1}\sqrt{F(b_2^{1/3})} \lesssim d^{-1}b_2^{1/3} = (r\mathrm{card}(\mathbb{J})h^{-1}d^{-2})^{1/3} \tag{6.17}$$

and the estimate (6.4) is established. □

7. Two elementary lemmata

We prove two elementary lemmata from plane geometry.

Lemma 7.1. *Let $\mathbb{S}$ be any set of N noncollinear lattice points in the plane and let $\mathcal{H}$ be the convex hull of $\mathbb{S}$. Then*

$$N \leq 2 + 2\text{area}\,(\mathcal{H})\,. \tag{7.1}$$

In particular, $N \leq 6$ area $(\mathcal{H})$.

Proof. We necessarily have $N \geq 3$ and the second conclusion follows from (7.1) since every lattice triangle has area at least $1/2$ thus area$(\mathcal{H}) \geq 1/2$.

To see (7.1) we argue by induction and the estimate is obviously true for $N = 3$. We now proceed by induction. Let $N \geq 4$ and assume that the estimate (7.1) holds for any collection of noncollinear lattice points with cardinality $< N$. Let $\mathbb{T}$ be a set of N noncollinear lattice points in the plane and let $\mathcal{H}$ be the convex hull of $\mathbb{T}$. There exists a pair of lattice points ℓ and m in $\mathbb{T}$ such that the line L joining ℓ and m has points of $\mathbb{T}$ lying to each side of it. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the two closed half-planes determined by L. Set $\mathbb{S}_i = \mathbb{T}\cap\mathcal{P}_i$ for $i = 1, 2$. Then each collection $\mathbb{S}_i$ is noncollinear. Let $N_i = \text{card}(\mathbb{S}_i)$. Since the two points ℓ and m lie in both $\mathbb{S}_1$ and $\mathbb{S}_2$, we have $N_1 + N_2 \geq N + 2$. Let $\mathcal{H}_i$ be the convex hull of $\mathbb{S}_i$ so that $\mathcal{H}_1 \cup \mathcal{H}_2 \subset \mathcal{H}$ and $\mathcal{H}_1 \cap \mathcal{H}_2$ is contained in a line. By the induction assumption on both $\mathbb{S}_1$ and $\mathbb{S}_2$ we have $N_i \leq 2 + 2\,\text{area}(\mathcal{H}_i)$, $i = 1, 2$, and thus $N \leq N_1 + N_2 - 2 \leq 2 + 2\text{area}\mathcal{H}$. □

Our next lemma is a standard result on lattice points on convex polygons.

Lemma 7.2. *Suppose we are given integer points $F_1, \dots, F_J$ in $\mathbb{Z}^2$, which are vertices of a convex polygonal curve; i.e. the interiors of the line segments $\overline{F_iF_{i+1}}$ are mutually disjoint and if $\overrightarrow{F_iF_{i+1}} = L_j(\cos\beta_i, \sin\beta_i)$, $i = 1, \dots, J-1$, then we have $L_j > 0$ and $\beta_{j+1} > \beta_j$. Suppose also that $\beta_{J-1} - \beta_1 \leq 2\pi$, and set $L = \sum_{i=1}^{J-1}$. Then*

$$J \leq 2 + (\beta_{J-1} - \beta_1)^{1/3} L^{2/3}.$$

Proof. Let $\Delta(A, B, C)$ denote the triangle with vertices A, B, C. Then we use again that the areaof $\Delta(A, B, C)$ is at least $1/2$ if A, B, C are noncollinear lattice points. Thus

$$\begin{aligned} J - 2 &\leq \sum_{j=1}^{J-2} \big(2\ \text{area}(\Delta(F_j, F_{j+1}, F_{j+2}))\big)^{1/3} \\ &= \sum_{j=1}^{J-2} \big(|\overrightarrow{F_jF_{j+1}}|\,|\overrightarrow{F_{j+1}F_{j+2}}|\,|\sin(\beta_{j+1} - \beta_j)|\big)^{1/3} \end{aligned}$$

and thus by Hölder's inequality

$$\begin{aligned} J-2 &\le \Big(\sum_{j=1}^{J-2}|\overrightarrow{F_jF_{j+1}}|\Big)^{1/3}\Big(\sum_{j=1}^{J-2}|\overrightarrow{F_{j+1}F_{j+2}}|\Big)^{1/3}\Big(\sum_{j=1}^{J-2}|\sin(\beta_{i+1}-\beta_i)|\Big)^{1/3} \\ &\le L^{2/3}(\beta_{J-1}-\beta_1)^{1/3}. \quad \square \end{aligned}$$

In particular if $\mathfrak{P}$ is a convex polygon of length L whose vertices are integer lattice points then the number of vertices of $\mathfrak{P}$ is $O(L^{2/3})$. This is a special case of Andrews' result [1].

8. Proof of Proposition 3.3

In what follows we fix r, R, h with $10 \le r \le R$, $R^{1/2} \le h \le R$. Let

$$\Omega^*_\pm = \left\{x \in \mathbb{R}^2 : \rho^*(x) < r \pm h^{-1}\right\}$$

and $\mathcal{A} = \Omega^*_+ \setminus \Omega^*_-$. Denote by $\mathbb{A} = \mathcal{A} \cap \mathbb{Z}^2$ the set of lattice points in the annulus $\mathcal{A}$. Let $\mathcal{H}$ be the convex hull of $\mathbb{A}$ and let $\mathbb{E} = \{E_j\}_{j=1}^J$ be the extreme points of $\mathcal{H}$ arranged in counterclockwise order around the origin. First observe that from Lemma 7.2 we have $J = \mathrm{card}(\mathbb{E}) \lesssim r^{2/3}$.

Recall our convention from the previous section that a *line segment* is a nontrivial segment I whose endpoints lie in the lattice $\mathbb{Z}^2$. Our second observation is that every lattice point in $\mathbb{A} \setminus \mathbb{E}$ belongs to some line segment I that contains an extreme point from $\mathbb{E}$ and lies entirely within the annulus $\mathcal{A}$. More precisely, let $\ell \in \mathbb{A}$ and suppose that for some $1 \le j \le J$, ℓ belongs to the closed triangular sector $\mathcal{S}_j = \Delta(E_j, E_{j+1}, 0)$ with vertices E_j, E_{j+1} and the origin. Then the convex set Ω^*_- cannot intersect both of the line segments $\overrightarrow{E_j\ell}$ and $\overrightarrow{E_{j+1}\ell}$, and hence at least one of them must lie in $\mathcal{A}$.

Thus for $1 \le j \le J$, $\mathbb{A}_j = \mathbb{A} \cap \mathcal{S}_j$ is contained in the union $\mathcal{I}_j$ of all maximal line segments I contained in $\mathcal{A} \cap \mathcal{S}_j$ having one endpoint that is either E_j or E_{j+1}. We further distinguish the segments I in $\mathcal{I}_j$ by letting $\left\{I^-_{j,n}\right\}_{n=1}^{N_j^-}$ be an enumeration of the line segments in $\mathcal{I}_j$ having E_j as an endpoint, and arranged clockwise about E_j as n increases from 1 to N_j^-. Similarly, $\left\{I^+_{j,n}\right\}_{n=1}^{N_j^+}$ is an enumeration of the line segments in $\mathcal{I}_j$ having E_{j+1} as an endpoint, and arranged counterclockwise about E_{j+1} as n increases from 1 to N_j^+. Note that $I^+_{j,N_j^+} = I^-_{j,N_j^-} = \overrightarrow{E_jE_{j+1}}$ is the line segment joining the consecutive extreme points E_j and E_{j+1}.

The next lemma implies that the total number of line segments in

$$\mathcal{I} := \cup_{j=1}^J \mathcal{I}_j$$

still does not exceed $Cr^{\frac{2}{3}}$ if we assume $r \le R$ and $h \ge R^{1/2}$. For this we define $L_j = |\overrightarrow{E_jE_{j+1}}|$ to be the length of the segment $\overrightarrow{E_jE_{j+1}}$, and Θ_j to be the (positive) change of angle for the normal direction to $\partial\Omega^*$ across the sector $\mathcal{S}_j$. Specifically

Lemma 8.1. *For each* $1 \le j \le J$*, we have*

$$N_j^- + N_j^+ \le C\big(1 + \text{area}(\mathcal{A} \cap \mathcal{S}_j) + L_j^{2/3}\Theta_j^{1/3}\big).$$

Proof. We will establish the indicated estimate for N_j^+, the case for N_j^- being similar. Fix j and let the segment $I_{j,n}^+$ have endpoints E_{j+1} and F_n so that $I_{j,n}^+ = \overrightarrow{E_{j+1}F_n}$. Consider the collection of segments $\{\overrightarrow{F_nF_{n+1}}\}_{n=1}^{N_j^+-1}$ and set

$$\begin{aligned} \mathfrak{T}_j &= \big\{n : 1 \le n < N_j^+ \text{ and } \overrightarrow{F_nF_{n+1}} \subseteq \mathcal{A} \cap \mathcal{S}_j\big\}, \\ \mathfrak{P}_j &= \big\{n : 1 \le n < N_j^+ \text{ and } \overrightarrow{F_nF_{n+1}} \nsubseteq \mathcal{A} \cap \mathcal{S}_j\big\}. \end{aligned}$$

We first note that the triangles $\Delta(E_{j+1}, F_n, F_{n+1})$ are pairwise disjoint and contained in $\mathcal{A} \cap \mathcal{S}_j$ for $n \in \mathfrak{T}$, and from Lemma 7.1 applied to each triangle we get

$$\text{card}(\mathfrak{T}_j) \le 6 \sum_{n \in \mathfrak{T}_j} \text{area}\big(\Delta(E_{j+1}, F_n, F_{n+1})\big) \le 6 \text{ area}(\mathcal{A} \cap \mathcal{S}_j). \tag{8.1}$$

Now the integers in $\mathfrak{P}_j$ can be written as a union of maximal chains $\mathcal{C}_i$ of consecutive integers as follows:

$$\mathfrak{P}_j = \cup_{i=1}^{M_j} \mathcal{C}_i, \qquad \mathcal{C}_i = \{n\}_{n=a_i}^{b_i},$$

where $a_i \le b_i$ and $b_i + 2 \le a_{i+1}$. Note that the number of chains M_j satisfies

$$M_j \le 1 + \text{card}(\mathfrak{T}_j) \le 1 + 6 \text{ area}\,(\mathcal{A} \cap \mathcal{S}_j)\,.$$

For each $n \in \mathfrak{P}_j$ we may write $\overrightarrow{F_nF_{n+1}} = \rho_n(\cos\alpha_n, \sin\alpha_n)$ where $\rho_n > 0$ and $\alpha_n > \alpha_m$ if $m > n$. In particular the lines associated to a fixed chain form a convex polygon.

; here we use that the convex set Ω_-^* intersects the line segments $\overrightarrow{F_nF_{n+1}}$ and $\overrightarrow{F_mF_{m+1}}$.

To see this, note that the convex set Ω_-^* intersects both of the line segments $\overline{F_nF_{n+1}}$ and $\overline{F_mF_{m+1}}$, and that the intersection with $\overline{F_mF_{m+1}}$ occurs to the clockwise side of the intersection with $\overline{F_nF_{n+1}}$ (we adopt an obvious convention regarding the use of the phrase "to the clockwise side of"). Thus the angle α_m of $\overline{F_mF_{m+1}}$ is less than the angle α_n of $\overline{F_nF_{n+1}}$. Moreover the sectors generated by $\{O, F_n, F_{n+1}\}$ and $\{O, F_m, F_{m+1}\}$ have disjoint interior so that

$$\sum_{n \in \mathfrak{P}_j} \big|\overrightarrow{F_nF_{n+1}}\big| \lesssim L_j.$$

Now consider a chain $\mathcal{C}_i = \{n\}_{n=a_i}^{b_i}$ of length $b_i - a_i + 1 \ge 3$ and the associated set of lattice points $\{F_n\}_{n=a_i}^{b_i}$.

By Lemma 7.2, we have

$$b_i - a_i + 1 \le 2 + \Big(\sum_{n=a_i}^{b_i-1} \big|\overrightarrow{F_nF_{n+1}}\big| \Big)^{\frac23} \big(\sum_{n=a_i}^{b_i-1} \alpha_{n+1} - \alpha_n \big)^{\frac13}$$

for all chains of length $b_i - a_i + 1 \ge 3$, and also trivially for chains of length 1 or 2 as well. Summing in i from 1 to M_j we thus obtain

$$\begin{aligned}\mathrm{card}(\mathfrak{P}_j) &= \sum_{i=1}^{M_j} (b_i - a_i + 1) \\ &\le 2M_j + \sum_{i=1}^{M_j} \Big\{ \big(\sum_{n=a_i}^{b_i-1} |\overrightarrow{F_nF_{n+1}}| \big)^{\frac23} \big(\sum_{n=a_i}^{b_i-1} \alpha_{n+1} - \alpha_n \big)^{\frac13} \Big\}\end{aligned}$$

and thus by Hölder's inequality

$$\begin{aligned}\mathrm{card}(\mathfrak{P}_j) &\le 2M_j + \Big(\sum_{i=1}^{M_j} \sum_{n=a_i}^{b_i-1} \big|\overrightarrow{F_nF_{n+1}}\big| \Big)^{\frac23} \Big(\sum_{i=1}^{M_j} \sum_{n=a_i}^{b_i-1} \alpha_{n+1} - \alpha_n \Big)^{\frac13} \\ &\le C\big(M_j + L_j^{2/3}\Theta_j^{1/3}\big) \le C'\big(1 + \mathrm{area}(\mathcal{A} \cap \mathcal{S}_j) + L_j^{2/3}\Theta_j^{1/3}\big).\end{aligned}$$

The Lemma follows by combining the inequalities for the cardinalities of $\mathfrak{T}_j$ and $\mathfrak{P}_j$. □

We now proceed with the proof of Proposition 3.3. First, by Lemma 8.1 we can estimate the cardinality of $\mathcal{I}$ by

$$\mathrm{card}(\mathcal{I}) \le \sum_{j=1}^{J} \mathrm{card}(\mathcal{I}_j) \lesssim \sum_{j=1}^{J} \{1 + \mathrm{area}(\mathcal{A} \cap \mathcal{S}_j) + L_j^{\frac23} \Theta_j^{\frac13}\} \tag{8.2}$$

$$\lesssim \Big(J + \mathrm{area}(\mathcal{A}) + \big(\sum_{j=1}^{J} L_j \big)^{\frac23} \big(\sum_{j=1}^{J} \Theta_j \big)^{\frac13} \Big) \tag{8.3}$$

$$\lesssim (r^{\frac23} + r/h) \lesssim r^{2/3} \tag{8.4}$$

since $h \ge R^{1/2} \ge r^{1/2}$.

Now consider the lattice line segments $\mathbb{I}^{\pm}_{j,n} = I^{\pm}_{j,n} \cap \mathbb{Z}^2$ consisting of the lattice points in $I^{\pm}_{j,n}$, and let $\mathbb{A}_j$ be the collection of lattice points which lie in $\mathcal{A} \cap \mathcal{S}_j$. We then have that for $1 \le j \le J$,

$$\mathbb{A}_j = \mathbb{A} \cap \mathcal{S}_j = \Big(\bigcup_{n=1}^{N_j^+} \mathbb{I}^+_{j,n} \Big) \cup \Big(\bigcup_{n=1}^{N_j^-} \mathbb{I}^-_{j,n} \Big) \tag{8.5}$$

where $\sum_{j=1}^{J} (N_j^+ + N_j^-) \lesssim r^{2/3}$.

We now wish to apply Lemma 6.1, to the intervals in $\mathcal{I}$; however the assumption that the ninefold dilates are still contained in the annulus $\mathcal{A}$ may not be satisfied. Therefore for every $I \in \mathcal{I}$ we decompose I in subintervals

$$I = \imath_+(I) \cup \imath_-(I) \cup \bigcup_{m=-N(I)}^{N(I)} \imath_m(I)$$

where $9\imath_m(I) \subset I$. Moreover if $\imath_m(\mathbb{I}) := \imath_m(I) \cap \mathbb{Z}^2$, and $\imath_\pm(\mathbb{I}) := \imath_\pm(I) \cap \mathbb{Z}^2$, and then $\operatorname{card}(\imath_m(\mathbb{I})) \lesssim 2^{-|m|}\operatorname{card}(\mathbb{I})$, $\operatorname{card}(\imath_\pm(\mathbb{I})) = O(1)$ and $N(I) \le C + \log_2(\operatorname{card}(\mathbb{I}))$.

We first have the trivial inequality

$$\sum_{I\in\mathcal{I}} \sum_{\pm} \sum_{\ell \in \imath_\pm(\mathbb{I})} \sqrt{Rr}\mu(\tfrac{\ell}{|\ell|}, \tfrac{1}{R|\ell|}) \lesssim \operatorname{card}(\mathcal{I}) \lesssim r^{2/3}. \tag{8.6}$$

Now let $\mathfrak{L}$ denote the set of all lattice line segments $\{\imath_m(\mathbb{I}) : |m| \le N(I), I \in \mathcal{I}\}$. We split $\mathfrak{L}$ into three subfamilies (here $\mathcal{T}$ is as defined in (6.3)).

(i) $\mathfrak{L}_1$ consists of all $\mathbb{J} \in \mathfrak{L}$ which satisfy $\operatorname{card}(\mathbb{J}) \le \mathcal{T}$

(ii) $\mathfrak{L}_2$ consists of all $\mathbb{J} \in \mathfrak{L}$ of the form $\imath_m(\mathbb{I})$ for suitable m, $\mathbb{I}$, where $\operatorname{card}(\mathbb{J}) > \mathcal{T}$ and $\operatorname{card}(\mathbb{I}) \le r^{1/3}$.

(iii) $\mathfrak{L}_3$ consists of all $\mathbb{J} \in \mathfrak{L}$ of the form $\imath_m(\mathbb{I})$ for suitable m, $\mathbb{I}$, where $\operatorname{card}(\mathbb{J}) > \mathcal{T}$ and $\operatorname{card}(\mathbb{I}) > r^{1/3}$.

Notice that for all I we have $N(I) \lesssim \log r$ so that $\operatorname{card}(\mathfrak{L}) \lesssim r^{2/3}\log r$. Thus from part (i) of Lemma 6.1 we get

$$\sum_{\mathbb{J}\in\mathfrak{L}_1} \sqrt{Rr} \sum_{\ell\in\mathbb{J}} \mu(\tfrac{\ell}{|\ell|}, \tfrac{1}{R|\ell|}) \lesssim r^{2/3}\log r (Rrh^{-2})^{1/4} \lesssim r^{11/12}\log r \tag{8.7}$$

since $h^2 \ge R \ge r$.

Next we consider the lattice line segments in $\mathfrak{L}_2$ and use the second part of Lemma 6.1. We obtain

$$\begin{aligned}
&\sum_{\mathbb{J}\in\mathfrak{L}_2} \sqrt{Rr} \sum_{\ell\in\mathbb{J}} \mu(\tfrac{\ell}{|\ell|}, \tfrac{1}{R|\ell|}) \\
&\quad \lesssim \sum_{\substack{I\in\mathcal{I}:\\ \operatorname{card}(\mathbb{I})\le r^{1/3}}} \sum_{\substack{m:\\ \imath_m(\mathbb{I})\in\mathfrak{L}_2}} (rh^{-1}\operatorname{card}(\imath_m(\mathbb{I})))^{1/3} \\
&\quad \lesssim \sum_{\substack{I\in\mathcal{I}:\\ \operatorname{card}(\mathbb{I})\le r^{1/3}}} \big(rh^{-1}\operatorname{card}(\mathbb{I})\big)^{1/3} \\
&\quad \lesssim r^{1/3+1/9}\operatorname{card}(\mathcal{I})h^{-1/3} \lesssim r^{17/18}
\end{aligned} \tag{8.8}$$

since $\operatorname{card}(\mathcal{I}) \lesssim r^{2/3}$ and $h \ge R^{1/2} \ge r^{1/2}$.

Finally for the lattice line segments in $\mathfrak{L}_3$ we use again the second part of Lemma 6.1 but estimate differently

$$\sum_{\mathbb{J}\in\mathfrak{L}_3}\sqrt{Rr}\sum_{\ell\in\mathbb{J}}\mu(\tfrac{\ell}{|\ell|},\tfrac{1}{R|\ell|})$$
$$\lesssim \sum_{\substack{I\in\mathcal{I}:\\ \mathrm{card}(\mathbb{I})>r^{1/3}}} \big(rh^{-1}\,\mathrm{card}(\mathbb{I})\big)^{1/3}$$
$$\lesssim \sum_{I\in\mathcal{I}}(r/h)^{1/3}\frac{\mathrm{card}(\mathbb{I})}{r^{2/9}}$$
$$\lesssim r^{1/3-2/9}h^{-1/3}\mathrm{card}(\mathbb{A})\lesssim r^{17/18} \tag{8.9}$$

since $h\ge r^{1/2}$ and $\mathrm{card}(\mathbb{A})\lesssim r$.

We combine estimates (8.6), (8.7), (8.8), and (8.9) and deduce the asserted bound. □

9. Discussion of sharpness

In this section we shall discuss the sharpness of Proposition 3.3 which implies the estimate

$$\mathcal{K}(R,h)\lesssim R^{-1},\quad h\ge R^{1/2} \tag{9.1}$$

for the quantity defined in (1.11). We show that the condition $h\ge R^{1/2}$ is needed in (9.1).

More specifically we show that there are positive constants c and C such that for every $\varepsilon>0$ and $R>C$, there exists an open convex bounded set Ω with curvature bounded uniformly below (in the sense of any of the equivalent definitions in the subsequent section) such that with $h=h(R)=R^{1/2}(\log R)^{-\varepsilon}$ and suitable C_0 the quantity $\mathcal{K}(R,h)$ is at least $c\varepsilon R^{-1}\log\log R$.

In what follows we let r be a large integer so that

$$r\lesssim(\log R)^{\epsilon/3}.$$

We will construct an open convex bounded set Ω, with curvature $\gtrsim 1/2$ everywhere so that

$$(R|(m,n)|)^{1/2}\mu(\tfrac{(m,n)}{|(m,n)|},\tfrac{1}{R|(m,n)|})=1 \tag{9.2}$$

for all $(m,n)\in\mathbb{Z}^2$ with $0<|n|\le m\le r$, and also so that

$$|\rho^*((m,n_1))-\rho^*((m,n_2))|\le 2R^{-1/2}(\log R)^{\varepsilon} \tag{9.3}$$

for $0\le|n_1|,|n_2|\le m\le r$. With this achieved, we restrict k and ℓ in the sum on the left side of (1.11) to lie in the triangle of lattice points $\mathbb{T}_r=\{(m,n):0\le|n|\le m\le r\}$. Writing $k=(m,n_1)$ and $\ell=(m,n_2)$, we obtain that with $h(R) = C_0^{-1} R^{\frac{1}{2}} (\log R)^{-\varepsilon}$,

$$\begin{aligned}\sum_{\substack{k,\ell\in\mathbb{Z}^2:|k|\le R,|\ell|\le R,\\ |\rho^*(\ell)-\rho^*(k)|\le h(R)^{-1}}} &|k|^{-2}\mu\big(\tfrac{k}{|k|},\tfrac{1}{R|k|}\big)\mu\big(\tfrac{\ell}{|\ell|},\tfrac{1}{R|\ell|}\big)\\ &\ge R^{-1}\sum_{\substack{k,\ell\in\mathbb{T}_r:\\ |\rho^*(\ell)-\rho^*(k)|\le h(R)^{-1}}} |k|^{-3} \ge R^{-1}\sum_{(m,n)\in\mathbb{T}_r} m^{-2}\\ &\ge cR^{-1}\log r = cR^{-1}\frac{\varepsilon}{3}\log\log R,\end{aligned}$$

the desired conclusion.

Now we give the details of the construction. Given $\varepsilon > 0$, $r \in \mathbb{N}$ large and $h > 1$, denote by $\mathcal{R}_{m,n}$ the ray from the origin $(0,0)$ through the point (m,n), and by K the circle of radius hr^2 centered at $\big(1-hr^2,0\big)$, so that K passes through the point $(1,0)$. We order the set of rays $\{\mathcal{R}_{m,n} : |n| \le m \le r, m > 0\}$ by increasing slope (so that the positive slopes form the Farey sequence of order r), and denote the resulting ordered sequence of rays by $\{\mathcal{L}_j\}_{j=-J}^{J}$. Let P_j be the intersection of $\mathcal{L}_j$ and K.

We now define a preliminary domain $\mathcal{D}_0$ with partially polygonal boundary, then smooth out the corners to get a domain $\mathcal{D}$ with bounded curvature, and we shall then take $\Omega = \mathcal{D}^*$ so that $\Omega^* = \mathcal{D}^{**} = \mathcal{D}$. The boundary of the set $\mathcal{D}_0$ is the polygon in the sector $\mathcal{S} = \{(x,y) : 0 \le |y| \le x\}$ whose edges are the segments $\overrightarrow{P_jP_{j+1}}$, $-J \le j < J$, and we let the boundary of $\mathcal{D}_0$ be a smooth curve of bounded curvature in the closure of the complement of $\mathcal{S}$. We note that with $P_j = (x_j, y_j)$, we have

$$1 - \frac{1}{2hr^2} \le x_j \le 1 \tag{9.4}$$

by a straightforward application of Pythagoras' theorem and the fact that $|P_j - (1,0)| \le 1$.

It follows that for this convex set $\mathcal{D}_0$, the defining functional ρ_0^* satisfies (9.3) if $h \ge R^{1/2}(\log R)^{-\varepsilon}$. Indeed, if $\mathcal{R}_{m,n_\sigma} = \mathcal{L}_{j_\sigma}$ for $\sigma = 1,2$, then $(m, n_\sigma) = \frac{m}{x_{j_\sigma}}(x_{j_\sigma}, y_{j_\sigma})$ and so by (9.4), we have

$$\begin{aligned}|\rho^*((m,n_1)) - \rho^*((m,n_2))| &= \Big|\frac{m}{x_{j_1}} - \frac{m}{x_{j_2}}\Big| = \frac{m|x_{j_1}-x_{j_2}|}{x_{j_1}x_{j_2}}\\ &\le 2m(hr^2)^{-1} < 2(hr)^{-1}.\end{aligned}$$

We now smooth out the corners to define the domain $\mathcal{D}_0$. We modify $\partial\mathcal{D}_0$ in a small neighbourhood of each P_j by inscribing a circle of radius 1 to be tangent to each edge incident with P_j, so that in this neighbourhood, $\partial\mathcal{D}$ is an arc Γ_j^* of a circle of radius 1, where the arc Γ_j^* is centered about the ray $\mathcal{L}_j$ and has diameter $\triangle_j$, where

$$c(hr^3)^{-1} \le \triangle_j \le C(hr^2)^{-1}, \tag{9.5}$$

depending on where in the Farey sequence the slope of $\mathcal{L}_j$ occurs. This modification is possible for $h \geq C_0$, where C_0 is a sufficiently large constant, since then by the second inequality in (9.5),

$$\triangle_j \ll r^{-2} \leq \left|\overrightarrow{P_j P_{j+1}}\right|.$$

It is easy to see that inequality (9.3) persists for this modification.

Now define $\Omega = \mathcal{D}^*$ and let ρ be the Minkowski functional of Ω. Let $\mathfrak{n}^*_{(m,n)}$ be the unit normal at the boundary point of $\mathcal{D}$ which lies on the ray determined by (m,n). If $\mathcal{R}_{m,n} = \mathcal{L}_j$ then the dual arc Γ_j of $\partial\Omega$ is centered at $\frac{\mathfrak{n}^*_{(m,n)}}{\rho\left(\mathfrak{n}^*_{(m,n)}\right)}$, has curvature 1 and diameter $\triangle_j$. Note that the point $\mathfrak{n}^*_{(m,n)}/\rho(\mathfrak{n}^*_{(m,n)})$ in $\partial\Omega$ has $P_j/|P_j|$ as unit normal. Thus by the first inequality in (9.5), the cap $\mathcal{C}(P_j/|P_j|,\delta)$ has diameter $\approx \delta^{\frac{1}{2}}$ for all $0 < \delta \leq c((hr^3)^{-2} \leq c' \triangle_j^2$, and so

$$(R|(m,n)|)^{1/2} \text{diam}\left(\mathcal{C}\left(\tfrac{P_{m,n}}{|P_{m,n}|}, \tfrac{1}{R|(m,n)|}\right)\right) \geq c$$

if $(R|(m,n)|)^{-1} \leq c(hr^3)^{-2}$, and in particular if $R \geq Ch^2r^6$. Thus if $\log R \geq Cr^{\frac{3}{\varepsilon}}$ and $R^{\frac{1}{2}}(\log R)^{-\varepsilon} \leq h \leq R^{\frac{1}{2}} r^{-3}$, we have both (9.2) and (9.3). □

10. On the notion of curvature bounded below for rough convex domains in the plane

Let Ω be an open bounded convex set in the plane $\mathbb{R}^2$. In the case $\partial\Omega$ is C^2, the classical curvature of $\partial\Omega$ is well-defined at every boundary point $P \in \partial\Omega$. If however Ω is an arbitrary bounded convex set, then $\partial\Omega$ may be only Lipschitz continuous, and the classical definition of curvature is no longer available. In this paper we needed to define the notion that a strictly convex set has *curvature bounded below* by a positive constant, and took for convenience the inequality (1.7) as a definition. Here we will consider three natural related definitions of this concept and demonstrate their equivalence, as well as their equivalence with the classical definition when $\partial\Omega$ is C^2. It turns out to be more convenient to work with the reciprocal of curvature, namely the radius of curvature, and we define instead the notion that a strictly convex set has *radius of curvature bounded above by a positive constant.*

Note that if Ω is bounded and strictly convex, then for every unit normal vector $\mathfrak{n}$, there is a unique point $P = P(\mathfrak{n}) \in \partial\Omega$ such that $\mathfrak{n}$ is an outer normal to $\partial\Omega$ at P. Given the unit vector $\mathfrak{n}$ and $\delta > 0$, we define the solid boundary cap $\mathfrak{C}(\mathfrak{n},\delta)$ associated to $\mathfrak{n}$

$$\mathfrak{C}_\Omega(\mathfrak{n},\delta) = \{x \in \Omega : 0 < \langle P - x, \mathfrak{n}\rangle < \delta\}. \tag{10.1}$$

It is easy to see that for any convex domain there is a constant $c = c_\Omega > 0$ so that

$$c\,\text{area}(\mathfrak{C}_\Omega) \leq \delta^{-1}\text{diam}(\mathcal{C}_\Omega(\mathfrak{n},\delta)) \leq c^{-1}\text{area}(\mathfrak{C}_\Omega); \tag{10.2}$$

here $\mathcal{C}_\Omega(\mathfrak{n},\delta)$ is the boundary cap as defined in (1.5). Given an interior point P_0 of Ω the constant c in (10.2) depends just on the ratio of radii of an inscribed and a circumscribed circle both centered at P_0.

Next, given three noncollinear points P,Q,R in the plane, denote by $\mathcal{R}(P,Q,R)$ the radius of curvature of the unique circle through P, Q and R, namely

$$\mathcal{R}(P,Q,R)=\frac{1}{2}\frac{|R-P|}{\sin(\measuredangle PQR)}=\frac{|P-Q|\,|Q-R|\,|R-P|}{4\text{area}(\triangle PQR)}. \tag{10.3}$$

Definition 10.1. Let Ω be a bounded open strictly convex set in the plane $\mathbb{R}^2$.

(1) We say that $\partial\Omega$ has radius of curvature bounded above by C_1 in the *disk sense* if for every point $P\in\partial\Omega$ and supporting L through P, there is an open disk D of radius C_1 such that

$$P\in\partial D,\ L \text{ is tangent to } \partial D \text{ and } \Omega\subset D. \tag{10.4}$$

(2) We say that $\partial\Omega$ has radius of curvature bounded above by C_2 in the *cap sense* if

$$\text{area}\big(\mathfrak{C}_\Omega(\mathfrak{n},\delta)\big)\le\sqrt{C_2}\,\delta^{3/2} \tag{10.5}$$

for all unit vectors $\mathfrak{n}$ and $\delta>0$.

(3) We say that $\partial\Omega$ has radius of curvature bounded above by C_3 in the *three-point sense* if

$$\mathcal{R}(P,Q,R)\le C_3 \tag{10.6}$$

for every triple of distinct points P,Q,R in $\partial\Omega$.

We remark that in view of (10.2) statement (2) in the definition is equivalent with the condition in (1.7) which we used earlier in the paper.

Lemma 10.2. *Let Ω be a bounded open strictly convex set in the plane $\mathbb{R}^2$. Then statements (1), (2), (3) of Definition 10.1 are equivalent; moreover for the infima of the constants in* (10.4)*,* (10.5) *and* (10.6) *we have*

$$\inf C_2\le 4\inf C_1,\ \ \inf C_3\le 8\inf C_2,\ \ \inf C_1\le\inf C_3.$$

Proof. ($1\Rightarrow 2$): Suppose that $\partial\Omega$ has radius of curvature bounded above by C_1 in the disk sense, and fix $\mathfrak{n}\in\mathbb{T}$ and $\delta>0$. Let P be the unique point in $\partial\Omega$ with $\mathfrak{n}$ as an outer normal, let L be a line through P perpendicular to $\mathfrak{n}$, and let D be the open disk of radius C_1 satisfying (10.4). Then $\mathfrak{C}_\Omega(\mathfrak{n},\delta)\subset\mathfrak{C}_D(\mathfrak{n},\delta)$, and since a cap of width β in the unit disk has area not more than $2\beta^{\frac{3}{2}}$, we have the estimate

$$\begin{aligned}\text{area}\big(\mathfrak{C}_\Omega(\mathfrak{n},\delta)\big)&\le\text{area}\big(\mathfrak{C}_D(\mathfrak{n},\delta)\big)=C_1^2\text{area}\big(\mathfrak{C}_{C_1^{-1}D}(\mathfrak{n},C_1^{-1}\delta)\big)\\&\le C_1^2 2\left(C_1^{-1}\delta\right)^{\frac{3}{2}}=\sqrt{4C_1}\,\delta^{\frac{3}{2}}.\end{aligned}$$

($2\Rightarrow 3$): Suppose now that $\partial\Omega$ has radius of curvature bounded above by C_2 in the cap sense, and let P,Q,R be a triple of distinct points in $\partial\Omega$ with largest side $\overline{PR}$. Consider the portion $\partial\Omega'$ of $\partial\Omega$ lying on the same side of the segment $\overline{PR}$ as the point Q, and let $Q' \in \partial\Omega'$ maximize the distance

$$\delta = dist\left(Q', \overline{PR}\right)$$

from Q' to $\overline{PR}$. Then with $\mathfrak{n}$ equal to the unit vector perpendicular to $\overline{PR}$ in the direction toward Q', we have from (10.1) that

$$\frac{1}{2}\delta\left|\overline{PR}\right| = \text{area}\left(\triangle PQ'R\right) \leq \text{area}\big(\mathfrak{C}_{\Omega}(\mathfrak{n},\delta)\big) \leq \sqrt{C_2}\delta^{\frac{3}{2}}.$$

Thus

$$\left|\overline{PQ'}\right|\left|\sin\left(\measuredangle Q'PR\right)\right| = dist\left(Q', \overline{PR}\right) = \delta \geq \frac{1}{4C_2}\left|\overline{PR}\right|^2,$$

and so by (10.3), we obtain

$$\mathcal{R}\left(P, Q', R\right) = \frac{1}{2}\frac{\left|\overline{PR}\right|}{\sin\left(\measuredangle PQ'R\right)} \leq 2C_2\frac{\left|\overline{PQ'}\right|}{\left|\overline{PR}\right|}.$$

In the case when $Q = Q'$, we then obtain

$$\mathcal{R}\left(P, Q, R\right) \leq 2C_2$$

since $\overline{PR}$ is the largest side of $\triangle\left(P, Q, R\right)$. Thus we have proved in this paragraph that $\mathcal{R}\left(P, Q, R\right) \leq 2C_2$ whenever $\overline{PR}$ is the largest side of $\triangle\left(P, Q, R\right)$ and Q has maximal distance on its side from the line through P and R.

Now let $\{P, Q, R\}$ be an arbitrary triple of points in $\partial\Omega$ with $\left|\overline{PR}\right| \geq \left|\overline{QR}\right| \geq \left|\overline{PQ}\right|$. Let L be the line through Q and R, and let $S \in \partial\Omega$ lie on the side of L opposite P maximize the distance from S to L. There are now two cases to consider: $\left|\overline{QS}\right| \leq \left|\overline{SR}\right|$ and $\left|\overline{QS}\right| > \left|\overline{SR}\right|$. In the first case, $\left|\overline{QS}\right| \leq \left|\overline{SR}\right|$ implies $\sin\left(\measuredangle SRQ\right) \leq \sin\left(\measuredangle SQR\right)$, and since we also have $\sin\left(\measuredangle SQR\right) \leq \sin\left(\measuredangle PQR\right)$, we conclude that

$$\sin\left(\measuredangle QSR\right) = \sin\left(\measuredangle SRQ + \measuredangle SQR\right) \leq 2\sin\left(\measuredangle SQR\right) \leq 2\sin\left(\measuredangle PQR\right).$$

Since also $\left|\overline{QR}\right| \geq \frac{1}{2}\left|\overline{PR}\right|$, it then follows from the previous paragraph that

$$\mathcal{R}\left(P, Q, R\right) = \frac{\left|\overline{PR}\right|}{2\sin\left(\measuredangle PQR\right)} \leq \frac{2\left|\overline{QR}\right|}{\sin\left(\measuredangle QSR\right)} = 4\mathcal{R}\left(Q, S, R\right) \leq 8C_2.$$

In the second case, we have $\left|\overline{QS}\right| \geq \frac{1}{2}\left|\overline{QR}\right| \geq \frac{1}{4}\left|\overline{PR}\right|$. Let M be the line through Q and S, and let $T \in \partial\Omega$ lie on the side of M opposite R maximize the distance from T to M. Since T is closer to the line L than S, it follows that we have the key property

$$\sin\left(\measuredangle QTS\right) \leq \sin\left(\measuredangle PQR\right).$$

From these inequalities we obtain

$$\mathcal{R}\left(P, Q, R\right) = \frac{\left|\overline{PR}\right|}{2\sin\left(\measuredangle PQR\right)} \leq \frac{4\left|\overline{QS}\right|}{2\sin\left(\measuredangle QTS\right)} = 4\mathcal{R}\left(Q, T, S\right) \leq 8C_2.$$

($3 \Rightarrow 1$): Now we suppose that $\partial\Omega$ has radius of curvature bounded above by C_3 in the three-point sense, and fix $P \in \partial\Omega$ and a supporting line

L at P. Consider the family of *closed* disks $\left\{\overline{D_r}\right\}_{r>0}$ where D_r is the open disk of radius r tangent to L at P and lying on the same side of L as Ω. Let $\overline{D_{r_0}}$ be the smallest such closed disk containing Ω, which exists since the intersection of any subfamily of $\left\{\overline{D_r}\right\}_{r>0}$ is either $\{P\}$ or a member of $\left\{\overline{D_r}\right\}_{r>0}$. If $\partial D_{r_0} \cap \partial\Omega$ contains at least three points, say P, Q, R, then $r_0 = \mathcal{R}(P, Q, R) \leq C_3$ and so (10.4) holds with $D = D_{r_0}$ and $C_1 = r_0 \leq C_3$ as required. The remaining cases where $\partial D_{r_0} \cap \partial\Omega$ contains exactly two points, or just the single point P, will now be handled separately.

In the case $\partial D_{r_0} \cap \partial\Omega = \{P, R\}$, consider the portion $\partial\Omega'$ of $\partial\Omega$ lying in the smaller of the two components of $D_{r_0} \setminus M$ where M is the line through P and R (if $\overline{PR}$ is a diameter of D_{r_0}, we can use either of the equally sized components of $D_{r_0} \setminus M$). Let $Q \in \partial\Omega'$ maximize the distance $\delta = dist\,(Q, M)$ from Q to the line M. Then again we have $r_0 \leq \mathcal{R}(P, Q, R) \leq C_3$ as required.

Finally we consider the case $\partial D_{r_0} \cap \partial\Omega = \{P\}$. Let $\eta = \frac{1}{10} diam\,(\Omega)$. The compact sets $\partial D_{r_0} \setminus \left\{x \in \mathbb{R}^2 : |x - P| < \eta\right\}$ and $\overline{\Omega}$ are disjoint, and so we can find $\dot{\varepsilon} > 0$ arbitrarily small such that both $\#\left(\partial D_{r_0 - \varepsilon} \cap \partial\Omega\right) > 1$ and

$$\partial D_{r_0 - \varepsilon} \cap \partial\Omega \subset \left\{x \in \mathbb{R}^2 : |x - P| < \eta\right\}.$$

If $\partial D_{r_0 - \varepsilon} \cap \partial\Omega$ contains at least three points, say P, Q, R, then

$$r_0 - \varepsilon = \mathcal{R}(P, Q, R) \leq C_3.$$

If $\partial D_{r_0 - \varepsilon} \cap \partial\Omega = \{P, R\}$, consider the component $\partial\Omega'$ of $\partial\Omega \setminus \{P, R\}$ lying in the ball $\left\{x \in \mathbb{R}^2 : |x - P| < \eta\right\}$. Then we have that $\partial\Omega' \subset D_{r_0} \setminus D_{r_0 - \varepsilon}$. We claim that a calculation using this, (10.3) and the fact that P and R lie on $\partial D_{r_0 - \varepsilon}$, yields

$$\lim_{Q \to P} \mathcal{R}(P, Q, R) = \lim_{Q \to P} \frac{|P - Q|\,|Q - R|\,|R - P|}{\text{area}\,(\triangle PQR)} = r_0 - \varepsilon,$$

where the limit is taken as Q tends to P along $\partial\Omega'$, or more generally along any path tangent to $\partial D_{r_0 - \varepsilon}$ at P. To simplify the calculation we let $r = r_0 - \varepsilon$ and take P to be the origin, L to be the $x - axis$, ∂D_r to be the circle $x^2 + (y - r)^2 = r^2$, $R = (u, v) \in \partial D_r$ and $Q = (x, y)$. Then by (10.3),

$$\begin{aligned} \mathcal{R}(P, Q, R) &= \frac{|P - Q|\,|Q - R|\,|R - P|}{4\text{area}\,(\triangle PQR)} \\ &= \frac{1}{2} \frac{\sqrt{x^2 + y^2}}{|xv - yu|} \sqrt{(x - u)^2 + (y - v)^2} \sqrt{u^2 + v^2}, \end{aligned}$$

which tends to $\frac{1}{2} \frac{u^2 + v^2}{|v|} = r$ as (x, y) approaches the origin along any path where $\left|\frac{y}{x}\right| \to 0$. Thus we have

$$r_0 - \varepsilon = \lim_{Q \to P} \mathcal{R}(P, Q, R) \leq C_3$$

in this case as well. Since $\varepsilon > 0$ can be made arbitrarily small, we have $r_0 \leq C_3$. Altogether then we see that (10.4) holds with $D = D_{r_0}$ and $C_1 = r_0 \leq C_3$, and this completes the proof of the lemma. □

Remark. If $\partial\Omega$ is C^2, then the curvature $\kappa(Q)$ of $\partial\Omega$ at the point $Q \in \partial\Omega$ satisfies

$$\frac{1}{\kappa(Q)} = \lim_{P,R\to Q} \mathcal{R}(P,Q,R),$$

where the limit is taken as distinct points P and R tend to Q along $\partial\Omega$. This shows that the curvature of a C^2, bounded and strictly convex set Ω is bounded below by $c > 0$ if and only if the radius of curvature is bounded above by c^{-1} in the three-point sense.

11. Appendix on generalized distances for lattice points in dimensions $d \geq 3$

Let ρ be the norm associated to a convex symmetric domain containing the origin, with smooth boundary and with the property that the Gaussian curvature of the boundary vanishes nowhere. Here we are interested in lower bounds for the distance sets

$$\Delta_K(E) = \{\rho(x-y) : x, y \in E\},$$

where E will be taken as $E(R) = \{k \in \mathbb{Z}^d, |k| \leq R\}$.

This can be considered as an instance of a problem by Erdős [7] who conjectured for K being the unit ball for the Euclidean metric that for *any* finite set $E \subset \mathbb{R}^d$ ($d \geq 2$) one should have the estimate

$$\text{card}(\Delta_K(E)) \geq C_\varepsilon(\text{card}(E))^{\frac{2}{d}-\varepsilon}; \tag{11.1}$$

this conjecture makes also sense for the more general metrics as described above and suggests the lower bound $\text{card}(\Delta_K(E(R))) \gtrsim R^{2-\varepsilon}$ in our special case, for all metrics as described above. The general conjecture is open in every dimension $d \geq 2$. For some of the best currently known partial results and a description of the relevant combinatorial techniques we refer to the survey [22] and other articles in the same volume.

For the case of the Euclidean metric (i.e. $K = \{|x| \leq 1\}$) the lower bound $R^{2-\varepsilon}$ for $\Delta_K(E(R))$ is well known and follows from properties of the number $r(n)$ of representations of an integer n as a sum of two squares (see Theorems 338 and 339 in [8]). For more general metrics we shall deduce the lower bound for $\text{card}(\Delta_K(E(R)))$ from mean discrepancy results in [13] (see also the previous work by W. Müller [17]). Unfortunately, in two dimensions these results do not seem to yield anything nontrivial for the distance problem.

Since the distance set $\Delta_K(E(R))$ contains the image of $E(R)$ under ρ it is sufficient to prove the lower bound $\text{card}(\rho(E(R))) \geq C_\epsilon R^{2-\epsilon}$, $\epsilon > 0$. We have the following more precise estimate.

Proposition 11.1. *Let $d \geq 3$ and let Ω be an open convex bounded set in $\mathbb{R}^d$ containing the origin in its interior. Suppose that the boundary $\partial\Omega$ is C^∞, with nonvanishing Gaussian curvature. Let ρ be the Minkowski-functional associated to Ω, let*

$$\mathbb{E}_R = \{k \in \mathbb{Z}^d : R/2 \leq |k| \leq R\}$$

and let $\rho(\mathbb{E}_R) = \{\rho(a) : a \in \mathbb{E}_R\}$. Then there exists a constant C_0 so that for all $R \geq C_0$ we have

$$\operatorname{card}(\rho(\mathbb{E}_R)) \geq \begin{cases} R^2 & \text{if } d \geq 4 \\ R^2/\log R & \text{if } d = 3. \end{cases}$$

Proof. Let $\alpha \geq 0$. We define for a finite set A the quantity

$$m_{\rho,\alpha}(A) = \max\{\operatorname{card}(F) : F \subset \rho(A), |s-t| > \alpha\} \text{ for all } s,t \in F\}.$$

In particular note that $m_{\rho,0}(A) = \rho(A)$; in fact in view of the finiteness of A we have $\rho(A) = m_{\rho,\varepsilon_0}(A)$ for some $\varepsilon_0 = \varepsilon_0(A) > 0$. Moreover we let for $\varepsilon \leq 1$, $R \geq 1$

$$S(\varepsilon, r; A) = \operatorname{card}\{k \in A; |\rho(k) - r| \leq \varepsilon\};$$
$$\sigma(A,\varepsilon) = \sum_{k\in A} S(2\varepsilon, \rho(k); A).$$

We first observe that for any finite set A (later $A = \mathbb{E}_R$), and $\varepsilon > 0$

$$\operatorname{card}(A) \leq \sqrt{m_{\rho,\varepsilon}(A)}\sqrt{3\sigma(A,\varepsilon)}. \tag{11.2}$$

Indeed if F is a subset of $\rho(A)$ for which the maximum in the definition of $m_{\rho,\varepsilon}$ is attained then the Cauchy-Schwarz inequality gives

$$\operatorname{card}(A) \leq \big(\operatorname{card} F\big)^{1/2}\Big(\sum_{t\in F}\big(S(\varepsilon,t;A)\big)^2\Big)^{1/2}.$$

Now for fixed k there are at most three intervals of the form $[s-\varepsilon, s+\varepsilon]$, $s \in F$ to which $\rho(k)$ can belong. Thus the right hand side of the last equation is dominated by $m_{\rho,\varepsilon}(A)^{1/2}(3\sum_{k\in A} S(2\varepsilon,\rho(k);A))^{1/2}$ which yields (11.2).

By estimates from [13] (namely the argument on p. 218/219 and the statement of Lemma 2.1 of that paper) we get for $\varepsilon \leq R^{-1}$

$$\Big(R^{-d}\sum_{k\in\mathbb{E}_R} S(2\varepsilon,\rho(k))^2\Big)^{1/2} \leq \Big(\sum_{k\in\mathbb{E}} S(2/R,\rho(k))^2\Big)^{1/2}$$
$$\leq C_1\Big(R^{-1}\int_{R/4}^{4R}|E(t)|^2 dt\Big)^{1/2} + C_2 R^{d-2}$$

where here of course $E(t) = \operatorname{card}(t\Omega \cap \mathbb{Z}^d) - t^d \operatorname{vol}(\Omega)$. The results on the mean square discrepancy in [13] imply that the first term is $O(R^{d-2})$ if $d \geq 4$ and $O(R\log R)$ if $d = 3$.

Now $\sigma(E_R,\varepsilon) \le CR^d(R^{-d}\sum_{k\in\mathbb{E}_R} S(2\varepsilon,\rho(k))^2)^{1/2}$ and thus

$$\sigma(E_R,\varepsilon) \le C \begin{cases} R^{2d-2} & \text{if } d \ge 4 \\ R^4 \log R & \text{if } d = 3 \end{cases}.$$

Since $\operatorname{card}(\mathbb{E}_R) \approx R^d$ we may use (11.2) to obtain for all $\varepsilon \in (0, 1/R)$ the lower bound $m_{\rho,\varepsilon}(A) \gtrsim R^2$ if $d \ge 4$ and $m_{\rho,\varepsilon}(A) \gtrsim R^2/\log R$ if $d = 3$ and this implies the asserted lower bounds for the cardinality of $\rho(\mathbb{E}_R)$. □

Remark: For the Euclidean ball $K = B_3$ in three dimensions one can use the mean discrepancy result by Jarník [15] to improve (in this very special case) the lower bound $R^2/\log R$ in Proposition 11.1 to $R^2/\sqrt{\log R}$.

References

[1] G. Andrews, *A lower bound for the volume of strictly convex bodies with many boundary lattice points,* Trans. Amer. Math. Soc. **106** (1963), 270-279.

[2] P. Bleher, *On the distribution of the number of lattice points inside a family of convex ovals*, Duke Math. J. **67** (1992), 461–481.

[3] L. Brandolini, L. Colzani, A. Iosevich, A. Podkorytov and G. Travaglini, *The geometry of the Gauss map and the distribution of lattice points in convex planar domains*, Mathematika, Mathematika **48** (2001), no. 1-2, 107–117 (2003).

[4] L. Brandolini, M. Rigoli and G. Travaglini, *Average decay of Fourier transforms and the geometry of convex sets*, Rev. Mat. Iberoamericana **14** (1998), 519–560

[5] J. Bruna, A. Nagel and S. Wainger, *Convex hypersurfaces and Fourier transform*, Ann. Math. **127** (1988), 333–365.

[6] Y. Colin de Verdière, *Nombre de points entiers dans une famille homothétique de domaines de* $\mathbb{R}^2$, Ann. Scient. Ec. Norm. Sup. **10** (1977), 559-575.

[7] P. Erdős, *On the set of distances of n points*, Amer. Math. Monthly, **53** (1946), 85–91.

[8] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers. Oxford Univ. Press.*

[9] E. Hlawka, *Über Integrale auf konvexen Körpern I*, Monatshefte Math. **54** (1950), 1–36; *II*, Monatshefte Math. **54** (1950), 81–99.

[10] L. Hörmander. *The analysis of linear partial differential operators Vol. I*, Springer-Verlag, New York, Berlin (1983).

[11] M. N. Huxley, *The mean lattice point discrepancy*, Proc. Edinburgh Math. Soc. **38** (1995), 523–531.

[12] M.N. Huxley, *Area, Lattice Points, and Exponentials Sums*, London Mathematical Society Monographs New Series 13, Oxford Univ. Press 1996

[13] A. Iosevich, E. Sawyer and A. Seeger, *Mean square discrepancy bounds for the number of lattice points in large convex bodies*, J. Anal. Math. **87** (2002), 209–230.

[14] V. Jarník, *Über die Gitterpunkte auf konvexen Kurven*, Math. Z. **24** (1925) 500–518.

[15] ———, *Über die Mittelwertsätze in der Gitterpunktlehre, V*, Časopis Pěst. Mat. **69** (1940), 148–174; *VI*, Časopis Pěst. Mat. **70** (1941), 89–103.

[16] E. Landau, *Zahlentheorie*, Hirzel, Leipzig (1927).

[17] W. Müller, *On the average order of the lattice rest of a convex body*, Acta Arithmetica **53** (1997), 89–100.

[18] A. Nagel, A. Seeger and S. Wainger, *Averages over convex hypersurfaces*, Amer. J. Math. **115** 903–927.

[19] W. G. Nowak, *On the average order of the lattice rest of a convex planar domain*, Proc. Camb. Phil. Soc. **98** (1985), 1–4.

[20] W. G. Nowak, *Zur Gitterpunktlehre in der Euklidischen Ebene, II*, Österreich. Akad. Wiss. Math. Natur. Kl. Sitzungsber. II **194** (1985) 31–37.
[21] W. G. Nowak, *On the mean lattice point discrepancy of a convex disc*, Arch. Math. **78** (2002). 241–248.
[22] J. Pach and M. Sharir, *Geometric incidences,* Towards a theory of geometric graphs, 185–223, Contemp. Math., **342**, Amer. Math. Soc., Providence, RI, 2004.
[23] A. Plagne, *A uniform version of Jarník's theorem*, Acta Arith. **87** (1999), 255–267.
[24] A. N. Podkorytov, *Fejér means in the two-dimensional case*, Vestnik Leningrad Univ. (Matem.) (1978), no. 13, 32–39, 155 (Russian); English translation in Vestnik Leningrad Univ. Math. **11** (1981).
[25] B. Randol, *A lattice point problem*, Trans. Amer. Math. Soc. **121** (1966), 257–268; *II*, Trans. Amer. Math. Soc. **125** 101–113.
[26] F. Riesz and B. Szökefalvi-Nagy, Functional Analysis, New York, 1955.
[27] A. Seeger and S. Ziesler, *Riesz means associated with convex domains in the plane*, Math. Z. **236** (2001), 643–676.
[28] E. M. Stein. *Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press (1993).
[29] E. M. Stein and N. J. Weiss, *On the convergence of Poisson integrals*, Trans. Amer. Math. Soc. **140** (1969) 34–54.

A. Iosevich, Mathematics Department, University of Missouri, Columbia, MO 65211, USA
E-mail address: iosevich@math.missouri.edu

E.T. Sawyer, Department of Mathematics and Statistics, 1280 Main Street West, Hamilton, Ontario L8S 4K1, Canada
E-mail address: sawyer@mcmaster.ca

A. Seeger, Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA
E-mail address: seeger@math.wisc.edu