SPECTRAL AND TILING PROPERTIES OF THE UNIT CUBE†

ALEX IOSEVICH AND STEEN PEDERSEN

Abstract. Let $\mathcal{Q} = [0,1)^d$ denote the unit cube in d-dimensional Euclidean space $\mathbb{R}^d$ and let $\mathcal{T}$ be a discrete subset of $\mathbb{R}^d$. We show that the exponentials $e_t(x) := exp(i2\pi tx)$, $t \in \mathcal{T}$ form an othonormal basis for $L^2(\mathcal{Q})$ if and only if the translates $\mathcal{Q}+t$, $t \in \mathcal{T}$ form a tiling of $\mathbb{R}^d$.

1. Introduction

Let $\mathcal{Q} := [0,1)^d$ denote the unit cube in d-dimensional Euclidean space $\mathbb{R}^d$. Let $\mathcal{T}$ be a discrete subset of $\mathbb{R}^d$. We say $\mathcal{T}$ is a *tiling set* for $\mathcal{Q}$, if each $x \in \mathbb{R}^d$ can be written uniquely as $x = q + t$, with $q \in \mathcal{Q}$ and $t \in \mathcal{T}$. We say $\mathcal{T}$ is a *spectrum* for $\mathcal{Q}$, if the exponentials

$$e_t(x) := e^{i2\pi tx}, t \in \mathcal{T}$$

form an orthonormal basis for $L^2(\mathcal{Q})$. Here juxtaposition tx of vectors t, x in $\mathbb{R}^d$ denote the usual inner product $tx = t_1x_1 + \cdots + t_dx_d$ in $\mathbb{R}^d$ and $L^2(\mathcal{Q})$ is equipped with the usual inner product, viz.,

$$\langle f, g \rangle := \int_{\mathcal{Q}} f\,\overline{g}\,dm$$

where m denotes Lebesgue measure. The main result proved in this paper is

Theorem 1.1. *Let $\mathcal{T}$ be a subset of $\mathbb{R}^d$. Then $\mathcal{T}$ is a spectrum for the unit cube $\mathcal{Q}$ if and only if $\mathcal{T}$ is a tiling set for the unit cube $\mathcal{Q}$.*

Remark 1.2. *As we shall discuss below there exists highly counter-intuitive cube-tilings in $\mathbb{R}^d$ for sufficiently large d. Those tilings can be much more complicated than lattice tilings. Theorem 1.1 is clear if $\mathcal{T}$ is a lattice. The point of Theorem 1.1 is that the result still holds even if the restrictive lattice assumption is dropped.*

Sets whose translates tile $\mathbb{R}^d$ and the corresponding tiling sets have been investigated intensively, see [GN], [LW1], [LW2] for some recent papers. Even the one-dimensional case

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$d = 1$ is non-trivial. The study of sets whose L^2-space admits orthogonal bases of exponentials was begun in [Fu]. Several papers have appeared recently, see e.g., [JP1], [Pe2], [LW3]. It was conjectured in [Fu] that a set admits a tiling set if and only if it admits a spectrum, i.e., the corresponding L^2-space admits an orthogonal basis of exponentials.

Cube tilings have a long history beginning with a conjecture due to Minkowski: in every lattice tiling of $\mathbb{R}^d$ by translates of $\mathcal{Q}$ some cubes must share a complete $(d-1)$-dimensional face. Minkowski's conjecture was proved in [Haj], see [SS] for a recent exposition. Keller [Kel] while working on Minkowski's conjecture made the stronger conjecture that one could omit the lattice assumption in Minkowski's conjecture. Using [Sza] and [CS] it was shown in [LS] that there are cube tilings in dimensions $d \geq 10$ not satisfying Keller's conjecture.

The study of the possible spectra for the unit cube was initiated in [JP2], where Theorem 1.1 was conjectured. Theorem 1.1 was proved in [JP2] if $d \leq 3$ and for any d if $\mathcal{T}$ is periodic.

The terminology *spectrum for* $\mathcal{Q}$ originates in a problem about the existence of certain commuting self-adjoint partial differential operators. We say that two self-adjoint operators *commute* if their spectral measures commute, see [RS] for an introduction to the theory of unbounded self-adjoint operators. The following result was proved in [Fu] under a mild regularity condition on the boundary, the regularity condition was removed in [Pe1].

Theorem 1.3. *Let Ω be a connected open subset of $\mathbb{R}^d$ with finite Lebesgue measure. There exists a set $\mathcal{T}$ so that the exponentials e_t, $t \in \mathcal{T}$ form an orthogonal basis for $L^2(\Omega)$ if and only if there exists commuting self-adjoint operators $H = (H_1, \ldots, H_d)$ so that each H_j is defined on $C_c^\infty(\Omega)$ and*

$$H_j f = \frac{1}{i2\pi} \frac{\partial f}{\partial x_j} \tag{1.1}$$

for any $f \in C_c^\infty(\Omega)$ and any $j = 1, \ldots, d$.

More precisely, if e_t, $t \in \mathcal{T}$ is an orthogonal basis for $L^2(\Omega)$ then a commuting tuple $H = (H_1, \ldots, H_d)$ of self-adjoint operators satisfying (1.1) is uniquely determined by $H_j e_t = t_j e_t$, $t \in \mathcal{T}$. Conversely, if $H = (H_1, \ldots, H_d)$ is a commuting tuple of self-adjoint operators satisfying (1.1) then the joint spectrum $\sigma(H)$ is discrete and each $t \in \sigma(H)$ is a simple eigen-value corresponding to the eigen-vector e_t, in particular, e_t, $t \in \sigma(H)$ is an orthogonal basis for $L^2(\Omega)$.

We prove any tiling set is a spectrum in Section 5, the converse is proved in Section 4. Key ideas in both proofs are that if (g_n) is an orthonormal family in $L^2(\mathcal{Q})$ and $f \in L^2(\mathcal{Q})$ then we have equality in Bessel's inequality

$$\sum |\langle f, g_n \rangle|^2 \leq ||f||^2$$

if and only if f is in the closed linear span of (g_n), and a sliding lemma (Lemma 3.3) showing that we may translate certain parts of a spectrum or tiling set while preserving the spectral respectively the tiling set property. In Section 3 we prove some elementary properties of spectra and tiling sets. For $t \in \mathbb{R}^d$ let $\mathcal{Q} + t := \{q + t : q \in \mathcal{Q}\}$ denote the translate of $\mathcal{Q}$ by the vector t. We say $(\mathcal{Q}, \mathcal{T})$ is *non-overlapping* if the cubes $\mathcal{Q} + t$ and $\mathcal{Q} + t'$ are disjoint for any $t, t' \in \mathcal{T}$. Note, $\mathcal{T}$ is a tiling set for $\mathcal{Q}$ if and only if $(\mathcal{Q}, \mathcal{T})$ is non-overlapping and $\mathbb{R}^d = \mathcal{Q}_{\mathcal{T}} := \bigcup_{t \in \mathcal{T}} (\mathcal{Q} + t)$. We say $(\mathcal{Q}, \mathcal{T})$ is *orthogonal*, if the exponentials $e_t, t \in \mathcal{T}$ are orthogonal in $L^2(\mathcal{Q})$. A set $\mathcal{T}$ is a spectrum for $\mathcal{Q}$ if and only if $(\mathcal{Q}, \mathcal{T})$ is orthogonal and

$$\sum_{t \in \mathcal{T}} |\langle e_n, e_t \rangle|^2 = 1$$

for all $n \in \mathbb{Z}^d$. Let $\mathbb{N}$ denote the positive integers $\{1, 2, 3, \ldots\}$ and let $\mathbb{Z}$ denote the set of all integers $\{\ldots, -1, 0, 1, 2, \ldots\}$.

As this paper was in the final stages of preparation, we received a preprint [LRW] by Lagarias, Reed and Wang proving our main result.Compared to [LRW] our proof that any spectrum is a tiling set uses completely different techniques, the proof that any tiling set is a spectrum is similar to the proof in [LRW] in that both proofs makes use of Keller's Theorem (Theorem 5.1) and an argument involving an inequality becoming equality. We wish to thank Lagarias for the preprint and useful remarks. Robert S. Strichartz helped us clarify the exposition.

2. Plan

Our plan is as follows. The basis property is equivalent to the statement that the sum

$$\sum_{t \in \mathcal{T}} |\langle e_x, e_t \rangle|^2 = 1 \tag{2.1}$$

for all $x \in \mathbb{R}^d$. It is easy to see that if $\mathcal{T}$ has the basis property then the cubes $\mathcal{Q}+t$, $t \in \mathcal{T}$ are non-overlapping. We show by a *geometric argument* that if the basis property holds and the tiling property does not hold then the sum in (2.1) is strictly less than one. Conversely, if $\mathcal{T}$

has the tiling property then the exponentials e_t, $t \in \mathcal{T}$ are orthogonal by Keller's Theorem, Plancerel's Theorem now implies that the sum in (2.1) is one. The geometric argument is based Lemma 3.3, an analoguous lemma was used by Perron [Per] in his proof of Keller's Theorem.

3. Spectral Properties

We begin by proving a simple result characterizing orthogonal subsets of $\mathbb{R}^d$. There is a corresponding (non-trivial) result for tilings, stated as Theorem 5.1 below.

Lemma 3.1 (Spectral version of Keller's theorem)**.** *Let $\mathcal{T}$ be a discrete subset of $\mathbb{R}^d$. The pair $(\mathcal{Q}, \mathcal{T})$ is orthogonal if and only if given any pair $t, t' \in \mathcal{T}$, with $t \neq t'$, there exists a $j \in \{1, \ldots, d\}$ so that $|t_j - t'_j| \in \mathbb{N}$.*

Proof. For $t, t' \in \mathbb{R}^d$ we have

$$\langle e_t, e_{t'} \rangle = \prod_{j=1}^{d} \phi(t_j - t'_j) \tag{3.1}$$

where for $x \in \mathbb{R}$

$$\phi(x) := \begin{cases} 1, & \text{if } x = 0; \\ \frac{e^{i2\pi x} - 1}{i2\pi x}, & \text{if } x \neq 0. \end{cases} \tag{3.2}$$

The lemma is now immediate. □

We can now state our first result showing that there is a connection between spectra and tiling sets for the unit cube.

Corollary 3.2. *Let $\mathcal{T}$ be a subset of $\mathbb{R}^d$. If $(\mathcal{Q}, \mathcal{T})$ is orthogonal, then $(\mathcal{Q}, \mathcal{T})$ is non-overlapping.*

A key technical lemma needed for our proofs of both implications in our main result is the following lemma. The lemma shows that a certain part of a spectrum (respectively tiling set) can be translated independently of its complement without destroying the spectral (respectively tiling set) property. The tiling set part of the lemma if taken from [Per].

Lemma 3.3. *Let $\mathcal{T}$ be a discrete subset of $\mathbb{R}^d$, fix $a, b \in \mathbb{R}$. Let $c := (b, 0, \dots, 0) \in \mathbb{R}^d$ and for $t \in \mathcal{T}$ let*

$$\alpha_{\mathcal{T},a,b}(t) := \begin{cases} t, & \text{if } t_1 - a \in \mathbb{Z}; \\ t + c, & \text{if } t_1 - a \notin \mathbb{Z}. \end{cases}$$

We have the following conclusions: (a) If $\mathcal{T}$ is a spectrum for $\mathcal{Q}$, so is $\alpha_{\mathcal{T},a,b}(\mathcal{T})$. (b) If $\mathcal{T}$ is a tiling set for $\mathcal{Q}$, so is $\alpha_{\mathcal{T},a,b}(\mathcal{T})$.

Proof. Suppose $\mathcal{T}$ is a spectrum for $\mathcal{Q}$. The orthogonality of $(\mathcal{Q}, \alpha_{\mathcal{T},a,b}(\mathcal{T}))$ is an easy consequence of Lemma 3.1. Let $A_{\mathcal{T},a,b}e_t := e_{\alpha_{\mathcal{T},a,b}(t)}$ for $t \in \mathcal{T}$. To simplify the notation we will write A_b in place of $A_{\mathcal{T},a,b}$. By orthogonality and linearity A_b extends to an isometry mapping $L^2(\mathcal{Q})$ into itself. We must show that the range $A_b L^2(\mathcal{Q})$ is all of $L^2(\mathcal{Q})$. Let $\mathcal{K}_+$ be the subspace of $L^2(\mathcal{Q})$ spanned by the exponentials e_t, $t \in \mathcal{T}$ with $t_1 - a \in \mathbb{Z}$ and let $\mathcal{K}_-$ be the subspace of $L^2(\mathcal{Q})$ spanned by the exponentials e_t, $t \in \mathcal{T}$ with $t_1 - a \notin \mathbb{Z}$. Then $A_b f = f$ for all $f \in \mathcal{K}_+$, so $A_b\mathcal{K}_+ = \mathcal{K}_+$. Since A_b preserves orthogonality, $A_b\mathcal{K}_- \subseteq \mathcal{K}_-$. We must show $A_b\mathcal{K}_- = \mathcal{K}_-$. Since $b \in \mathbb{R}$ is arbitrary, we also have that the map A_{-b} is an isometry mapping $\mathcal{K}_-$ into itself. By construction $A_b f = e_c f$ and $A_{-b} f = \overline{e_c} f$ for all $f \in \mathcal{K}_-$. It follows that $\mathcal{K}_- = A_b A_{-b}\mathcal{K}_- \subseteq A_b\mathcal{K}_- \subseteq \mathcal{K}_-$. Hence, $A_b\mathcal{K}_- = \mathcal{K}_-$ as desired. The proof that $\alpha_{\mathcal{T},a,b}(\mathcal{T})$ is a tiling set provided $\mathcal{T}$ is, follows from the last part of the proof of Theorem 5.1 below. $\square$

4. Any spectrum is a tiling set

For $n' \in \mathbb{Z}^{d-1}$ let $\ell_{n'}$ be the line in $\mathbb{R}^d$ given by $\{(x, n'),\ x \in \mathbb{R}\}$. The idea of our proof that any spectrum for $\mathcal{Q}$ must be a tiling set for $\mathcal{Q}$ is as follows. Suppose $\mathcal{T}$ is a spectrum but not a tiling set. Fix $n' \in \mathbb{Z}^{d-1}$ and pick a $t \in \mathcal{T}$ (if any) so that $\mathcal{Q} + t$ intersects the line $\ell_{n'}$ applying Lemma 3.3 we can insure that $t_1 \in \mathbb{Z}$. Repeating this for each $n' \in \mathbb{Z}^{d-1}$ we can ensure $t_1 \in \mathbb{Z}$ for any $t \in \mathcal{T}^{\text{new}}$. Considering each of the remaining coordinate directions we end up with $\mathcal{T}^{\text{new}}$ being a subset of $\mathbb{Z}^d$. (The meaning of $\mathcal{T}^{\text{new}}$ changes with each application of Lemma 3.3.) By Lemma 4.2 $\mathcal{T}^{\text{new}}$ is not a tiling set for $\mathcal{Q}$ since $\mathcal{T}$ was not a tiling set, so $\mathcal{T}^{\text{new}}$ is a proper subset of $\mathbb{Z}^d$, contradicting the basis property. The difficulty with this outline is that after we apply Lemma 3.3 an infinite number of times the basis property may not hold. In fact, associated to each application of Lemma 3.3 is an isometric isomorphism A_{b_n}. Without restrictions on the sequence (b_n) the infinite product $\prod_{n=1}^{\infty} A_{b_n}$ need not be convergent (e.g., with respect to the weak operator topology). Even if the infinite product $\prod_{n=1}^{\infty} A_{b_n}$ is convergent, the limit may be a non-surjective isometry.

It turns out that if we use Lemma 3.3 to put a large finite part of $\mathcal{T}$ into $\mathbb{Z}^d$ then we can use decay properties of the Fourier transform of the characteristic function of the cube $\mathcal{Q}$ to contradict (2.1).

The following lemma shows that sums of the Fourier transform of the characteristic function of the cube $\mathcal{Q}$ over certain discrete sets has uniform decay properties.

Lemma 4.1. *Let ϕ be given by (3.2). There exists a constant $C > 0$ so that*

$$\sum_{t\in\mathcal{T}_N} \prod_{j=1}^{d} |\phi(t_j)|^2 \leq \frac{C}{N}$$

for any $N > 1$, whenever $\mathcal{T} \subset \mathbb{R}^d$ is a spectrum for the unit cube $\mathcal{Q}$. Here $\mathcal{T}_N$ is the set of $t \in \mathcal{T}$ for which $|t_j| > N$, for at least one j. Note, the constant C is uniform over all spectra $\mathcal{T}$ for the unit cube $\mathcal{Q}$ and all $N > 1$.

Proof. Let $\mathcal{T}$ be a spectrum for $\mathcal{Q}$. For any partition $P = \{I, II, III, IV\}$ of $\{1, \dots, d\}$, let $\mathcal{T}_{N,P}$ denote the set of $t \in \mathcal{T}_N$ for so that $t_j > N$ for $j \in I$; $t_j < -N$ for $j \in II$; $0 \leq t_j \leq N$ for $j \in III$ and $-N \leq t_j < 0$ for $j \in IV$. Note $\mathcal{T}_{N,P}$ is empty unless $I \cup II$ is non-empty. For $x \in \mathbb{R}$ let $\psi(x) = 1$, if $-1 < x < 1$ and let $\psi(x) = x^{-2}$ if $|x| \geq 1$. Then for $t \in \mathcal{T}_{N,P}$,

$$\prod_{j=1}^{d} |\phi(t_j)|^2 \leq \prod_{j=1}^{d} \psi(s_j) \tag{4.1}$$

for any $s = (s_1, \dots, s_d)$ in the cube $X_{t,P}$ given by $t_j - 1 \leq s_j < t_j$ if $j \in I \cup III$, and $t_j \leq s_j < t_j + 1$ if $j \in II \cup IV$. It follows from (4.1) and disjointness (Lemma 3.1) of the cubes $X_{t,P}$, $t \in \mathcal{T}_{N,P}$ that

$$\sum_{t\in\mathcal{T}_{N,P}} \prod_{j=1}^{d} |\phi(t_j)|^2 \leq \sum_{t\in\mathcal{T}_{N,P}} \int_{X_{t,P}} \prod_{j=1}^{d} \psi(s_j)\, ds \leq \int_{Y_{N,P}} \prod_{j=1}^{d} \psi(s_j)\, ds,$$

where $Y_{N,P}$ is the set of $y \in \mathbb{R}^d$ for which $N - 1 < y_j$ for $j \in I$, $y_j < -N + 1$ for $j \in II$, $-1 < y_j < N$ for $j \in III$, and $-N < y_j < 1$ for $j \in IV$. By definition of ψ we have

$$\int_{Y_{N,P}} \prod_{j=1}^{d} \psi(s_j)\, ds \leq 3^{d-n} \frac{1}{(N-1)^n},$$

where $n > 0$ is the cardinality of $I \cup II$. Since the number of possible partitions $P = \{I, II, III, IV\}$ only depends on the dimension d of $\mathbb{R}^d$, the proof is complete $\square$

The following lemma shows that if $\mathcal{T}$ is a spectrum but not a tiling set for $\mathcal{Q}$ then the set constructed in Lemma 3.3 is also not a tiling set for $\mathcal{Q}$. It is needed because the inverse of the transformation in Lemma 3.3 is not of the same form.

Lemma 4.2. *If $\mathcal{T}$ is a spetrum for $\mathcal{Q}$ but not a tiling set for $\mathcal{Q}$, then $\alpha_{\mathcal{T},a,b}(\mathcal{T})$ is not a tiling set for $\mathcal{Q}$.*

Proof. Suppose $\mathcal{T}$ is a spectrum for $\mathcal{Q}$ but not a tiling set for $\mathcal{Q}$. Let $g \notin \mathcal{Q}_{\mathcal{T}}$. Let $\ell := \{(x, g_2, g_3, \ldots, g_d\}$. If $r, s \in \mathcal{T}$ are so that $\mathcal{Q} + r$ and $\mathcal{Q} + s$ intersect ℓ then it follows from Lemma 3.1 that $s_1 - r_1$ is an integer, since $|s_j - t_j| < 1$ for $j \neq 1$ because $\mathcal{Q} + r$ and $\mathcal{Q} + s$ intersect ℓ. So either $t_1 - a \in \mathbb{Z}$ for all $t \in \mathcal{T}$ so that $\mathcal{Q} + t$ intersects ℓ or $t_1 - a \notin \mathbb{Z}$ for all $t \in \mathcal{T}$ so that $\mathcal{Q} + t$ intersects ℓ. In the first case $g \notin \mathcal{Q}_{\alpha_{\mathcal{T},a,b}(\mathcal{T})}$ in the second case $g + c \notin \mathcal{Q}_{\alpha_{\mathcal{T},a,b}(\mathcal{T})}$. $\square$

Proof of basis implies tiling. Suppose $\mathcal{T}$ is a spectrum for the unit cube $\mathcal{Q}$. By Corollary 3.2 the pair $(\mathcal{Q}, \mathcal{T})$ is non-overlapping. We must show that the union $\mathcal{Q}_{\mathcal{T}} = \cup_{t \in \mathcal{T}}(\mathcal{Q} + t)$ is all of $\mathbb{R}^d$. To get a contradiction suppose $g \notin \mathcal{Q}_{\mathcal{T}}$. Let N be so large that $g \in (-N + 2, N - 2)^d$. Let $\mathcal{T}(N) := \mathcal{T} \cap (-N - 1, N + 1)^d$.

Let $n_1' := (-N, -N, \ldots, -N) \in \mathbb{Z}^{d-1}$. Pick $t \in \mathcal{T}(N)$ so that $\mathcal{Q} + t$ intersects $\ell_{n_1'}$ (if such a t exists). Use Lemma 3.3 with $a = 0$ and $b = b_1 := t_1 - \lfloor t_1 \rfloor$ to conclude $\mathcal{T}_1 := \alpha_{\mathcal{T},a,b}(\mathcal{T})$ has the basis property. It follows from Lemma 3.1 that $t_1 \in \mathbb{Z}$ for any $t \in \mathcal{T}_1$ so that $\mathcal{Q} + t$ intersects $\ell_{n_1'}$.

Let $n_2' := (-N, -N, \ldots, -N, -N + 1) \in \mathbb{Z}^{d-1}$. Pick $t \in \mathcal{T}_1(N)$ so that $\mathcal{Q} + t$ intersects $\ell_{n_2'}$ (if such a t exists). Use Lemma 3.3 with $a = 0$ and $b = b_2 := t_1 - \lfloor t_1 \rfloor$ if $b_1 + t_1 - \lfloor t_1 \rfloor \geq 1$ and $b = b_2 := t_1 - \lfloor t_1 \rfloor - 1$ if $b_1 + t_1 - \lfloor t_1 \rfloor < 1$ to conclude $\mathcal{T}_2 := \alpha_{\mathcal{T}_1,a,b}(\mathcal{T}_1)$ has the basis property. It follows from Lemma 3.1 that $t_1 \in \mathbb{Z}$ for any $t \in \mathcal{T}_2$ so that $\mathcal{Q} + t$ intersects $\ell_{n_2'}$. Note we did not move any of the cubes in $\mathcal{T}_1$ with $-N - 1 < t_j \leq -N$, for $j = 2, \ldots, d$.

Continuing in this manner, we end up with $\mathcal{T}'$ having the basis property so that $t_1 \in \mathbb{Z}$ for any $t \in \mathcal{T}'$ with $-N - 1 < t_j < N + 1$ for $j = 2, \ldots, d$. Note $-1 < \sum_1^n b_j < 1$ for any n. So if at some stage $t \in \mathcal{T}_n$ is derived from $t^{\text{original}} \in \mathcal{T}$ then we have $t_1^{\text{original}} - 1 < t_1 < t_1^{\text{original}} + 1$.

Repeating this process for each of the other coordinate directions we end up with $\mathcal{T}^{\text{new}}$ so that $\mathcal{T}^{\text{new}}(N - 1)$ is a subset of the integer lattice $\mathbb{Z}^d$, any $t \in \mathcal{T}^{\text{new}}(N - 1)$ is obtained from some $t^{\text{original}} \in \mathcal{T}(N)$, and any $t^{\text{original}} \in \mathcal{T}(N-1)$ is translated onto some $\mathcal{T}^{\text{new}}(N)$. In short, we did not move any point in $\mathcal{T}$ very much. By Lemma 4.2 it follows that $(-N,N)^d \setminus \mathcal{Q}_{\mathcal{T}^{\text{new}}}$ is non-empty, hence there exists $g^{\text{new}} \in \mathbb{Z}^d$, so that $g^{\text{new}} \in (-N,N)^d \setminus \mathcal{T}^{\text{new}}$. Replacing $\mathcal{T}^{\text{new}}$ by $\mathcal{T}^{\text{new}} - g^{\text{new}}$, if necessary, and applying the process described above we may assume $g^{\text{new}} = 0$. To simplify the notation let $\mathcal{T} = \mathcal{T}^{\text{new}}$. We have

$$1 = \sum_{t \in \mathcal{T}} |\langle e_t, e_0 \rangle|^2 = \sum_{t \in \mathcal{T}(N)} |\langle e_t, e_0 \rangle|^2 + \sum_{t \in \mathcal{T}_N} |\langle e_t, e_0 \rangle|^2.$$

The first sum $= 0$ since $\mathcal{T}(N) \subset \mathbb{Z}^d$ and $0 \notin \mathcal{T}(N)$, the second sum is < 1 for N sufficiently large by Lemma 4.1. This contradiction completes the proof. □

5. Any tiling set is a spectrum

The following result (due to [Kel]) shows that any tiling set for the cube is orthogonal. It is a key step in our proof that any tiling set for the cube must be a spectrum for the cube and should be compared with Lemma 3.1 above. The proof is essentially taken from [Per].

Theorem 5.1 (Keller's Theorem). *If $\mathcal{T}$ is a tiling set for $\mathcal{Q}$, then given any pair $t, t' \in \mathcal{T}$, with $t \neq t'$, there exists a $j \in \{1, \ldots, d\}$ so that $|t_j - t'_j| \in \mathbb{N}$.*

Proof. Let $\mathcal{T}$ be a tiling set for $\mathcal{Q}$. Suppose $t, t' \in \mathcal{T}$. The proof is by induction on the number of j's for which $|t_j - t'_j| \geq 1$. Suppose that $|t_j - t'_j| < 1$ for all but one $j \in \{1, \ldots, d\}$. Let j_0 be the exceptional j, then $|t_{j_0} - t'_{j_0}| \geq 1$. Fix x_j, $j \neq j_0$ so that the line $\ell_{j_0} := \{(x_1, \ldots, x_d) : x_{j_0} \in \mathbb{R}\}$ passes through both of the cubes $\mathcal{Q} + t$ and $\mathcal{Q} + t'$. Considering the cubes $Q + t$, $t \in \mathcal{T}$ that intersect ℓ_{j_0} it is immediate that $|t_{j_0} - t'_{j_0}| \in \mathbb{N}$.

For the inductive step, suppose $|t_j - t'_j| < 1$ for k values of j and $|t_j - t'_j| \geq 1$ for the remaining $d-k$ values of j implies $|t_{j_0} - t'_{j_0}| \in \mathbb{N}$ for some j_0. Let $t, t' \in \mathcal{T}$ be so that $|t_j - t'_j| < 1$ for $k-1$ values of j and $|t_j - t'_j| \geq 1$ for the remaining $d-k+1$ values of j. Interchanging the coordinate axes, if necessary, we may assume

$$|t_j - t'_j| \geq 1, \text{ for } j = 1, \ldots, d-k+1$$
$$|t_j - t'_j| < 1, \text{ for } j = d-k+2, \ldots, d.$$

If $t_1 - t_1'$ is an integer, then there we are done. Assume $t_1 - t_1' \notin \mathbb{Z}$. Let $c := (t_1 - t_1', 0, \ldots, 0)$, and for $\tilde{t} \in \mathcal{T}$ let

$$s(\tilde{t}) := \begin{cases} \tilde{t} - c, & \text{if } \tilde{t}_1 - t_1 \in \mathbb{Z} \\ \tilde{t}, & \text{if } \tilde{t}_1 - t_1 \notin \mathbb{Z}. \end{cases}$$

In particular, $s(t) = t - c$ and $s(t') = t'$. We claim the set $\mathcal{S} := \{s(\tilde{t}) : \tilde{t} \in \mathcal{T}\}$ is a tiling set for $\mathcal{Q}$. Assuming, for a moment, that the claim is valid, we can easily complete the proof. In fact, $|s(t)_1 - s(t')_1| = 0$ and $|s(t)_j - s(t')_j| < 1$ for $j = d-k+2, \ldots, d$, so by the inductive hypothesis one of the numbers $t_j - t_j' = s(t)_j - s(t')_j$, $j = 2, \ldots, d-k+1$ is a non-zero integer.

It remains to prove that $\mathcal{S}$ is a tiling set for $\mathcal{Q}$. We must show that $\mathcal{Q}_{\mathcal{S}}$ is non-overlapping and that $\mathbb{R}^d \subset \mathcal{Q}_{\mathcal{S}}$. First we dispense with the non-overlapping part. Let a, a' be distinct points in $\mathcal{T}$. Suppose x is a point in the intersection $(\mathcal{Q} + s(a)) \cap (\mathcal{Q} + s(a'))$, then $x - s(a), x - s(a') \in \mathcal{Q}$, in particular, $0 \leq x_j - a_j < 1$ and $0 \leq x_j - a_j' < 1$ for $j = 2, \ldots, d$. It follows that $|a_j - a_j'| < 1$ for $j = 2, \ldots, d$, so first paragraph of the proof shows that $|a_1 - a_1'| \in \mathbb{N}$, hence either $a_1 - t_1, a_1' - t_1 \in \mathbb{Z}$ or $a_1 - t_1, a_1' - t_1 \notin \mathbb{Z}$. In both cases we get a contradiction to the non-overlapping property of $\mathcal{Q}_{\mathcal{T}}$. In fact, if $a_1 - t_1, a' - t_1 \in \mathbb{Z}$, then $(\mathcal{Q} + s(a)) \cap (\mathcal{Q} + s(a')) = ((\mathcal{Q} + a) \cap (\mathcal{Q} + a')) - c = \emptyset$. If $a_1 - t_1, a_1' - t_1 \notin \mathbb{Z}$, then $(\mathcal{Q} + s(a)) \cap (\mathcal{Q} + s(a')) = ((\mathcal{Q} + a) \cap (\mathcal{Q} + a')) = \emptyset$.

Let $x \in \mathbb{R}^d$ be an arbitrary point, then $x \in \mathcal{Q}_{\mathcal{T}}$. If $x \in (\mathcal{Q} + a)$ for some $a \in \mathcal{T}$ with $a_1 - t_1 \notin \mathbb{Z}$ then there is nothing to prove. Assume $x \in (\mathcal{Q} + a)$ for some $a \in \mathcal{T}$ with $a_1 - t_1 \in \mathbb{Z}$. The point $x + c$ is in $\mathcal{Q} + b$ for some $b \in \mathcal{T}$. First we show that $|a_1 - b_1| \in \mathbb{N}$. Since $x \in \mathcal{Q} + a$ and $x + c \in \mathcal{Q} + b$ we have

$$0 \leq x_j - a_j < 1, \quad 0 \leq x_j - b_j + c_j < 1, \tag{5.1}$$

for $j = 1, \ldots, d$; so using $c_j = 0$, for $j = 2, \ldots, d$, it follows that $|a_j - b_j| < 1$, for $j = 2, \ldots, d$; an application for the first paragraph of the proof yields the desired result that $|a_1 - b_1| \in \mathbb{N}$. Using $a_1 - t_1 \in \mathbb{Z}$ we conclude $b_1 - t_1 \in \mathbb{Z}$; so using the second half of (5.1) and the definition of $s(b)$ we have $x \in \mathcal{Q} + s(b)$ as needed. $\square$

Corollary 5.2. *If $\mathcal{T}$ is a tiling set for $\mathcal{Q}$, then $(\mathcal{Q}, \mathcal{T})$ is orthogonal.*

Proof. This is a direct consequence of Keller's Theorem and Lemma 3.1. $\square$

It is now easy to complete the proof that any tiling set for the unit cube $\mathcal{Q}$ must be a spectrum for $\mathcal{Q}$.

Proof of tiling implies basis. Suppose $\mathcal{T}$ is a tiling set for $\mathcal{Q}$. By Keller's Theorem $\{e_t : t \in \mathcal{T}\}$ is an orthogonal set of unit vectors in $L^2(\mathcal{Q})$, so by Bessel's inequality

$$\sum_{t\in\mathcal{T}} |\langle e_s, e_t\rangle|^2 \leq 1 \tag{5.2}$$

for any $s \in \mathbb{R}^d$. Note that $\langle e_s, e_t\rangle$ is the Fourier transform of the characteristic function of the cube $\mathcal{Q}$ at the point $s-t$. For any $r \in \mathbb{R}^d$ we have

$$1 = \int_{\mathbb{R}^d} |\langle e_y, e_0\rangle|^2 \, dy = \int_{\mathcal{Q}+r} \sum_{t\in\mathcal{T}} |\langle e_x, e_t\rangle|^2 \, dx \leq \int_{\mathcal{Q}+r} 1 \, dy = 1,$$

where we used Plancherel's Theorem, the tiling property, and Bessel's inequality (5.2). It follows that

$$\sum_{t\in\mathcal{T}} |\langle e_s, e_t\rangle|^2 = 1 \tag{5.3}$$

for almost every s in $\mathcal{Q}+r$, and since r is arbitrary, for almost every s in $\mathbb{R}^d$. Hence for almost every $s \in \mathbb{R}^d$ the exponential e_s is in the closed span of the e_t, $t \in \mathcal{T}$. This completes the proof. ☐

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Department of Mathematics, Wright State University, Dayton OH 45435, USA

E-mail address: `iosevich@zara.math.wright.edu`

E-mail address: `steen@math.wright.edu`