A RESTRICTION THEOREM FOR FLAT MANIFOLDS OF CODIMENSION TWO

Laura De Carli and Alex Iosevich

Introduction: Let M denote a submanifold of $\mathbb{R}^{n+2}$ of codimension 2. Let $\mathcal{R}$ denote a restriction operator

$$\mathcal{R}f(\eta) = \int e^{-i\langle x,\,\eta\rangle} f(x)dx, \quad \eta \in M, \quad f \in \mathcal{S}(\mathbb{R}^{n+2}). \tag{(1.1)r}$$

We wish to find an optimal range of exponents p such that

$$\|\mathcal{R}f_{L^2(M,\,d\sigma)}\| \leq C_p\|f\|_{L^p(\mathbb{R}^{n+2})}, \tag{(1.2)}$$

where $d\sigma$ is a compactly supported measure on M.

Let $\mathcal{F}[d\sigma]$ denote the Fourier transform of $d\sigma$. By a theorem of Greenleaf (see [G]), the inequality ((1.2)) holds for $p = \dfrac{2(2+\gamma)}{4+\gamma}$ if

$$|\mathcal{F}[d\sigma](R\zeta)\| \leq C(1+R)^{-\gamma}, \qquad \zeta \in S^{n+1}. \tag{(1.3)}$$

The purpose of this paper is to use Greenleaf's result to establish a restriction theorem for a class of degenerate submanifolds of $\mathbb{R}^{n+2}$ of codimension 2. We shall assume that our manifold is given as a joint graph of two homogeneous functions, where the first graphing function is homogeneous of degree 1 and the second graphing function is homogeneous of degree m. Under the appropriate curvature assumption we will show that ((1.3)) holds with $\gamma = \frac{n}{m}$.

An application of Greenleaf's result yields a restriction theorem with $p = \dfrac{2(2m+n)}{4m+n}$.

We shall need the following definitions.

Nonvanishing Gaussian curvature: Let Σ be a submanifold of $\mathbb{R}^{N+1}$ of codimension 1 equiped with a smooth compactly supported measure $d\mu$. Let $J : \Sigma \to S^N$ be the usual Gauss map taking each point on Σ to the outward unit normal at that point. We say that Σ has everywhere nonvanishing Gaussian curvature if the differential of the Gauss map dJ is always nonsingular.

Strong curvature condition: Let S be a submanifold of $\mathbb{R}^{N+2}$ of codimension 2 equiped with a smooth compactly supported measure $d\mu$. Suppose that S is a joint graph of smooth functions g_1 and g_2, where $g_j : \mathbb{R}^N \to \mathbb{R}$. Let $\mathcal{N}_{x_0}(S)$ denote the two dimensional space of normals to S at a point x_0. We say that S satisfies the strong curvature condition (SCC) if for all $x_0 \in S$ in some neighborhood of support$(d\mu)$,

$$\det D^2(\nu_1 g_1(x) + \nu_2 g_2(x)) \neq 0, \qquad \forall \nu \in \mathcal{N}_{x_0}$$

, where D^2 denotes the Hessian matrix.

One can check that the above definitions are independent of the parametrization. Our main result is the following:

Main Theorem. *Let $M = \{(x, x_{n+1}, x_{n+2}) \in \mathbb{R}^{n+2} : x_{n+1} = \phi_1(x),\ x_{n+2} = \phi_2(x)\}$, $n \geq 2$, where $\phi_i \in \mathcal{C}_0^\infty(\mathbb{R}^n \backslash \{0\})$, ϕ_1 is homogeneous of degree 1, and ϕ_2 is homogeneous of degree $m \geq 2$. Let $\Sigma_j = \{x : \phi_j(x) = 1\}$. Assume also that ϕ_2 only vanishes at the origin and that Σ_2 has everywhere nonvanishing Gaussian curvature. Let*

$$F(\xi, \lambda_1, \lambda_2) = \int_{\mathbb{R}^n} e^{i(\langle \xi, x\rangle + \lambda_1\phi_1(x) + \lambda_2\phi_2(x))}\chi(x)\, dx,$$

where $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

a) Suppose that the restriction of ϕ_1 to the set where $\phi_2 = 1$, $\phi_{1|_{\Sigma_2}}$, is constant. Then

$$|F(\xi, \lambda_1, \lambda_2)| \leq C(|\,\xi\,| + |\,\lambda\,|_1 + |\,\lambda\,|_2)^{-\frac{n}{m}} \tag{(1.4)}$$

when $m \geq 2n$.

b) Let $M_{|_{\{x_{n+2}=1\}}}$ denote the restriction of M to the hyperplane $\{x_{n+2} = 1\}$. If $M_{|_{\{x_{n+2}=1\}}}$ (wiewed as a submanifold of codimension 2 of $\{x_{n+2} = 1\}$) satisfies the strong curvature condition, then ((1.4)) holds for $m \geq 2$.

The conclusions of part (a) do not in general hold if $m < 2n$. Let $\phi_1(x) = |\,x\,|$, $\phi_2(x) = |\,x\,|^m$. Let $\xi = (0, 0, \cdots, 0)$. Then, in polar coordinates,

$$F(0, \lambda_1, \lambda_2) = C\int_0^\infty e^{i(\lambda_1 r + \lambda_2 r^m)} r^{n-1}\chi(r)\, dr.$$

It is not hard to see that the best isotropic decay for this integral cannot exceed $O\left(\left(\sqrt{\lambda_1^2 + \lambda_2^2}\right)^{-\frac{1}{2}}\right)$. Hence the restriction $m \geq 2n$ is necessary.

Remarks: (1) It is known that isotropic decay estimates for the Fourier transform of the surface-carried measure cannot be expected to yield an optimal restriction theorem (see e.g. [C]). We shall apply a homogeneity argument due to Knapp to the class of manifolds considered in the theorem above.

Let $\mathcal{R}$ denote the restriction operator defined above. Let $\hat{f}_\delta = h$, where h is the characteristic function of a rectangle in $\mathbb{R}^{n+2}$ with sides of lengths $(1, 1, \cdots 1, C, C)$, C large.

Then

$$\|f_\delta\|_p \approx \delta^{(1-1/p)(n+m+1)} \qquad \text{and} \qquad \|\mathcal{R} f_\delta\|_p \approx \delta^{n/2}.$$

Hence ((1.2)) can only hold if $p \leq \dfrac{2(n+m+1)}{n+2(m+1)}$.

If we apply Greenleaf's result ((1.3)) to the Main Theorem, we see that ((1.2)) holds for $p \leq \dfrac{2(2m+n)}{4m+n}$.

The gap between this exponent and the exponent given by Knapp's homogeneity argument suggests that the restriction theorem (1.2) may hold for a wider range of exponents. The result obtained using the Main Theorem is not sharp. In order to obtain a sharp result one would probably have to obtain precise non-isotropic estimates for the Fourier transform of the surface carried measure using the techniques of M. Christ (see [C]).

(2) The curvature conditions of the Main Theorem are not entirely satisfying because there is no natural transition between parts (a) and (b).

We hope to address these difficulties in a subsequent paper.

Proof of the main result:

Notation:

1. Given $a, b > 0$ we say that $a \approx b$ (a comparable to b) if there exist $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. We say that $a >> b$ (a much larger than b) if the inequality $a \leq Cb$ is not satisfied for any $C > 0$. The notion $a << b$ is defined similarly.
2. We denote by C a generic constant which may change from line to line.

Proof of part (a) of the Main Theorem. Let $\Psi(x) = \langle \xi, x\rangle + \lambda_1\phi_1(x) + \lambda_2\phi_2(x)$. Then $\bigtriangledown\Psi(x) = \xi + \lambda_1 \bigtriangledown \phi_1(x) + \lambda_2 \bigtriangledown \phi_2(x)$. Since ${\phi_1}_{|_{\Sigma_2}}$ is constant by assumption, then $\phi_1 \neq 0$ away from the origin. Hence, $\bigtriangledown\phi_1(x) \neq 0$ away from the origin by the Euler homogeneity relation, and since every component of $\bigtriangledown\phi_1(x)$ is homogeneous of degree zero, we have $|\bigtriangledown\phi_1(x)| \geq C$ for all $x \in \text{support}(\phi_1)$.

Suppose that $|\xi| << |\lambda_2| << |\lambda_1|$ or $|\lambda_2| << |\xi| << |\lambda_1|$. Then $|\bigtriangledown\Psi(x)| \geq C|\lambda_1|$ and so an integration by parts argument (see theorem (1) in the appendix) shows that $|F(\xi, \lambda_1, \lambda_2)| \leq C(1+|\lambda_1|)^{-N} \quad \forall N > 0$. Similarly, if $|\lambda_1| << |\lambda_2| << |\xi|$, or $|\lambda_1| \approx |\lambda_2| < < |\xi|$, then $|F(\xi, \lambda_1, \lambda_2)| \leq C(1+|\xi|)^{-N} \quad \forall N > 0$.

If we rewrite F using polar coordinates with respect to Σ_2 and assume that χ is radial with respect to $\Sigma_2 igmsa$, we get

$$F(\xi, \lambda_1, \lambda_2) = \int_0^{+\infty} r^{n-1}\chi(r) \int_{\Sigma_2} e^{i(r\langle\xi, \omega\rangle + r\lambda_1 + r^m\lambda_2)}\, d\sigma(\omega)\, dr,$$

where $d\sigma$ is the Lebesgue measure carried by Σ_2. Let $I(\xi)$ denote the Fourier transform of the surface-carried measure on Σ_2,

$$I(\xi) = \int_{\Sigma_2} e^{i\langle \xi, \omega\rangle}\, d\sigma(\omega).$$

Since the Gaussian curvature on Σ_2 never vanishes, we can use the method of stationary phase (see theorem (3) in the appendix) to write $I(\xi) = b(\xi)e^{iq(\xi)}$, where ξ belongs to a cone Γ containing the normal directions to Σ_2 on the support of $d\sigma$, and where $b(\xi)$ is a symbol of order $-\frac{n-1}{2}$, $q(\xi)$ is homogeneous of degree 1, and $q(\xi) \approx |\,\xi\,|$. Away from Γ, $I(\xi)$ decays rapidly in $|\,\xi\,|$.

Suppose that we are in one of the cases where $|\,\xi\,|$ dominates:

(1) $|\lambda_2| << |\lambda_1| \approx |\xi|$,
(2) $|\lambda_1| << |\lambda_2| \approx |\xi|$,
(3) $|\lambda_1| << |\lambda_2| << |\xi|$,
(4) $|\lambda_2| << |\lambda_1| << |\xi|$,
(5) $|\lambda_1| \approx |\lambda_2| \approx |\xi|$.

Using our observation about $I(\xi)$, we write

$$F(\xi, \lambda_1, \lambda_2) = \int_0^{+\infty} r^{n-1} e^{i(rq(\xi)+r\lambda_1+r^m\lambda_2)} b(r\xi)\chi(r)dr\ .$$

Then

$$|F(\xi, \lambda_1, \lambda_2)| \le C\int_0^2 r^{n-1}\,|\,b(r\xi)\,|\,dr.$$

Let $s = r\,|\,\xi\,|$, and define $\tilde{\xi} = \xi|\xi|^{-1}$. The integral above is bounded by

$$C|\xi|^{-n}\int_0^{2|\,\xi\,|} s^{n-1}|b(s\tilde{\xi})|ds =$$

$$= C|\xi|^{-n}\int_0^N s^{n-1}|b(s\tilde{\xi})|ds + C|\xi|^{-n}\int_N^{2|\xi|} s^{n-1}|b(s\tilde{\xi})|ds,$$

where N is large. The first integral is $O(|\xi|^{-n})$ and the second integral is bounded by

$$C\,|\,\xi\,|^{-n}\int_N^{2|\,\xi\,|} s^{\frac{n-1}{2}} ds \le C(1+|\,\xi\,|)^{-\frac{n-1}{2}}.$$

Note that $\dfrac{n-1}{2} \ge \dfrac{n}{m}$ when $m \ge \dfrac{2n}{n-1}$.

We are left to consider the cases where λ_2 dominates:

(1) $|\,\xi\,| \approx |\lambda_1| << |\lambda_2|$,
(2) $|\,\xi\,| << |\lambda_1| \approx |\lambda_2|$,
(3) $|\,\xi\,| << |\lambda_1| << |\lambda_2|$,
(4) $|\lambda_1| << |\,\xi\,| << |\lambda_2|$.

As before, let

$$F(\xi,\,\lambda_1,\,\lambda_2)=\int_0^{+\infty} r^{n-1}e^{i(rq(\xi)+r\lambda_1+r^m\lambda_2)}b(r\xi)\chi(r)dr.$$

Let $s\lambda_2^{-1/m}=r$. Then

$$F(\xi,\,\lambda_1,\,\lambda_2)=\lambda_2^{-\frac{n}{m}}\int_0^{+\infty} s^{n-1}e^{i(q(s\lambda_2^{-1/m}\xi)+s\lambda_2^{-1/m}\lambda_1+s^m)}b(s\lambda_2^{-1/m}\xi)\chi(s\lambda_2^{-1/m})ds.$$

Let

$$G(\xi,\,\lambda_1,\,\lambda_2)=\int_0^{+\infty} s^{n-1}e^{i(\lambda_2^{-1/m}sq(\xi)+s\lambda_2^{-1/m}\lambda_1+s^m)}b(s\lambda_2^{-1/m}\xi)\chi(s\lambda_2^{-1/m})ds.$$

It suffices to show that $|G(\xi,\,\lambda_1,\,\lambda_2)|$ is uniformly bounded. When $|\frac{\lambda_1+|\xi|}{\lambda_2{}^{\frac{1}{m}}}|$ is sufficiently small, then $|G|$ is bounded by $C|\int_0^{+\infty} e^{it^m}t^{n-1}dt|$. An integration by parts argument shows that this integral converges. In particular the above integral equals $e^{\frac{2\pi i}{m}}\frac{1}{m}\Gamma\left(\frac{n}{m}\right)$. Thus we may assume that $|\frac{\lambda_1+|\xi|}{\lambda_2{}^{\frac{1}{m}}}|\geq C$.

Let $\Phi(s)=s\frac{\lambda_1+q(\xi)}{\lambda_2{}^{\frac{1}{m}}}+s^m$. Then $\Phi'(s)=0$ if $s=C\left(\frac{\lambda_1+q(\xi)}{\lambda_2{}^{\frac{1}{m}}}\right)^{\frac{1}{m-1}}$, and $\Phi''(s)=m(m-1)s^{m-2}$.

If we apply the van der Corput Lemma (see theorem (2) in the appendix) in the case $k=2$, and recall that in particular $|b|$ is uniformly bounded, we see that $|G|$ is bounded by

$$C|\frac{\lambda_1+|\xi|}{\lambda_2{}^{\frac{1}{m}}}^{-\frac{(m-2)}{2(m-1)}+\frac{n-1}{m-1}}|.$$

The power of $|\frac{\lambda_1+|\xi|}{\lambda_2{}^{\frac{1}{m}}}|$ in the expression above is non-positive if $m\geq 2n$, and so $G(\xi,\,\lambda_1,\,\lambda_2)$ is uniformly bounded. This completes the proof of part (a) of the Main Theorem.

Proof of part (b) of the Main Theorem. As before, we rewrite F using polar coordinateds associated to Σ_2. We get

$$F(\xi,\,\lambda_1,\,\lambda_2)=\int_0^{+\infty}\int_{\Sigma_2} e^{i(r\langle\omega,\,\xi\rangle+r\lambda_1\phi_1(\omega)+\lambda_2 r^m)}r^{n-1}\chi(r)d\omega dr,$$

where, as before, χ is a smooth cutoff function which is radial with respect to the polar coordinates associated to Σ_2. Let

$$I(\xi,\,\lambda_1)=\int_{\Sigma_2} e^{i(\langle\omega,\,\xi\rangle+\lambda_1\phi_1(\omega))}d\omega.$$

Using the implicit function theorem we can parametrize Σ_2 near a point s_0 by a smooth function $\psi:\mathbb{R}^{n-1}\to\mathbb{R}$. Without loss of generality, we can assume that $\bigtriangledown\phi_1(s_0)=0$ and that $\bigtriangledown\phi_2(s_0)=(0,\,0,\,\cdots,\,0,\,1)$. Thus, we can locally write $\Sigma_2=\{(\omega',\,\omega_n)\,:\,\omega_n=\psi(\omega')\}$. The restriction of M to the hyperplane $\{x_{n+2}=1\}$ can thus be locally parametrized by the functions $\psi(\omega')$ and $\phi_1(\omega',\,\psi(\omega'))$. If we let $\xi=(\xi',\,\xi_n)$, we can write $I(\xi,\,\lambda_1)$ as a finite sum of terms of the form

$$\int_{\mathbb{R}^{n-1}} e^{i(\langle\omega',\,\xi'\rangle+\xi_n\psi(\omega')+\lambda_1\phi_1(\omega',\,\psi(\omega')))}\chi_1(\omega')d\omega', \tag{(1.5)}$$

where χ_1 is a smooth cutoff function supported in a neighborhood of s_0. It was observed by M. Christ (see [C]) that the strong curvature condition (see the introduction) implies the following result.

Lemma. *Let Ω be a submanifold of $\mathbb{R}^{N+2}$ of codimension 2 locally parametrized by smooth functions g_1 and g_2, where $g_j:\mathbb{R}^N\to\mathbb{R}$. Let $d\sigma$ denote a smooth measure on Ω. Suppose that Ω satisfies the strong curvature condition. Then*

$$|\,\mathcal{F}[d\sigma](R\eta)\,|\le C(1+R)^{-\frac{N}{2}}.$$

The proof of the lemma shows that the integral in ((1.5)) can be written as $b(\xi,\,\lambda_1)e^{iq(\xi,\,\lambda_1)}$, where (ξ,λ_1) belongs to a cone containing the normal directions to $M_{|\{x_{n+2}=1\}}$ on the support of $d\sigma$, $b(\xi,\,\lambda_1)$ is a symbol of order $-\frac{n-1}{2}$, $q(\xi,\,\lambda_1)$ is homogeneous of degree 1, and $|\,q(\xi,\,\lambda_1)\,|\approx(|\,\xi\,|+|\,\lambda_1\,|)$.

We must analyze the following integral:

$$\int_0^{+\infty} r^{n-1}e^{i(rq(\xi,\,\lambda_1)+r^m\lambda_2)}b(r\xi,\,r\lambda_1)\chi(r)dr. \tag{(1. 6)}$$

We may assume that $|\,q(\xi,\,\lambda_1)\,|\le C\,|\,\lambda_2\,|$, since if $|\,q(\xi,\,\lambda_1)\,|\ge c\,|\,\lambda_2\,|$ for a sufficiently large $c>0$, then the integral in ((1. 6)) decays rapidly in $|\,\xi\,|+|\,\lambda_1\,|$. (See theorem (1) in the appendix.)

Let $s=r\lambda_2^{\frac{1}{m}}$. Then, the integral in (1. 6) can be written as

$$\lambda_2^{-\frac{n}{m}}\int_0^{+\infty} s^{n-1}e^{i(s\lambda_2^{\frac{1}{m}}q(\xi,\,\lambda_1)+s^m)}b(s\lambda_2^{\frac{1}{m}}\xi,\,s\lambda_2^{\frac{1}{m}}\lambda_1)\chi(s\lambda_2^{\frac{1}{m}})dr.$$

Let

$$G(\xi,\,\lambda_1,\,\lambda_2)=\int_0^{+\infty} s^{n-1}e^{i(s\lambda_2^{\frac{1}{m}}q(\xi,\,\lambda_1)+s^m)}b(s\lambda_2^{\frac{1}{m}}\xi,\,s\lambda_2^{\frac{1}{m}}\lambda_1)\chi(s\lambda_2^{\frac{1}{m}})dr.$$

As before, it suffices to show that $|\,G(\xi,\,\lambda_1,\,\lambda_2)\,|$ is uniformly bounded . When $|\frac{|\,\lambda_1\,|+|\,\xi\,|}{\lambda_2{}^{\frac{1}{m}}}|$ is sufficiently small, then $|\,G\,|$ is bounded by $C\left|\int_0^{+\infty} e^{it^m}t^{n-1}dt\right|$. Hence we can assume $|\frac{|\,\lambda_1\,|+|\,\xi\,|}{\lambda_2{}^{\frac{1}{m}}}|\geq C$. We can write $G(\xi,\,\lambda_1,\,\lambda_2)=\int_0^N+\int_N^{C|\,\lambda_2\,|^{\frac{1}{m}}}$, N large. The first integral is uniformly bounded. In order to handle the second integral let $\Phi(s)=s\lambda_2^{\frac{1}{m}}q(\xi,\,\lambda_1)+s^m$. Then $\Phi'(s)=0$ if $s=c_m\left(\lambda_2{}^{-\frac{1}{m}}q(\xi,\,\lambda_1)\right)^{\frac{1}{m-1}}$, and $\Phi''(s)=m(m-1)s^{m-2}$. If the critical point is smaller than N the integral has rapid decay, so we may assume that $\left|\,\lambda_2{}^{-\frac{1}{m}}q(\xi,\,\lambda_1)\,\right|$ is large. If we recall that $|\,q(\xi,\,\lambda_1)\,|\approx|\,\xi\,|+|\,\lambda_1\,|$, then by the van der Corput lemma (see theorem (2) in the appendix) we get

$$\text{((1. 7))}\qquad \int_N^{C|\,\lambda_2\,|^{\frac{1}{m}}}\leq\left|\frac{|\,\lambda_1\,|+|\,\xi\,|}{\lambda_2{}^{\frac{1}{m}}}\right|^{-\frac{(m-2)}{2(m-1)}+\frac{n-1}{m-1}-\frac{n-1}{2(m-1)}-\frac{n-1}{2}}.$$

Note that the third and the fourth terms in the power of $\left|\frac{|\,\lambda_1\,|+|\,\xi\,|}{\lambda_2{}^{\frac{1}{m}}}\right|$ arise from the fact that b is a symbol of order $-\frac{n-1}{2}$, and $|\frac{|\,\lambda_1\,|+|\,\xi\,|}{\lambda_2{}^{\frac{1}{m}}}|$ is large.

The power of $|\frac{|\,\lambda_1\,|+|\,\xi\,|}{\lambda_2{}^{\frac{1}{m}}}|$ in ((1. 7)) is nonnegative provided that $m\geq 2$. Hence, $|G(\xi,\,\lambda_1,\,\lambda_2)|$ is bounded and the proof is complete.

Appendix

In this section we recall a few classical results that we used to prove the Main Theorem. The first two theorems, which deal with oscillatory integrals, can be found e.g. in [St].

Theorem 1. *Suppose $\phi\in\mathcal{C}_0^\infty\mathbb{R}^n$ and suppose that ψ is a real-valued and smooth function which has no critical points on the support of ϕ. Then*

$$\left|\int_{\mathbb{R}^n} e^{i\lambda\psi(x)}\phi(x)dx\right|=O(\lambda^{-N})$$

as $\lambda\to\infty$, for every $N\geq 0$.

Theorem 2. *Suppose that ψ is real-valued and smooth and that ϕ is complex-valued and smooth in $[a, b]$. If $\left| \psi^{(k)}(x) \right| \geq 1$, then*

$$\left| \int_a^b e^{i\lambda\psi(x)}\phi(x)dx \right| \leq C_k \lambda^{-\frac{1}{k}} \left[|\, \phi(b) \,| + \int_a^b |\, \phi'(t) \,|\, dt \right]$$

holds when

(1) $k \geq 2$

(2) *or $k = 1$, if in addition it is assumed that $\psi'(x)$ is monotonic.*

Theorem 3. *Let S be a smooth hypersurface in $\mathbb{R}^n$ with nonvanishing Gaussian curvature, and let $d\sigma$ be a $\mathcal{C}^\infty$ measure on S. Then*

$$\left| \widehat{d\mu}(\xi) \right| \leq C(1 + |\, \xi \,|)^{-\frac{n-1}{2}}.$$

Moreover suppose that $\Gamma \subset \mathbb{R}^n \backslash \{0\}$ is the cone consisting of all ξ which are normal to some point $x \in S$ belonging to some compact neighborhood $\mathcal{N}$ of support$(d\mu)$. Then,

$$\frac{\partial^\alpha}{\partial \xi} \widehat{d\mu}(\xi) = O\left((1 + |\, \xi \,|)^{-N} \right), \qquad \forall N, \ \textit{if } \xi \notin \Gamma,$$

$$\widehat{d\mu}(\xi) = \sum = a_j(\xi) e^{i\langle x_j, \xi \rangle}, \qquad \textit{if } \xi \in \Gamma,$$

where the finite sum is taken over all points $x_j \in \mathcal{N}$ having ξ as a normal and

$$|\frac{\partial^{(\alpha)}}{\partial \xi} \widehat{d\mu}(\xi)| \leq C_\alpha (1 + |\, \xi \,|)^{-\frac{n-1}{2} - |\, \alpha \,|}.$$

Proof. See [So] pag. 50-51.

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