MATH 238, FALL 2023, HOMEWORK #2

ALEX IOSEVICH

Problem #1: Show that a connected planar graph satisfies

$$n - e + f = 2,$$

where n is the number of vertices, e is the number of edges, and f is the number of faces.

Problem #2: Show that in a planar connected graph,

$$3f \leq 2e.$$

Problem #3: Show that the lower bound on the crossing number of a connected graph with $e \geq 4n$ edges,

$$cr(G) \geq c\frac{e^3}{n^2}$$

cannot, in general, be improved.

Problem #4: Prove that the upper bound on the number of incidences between n points and m lines,

$$I(P, L) \leq 4(n + m + (nm)^{\frac{2}{3}})$$

cannot, in general, be improved (up to constants).

Problem #5: Prove that the bound in the previous problem still holds, possibly with a bigger constant, if lines are replaced by translates of a fixed parabola, or translates of a fixed circle.

Problem #6: Prove that $\mathbb{Z}_p$, p prime, is a field.

Problem #7: Prove that if $x \in \mathbb{Z}_p^d$, p prime, $||x|| = x_1^2 + \cdots + x_d^2$, and A is an orthogonal d by d matrix over $\mathbb{Z}_p$, then

$$||Ax|| = ||x||.$$

Problem #8: Prove that if $p \equiv 1 \mod 4$, then there exists $i \in \mathbb{Z}_p$, p prime, such that $i^2 = -1$.

Problem #9: i) Construct a subset $\mathbb{Z}_p^d$, p prime, such that

$$\mathcal{E}(E) \equiv |\{(x, y, x', y') \in E^4 : x + y + x' + y'\}| \leq C|E|^2,$$

where C is independent of p.

ii) Construct a subset $\mathbb{Z}_p^d$, p prime, such that

$$\mathcal{E}(E) \geq c|E|^3$$

with c independent of p.

iii) Given any $a \in [2,3]$, construct a subset $\mathbb{Z}_p^d$, p prime, such that

$$\frac{1}{C}|E|^a \leq \mathcal{E}(E) \leq C|E|^a$$

with C, c independent of p.

Problem #10: Show that there exists $E \subset \mathbb{Z}_p^d$, p prime, such that

$$|\widehat{E}(m)| \leq Cp^{-d}|E|^{\frac{1}{2}}$$

with C independent of p DOES NOT hold, yet

$$\mathcal{E}(E) \leq C|E|^2$$

with C independent of p.

Problem #11: Prove that if H is a k-dimensional subspace of $\mathbb{Z}_p^d$, p prime, then

$$\widehat{H}(m) = q^{-(d-k)}H^{\perp}(m),$$

where $H^{\perp}$ is the perpendicular subspace.

Problem #12: i) Prove that if $E \subset \mathbb{Z}_p^d$, $d \geq 2$, and

$$P = \{x \in \mathbb{Z}_p^d : x_d = x_1^2 + \cdots + x_{d-1}^2\},$$

then

$$|\{(x,y) \in E \times E : x - y \in P\}| = |E|^2 p^{-1} + R,$$

where

$$|R| \leq p^{\frac{d-1}{2}}|E|.$$

Conclude that if $|E| > p^{\frac{d+1}{2}}$, then $(E - E) \cap P$ is not empty.

ii) Replace the paraboloid P by a $(d-1)$-dimensional hyper-plane H. Is it still true that if $|E| > p^{\frac{d+1}{2}}$, then $(E - E) \cap H$ is not empty? Prove or disprove.