THREE PROBLEMS MOTIVATED BY THE AVERAGE DECAY OF THE FOURIER TRANSFORM

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Abstract. Let $E_d = \{x = r\omega \in \mathbb{R}^d : r \in E\}$, where E is a compact one-dimensional set of Hasudorff dimension α with some appropriate metric uniformity assumptions. Let $A_t f(x) = \int f(x - ty) d\mu_d(y)$, and let $\mathcal{A} f(x) = \sup_{t>0} |A_t f(x)|$, where μ is a measure on E and μ_d is its "rotated" analog. We prove that $\mathcal{A} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ for $p > \frac{d-\alpha}{d-1}$, the range that "interpolates" between the Hardy-Littlewood maximal function and Stein's spherical maximal operator. The result holds even though the Fourier transform $\widehat{\mu}$ may have no decay at infinity, and, consequently, the decay of $\widehat{\mu_d}$ is no better than that of the Fourier transform of the Lebesgue measure on the $d-1$-dimensional sphere. In the second part of the paper, we prove that $\mathcal{M}_d f(x) = \sup_{t \in [1,2]} \int_Q f(x' - ty', x_d - t^2 y_d) dy$, where Q is an appropriately rotated unit cube, is bounded on $L^p(\mathbb{R}^d)$ with constants independent of d for $p > \frac{4}{3}$. This is in contrast to the range $p > \frac{3}{2}$ obtained by Bourgain and Carbery for the usual isotropic dilation. In the final section of this paper we show that there exist compact hyper-surfaces in $\mathbb{R}^d$ with everywhere positive and bounded Gaussian curvature such that $\int_S e^{-2\pi i x \cdot \xi} d\sigma(x) = O(|\xi|^{-1})$ and no better.

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Section 0: Maximal averages over fractals

Let S be a smooth hyper-surface in $\mathbb{R}^d$. Let

$$A_t f(x) = \int_S f(x-ty)d\sigma(y), \tag{0.1}$$

where $d\sigma$ denotes the Lebesgue measure on S, and let

$$\mathcal{A}f(x) = \sup_{t>0} |A_t f(x)|. \tag{0.2}$$

The basic unsolved problem is, given a hyper-surface S, what is the sharp range of exponents so that

$$||\mathcal{A}f||_{L^p(\mathbb{R}^d)} \leq C_p ||f||_{L^p(\mathbb{R}^d)}. \tag{0.3}$$

If S has everywhere non-vanishing Gaussian curvature then the estimate (0.3) holds if and only if $p > \frac{d}{d-1}$. See [St], [Gr]. If the Gaussian curvature is allowed to vanish, numerous results are available, though the general picture is far from complete. See, for example, [CoMa], [SoSt], [NSW], [I1], [I2], [IS1], [IS2], [ISS].

More generally we can define

$$A_t f(x) = \int_E f(x-ty)d\nu(y), \tag{0.4}$$

where E is a subset of $\mathbb{R}^d$ of Hausdorff dimension α, not necessarily an integer, and ν is a measure carried by E. Suppose that $0 < \nu(E) < \infty$ and re-scale ν so that $\nu(E) = 1$. Let $\mathcal{A}f(x)$ be the corresponding maximal operator as in (0.2) above. As before, we ask for the sharp range of exponents such that the estimate (0.3) holds.

Definition. ([Str]) Let (E,μ) be as in the first paragraph. We say that μ is *locally uniformly α-dimensional* if $\mu(B_R(x)) \leq CR^\alpha$, where $B_R(x)$ is a ball of radius $R \leq 1$ centered at $x \in E$ and C is independent of x and R.

Definition. Let μ denote a Hausdorff measure on $E \subset [1,2]$. Let σ_r denote the rotationally invariant probability measure on the sphere of radius r. Let $\mu_d = \int_0^1 \sigma_r d\mu(r)$ denote the corresponding rotationally invariant measure on the set $E_d = \{x \in \mathbb{R}^d : |x| \in E\}$.

Theorem 0.1. *Suppose that μ is locally uniformly α-dimensional. Let*

$$A_t f(x) = \int f(x-ty)d\mu_d(y), \tag{0.5}$$

and let $\mathcal{A}f(x)$ be defined as in (0.2)*.*

Then

$$||\mathcal{A}f||_{L^p(\mathbb{R}^2)} \leq C_p||f||_{L^p(\mathbb{R}^2)} \quad for \quad p > \frac{d-\alpha}{d-1}. \tag{0.6}$$

When $\alpha = 1$, this is precisely the range of boundedness of the Hardy-Littlewood maximal operator. When $\alpha = 0$, this is the range of boundedness of Stein's spherical maximal operator.

Taking

$$F(x) = |x|^{-d+1+\alpha} \log^{-1}\left(\frac{1}{|x|}\right) \tag{0.7}$$

times the characteristic function of the ball of radius $\frac{1}{2}$ shows that $\mathcal{A}$ is unbounded on $L^p(\mathbb{R}^d)$ if $p \leq \frac{d}{d-1+\alpha}$. The two estimates agree if $\alpha = 0$ or 1, but the $p > \frac{d-\alpha}{d-1}$ result is worse for other values of α. A gap remains. The possible nature of this gap can be illustrated as follows. Consider the maximal operator associated to the analytic family of measures given by

$$\mu_\alpha = \frac{1}{\Gamma(\alpha)}(1-|x|^2)_+^{\alpha-1}. \tag{0.8}$$

If $\alpha = 0$, we get the spherical maximal operator. It is bounded on $L^p(\mathbb{R}^d)$ for $p > \frac{d}{d-1}$ as we mentioned above. If $\alpha = 1$, we get the Hardy-Littlewood maximal operator. It is bounded on $L^p(\mathbb{R}^d)$ for $p > 1$. Interpolating we see that for $0 < Re(\alpha) < 1$, the maximal averaging operator is bounded on $L^p(\mathbb{R}^d)$ for $p > \frac{d}{d-1+\alpha}$.

It is not hard to check that μ_α is α-dimensional, and by [SW],

$$|\widehat{\mu}_\alpha(\xi)| \leq C(1+|\xi|)^{-\alpha}. \tag{0.9}$$

On the other hand, let E_m denote the Cantor-like subset of $[0,1]$ consisting of real numbers whose base m, $m > 2$, expansions have only 0's and 1's. Let μ_m denote the probability measure on E_m. One can check (see e.g. [Z]) that

$$\widehat{\mu_m}(\xi) = \Pi_{j=0}^{\infty} \cos\left(\frac{\pi\xi}{2 \cdot m^j}\right). \tag{0.10}$$

Taking the sequence $\xi_n = m^n$ of powers of m, we see that $\widehat{\mu}_m(\xi_n)$ does not tend to 0 as $n \to \infty$. However, we are saved by the fact that the proof of L^2 bounded-ness of maximal averaging operators, see Lemma 1.3 below, does not use the decay rate of $\widehat{\mu}$, but rather the "average"

$$\left(\int_1^2 |\widehat{\mu}(t\xi)|^2 dt\right)^{\frac{1}{2}}. \tag{0.11}$$

In the case $\mu = \mu_m$, see Lemma 1.1 below, one can show that (0.12) is bounded by

$$C|\xi|^{-\frac{\alpha_m}{2}}, \tag{0.12}$$

where

$$\alpha_m = \frac{\log(2)}{\log(m)}, \tag{0.13}$$

the Hausdorff dimension of E_m.

For more general Hausdorff measures, see Lemma 1.1 below, the same result holds with α_m replaced by the Hausdorff dimension of the corresponding set.

It is well known (see, for example, [Wolff], that the power of ξ in (0.12) cannot be improved beyond the index in (0.13) since the dimension of E_m is precisely α_m. This means that in order to improve the $\frac{d-\alpha}{d-1}$ index in Theorem 0.1, we must use something more than just the decay of (0.11).

Note that the phenomenon we are exploiting to obtain Theorem 0.1 is purely fractal in nature. We are using the fact that even though $\widehat{\mu_d}$ decays only of order $-\frac{d-1}{2}$ at infinity, the square function

$$\left(\int_1^2 \left|\widehat{\mu_d}(t\xi)\right|^2 dt\right)^{\frac{1}{2}} \tag{0.14}$$

decays of order $-\frac{d-1+\alpha}{2}$, where α is the Hausdorff dimension of μ. This phenomenon cannot take place if we consider the standard maximal averaging operator

$$\sup_{t>0} \left| \int_S f(x-ty)d\sigma(y) \right|, \tag{0.15}$$

where S is a smooth compact hyper-surface in $\mathbb{R}^d$ and $d\sigma$ is the restriction of Lebesgue measure to S. By the method of stationary phase, $\widehat{d\sigma}$ decays slowly in the directions normal to the points in S where the Gaussian curvature vanishes. This means, in particular, that multiplying by t and averaging as in (0.14) cannot possibly improve the decay. One can improve the decay by using a non-isotropic dilation, for example, $\rho_t f(x) = f(tx_1, \dots, tx_{d-1}, t^2 x_d)$. See [IS2] and Section III below. The idea here is that non-isotropic dilations "rotate" the surface in such a way that "bad" normals are encountered very infrequently. In contrast, the improved estimate for the square function in (0.14) is purely metric in nature.

Section I: Proof of Theorem 0.1

The proof of Theorem 0.1 is based on the following sequence of lemmas.

Lemma 1.1. *Let (E, μ) be as above. Then*

$$\left(\int_1^2 |\widehat{\mu}(t\xi)|^2 dt\right)^{\frac{1}{2}} \le C(1+|\xi|)^{-\frac{\alpha}{2}}, \tag{1.1}$$

and, consequently,

$$\left(\int_1^2 |\widehat{\mu_d}(t\xi)|^2 dt\right)^{\frac{1}{2}} \le C(1+|\xi|)^{-\frac{d+1+\alpha}{2}}. \tag{1.2}$$

Moreover, the same estimates hold if $\widehat{\mu}(t\xi)$ is replaced by $\nabla\widehat{\mu}(t\xi)$.

A more general version of (1.1) follows from the main result in [Str].

Lemma 1.2. *Let μ_d^k be defined by the formula $\widehat{\mu_d^k}(\xi) = \widehat{\mu_d}(\xi)\phi_k(\xi)$, where ϕ_k is the usual Littlewood-Paley cutoff, and μ_d is as in the statement of the Theorem 0.1. Let*

$$A_t^k f(x) = \int f(x-ty)d\mu_d^k(y), \tag{1.3}$$

and

$$\mathcal{A}^k f(x) = \sup_{t>0} A_t^k f(x). \tag{1.4}$$

Then

$$||\mathcal{A}^k f||_{L^p(\mathbb{R}^2)} \le C2^{k(1-\alpha)}||f||_{L^p(\mathbb{R}^2)} \quad \textit{for} \quad p>1. \tag{1.5}$$

The following result was proved in [IS], Theorem 8. See also, [SoSt] and [CoMa].

Lemma 1.3. *Suppose that τ is a distribution supported in a ball B of radius C_1 with $|\hat{\tau}(\xi)| \le C_1$ and $\max\{|x| : x \in supp\ \tau\} \le C_2$. Suppose, moreover, that*

$$\left\{\int_1^2 |\hat{\tau}(t\xi)|^2 dt\right\}^{\frac{1}{2}} \le C_1(1+|\xi|)^{-\frac{1}{2}}\gamma(\xi), \tag{1.6}$$

and

$$\left\{\int_1^2 |\nabla\hat{\tau}(t\xi)|^2 dt\right\}^{\frac{1}{2}} \le C_1(1+|\xi|)^{-\frac{1}{2}}\gamma(\xi), \tag{1.7}$$

where γ is bounded and non-increasing on $[0,\infty)$, and $\sum_{n=0}^{\infty}\gamma(2^n) < \infty$. Let $\hat{\tau}_t(\xi) = \hat{\tau}(t\xi)$. Let

$$\mathcal{M}_\tau f(x) = \sup_{t>0} |f * \tau_t(x)|. \tag{1.8}$$

Then

$$||\mathcal{M}_\tau f||_{L^2(\mathbb{R}^d)} \le C\sqrt{C_1C_2}||f||_{L^2(\mathbb{R}^d)} \tag{1.9}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$, the usual Schwartz class.

Moreover, the same conclusions hold if the dilation $t\xi$ is replaced by a non-isotropic dilation $(t\xi', t^\beta\xi_d)$, $\beta > 0$.

Lemma 1.4. *With the notation of Lemma 0.2, we have*

$$||\mathcal{A}^k f||_{L^2(\mathbb{R}^2)} \leq C 2^{-k\left(\frac{d-2+\alpha}{2}\right)} ||f||_{L^2(\mathbb{R}^2)}. \tag{1.10}$$

Synthesis. Lemma 1.1 and Lemma 1.3 combine to imply Lemma 1.4. Interpolating Lemma 1.2 and Lemma 1.4 and summing up the geometric series we obtain the conclusion of Theorem 0.1 unless $d = 2$ and $\alpha = 0$. This is the only case when the exponent in (1.10) is not negative and our interpolation scheme does not work. This makes sense because a special case of the situation where $d = 2$, $\alpha = 0$, is Bourgain's circular maximal operator. The proof that this operator is bounded for $p > 2$ certainly cannot be carried out using the methods of this paper.

In order to handle the case $d = 2$, $\alpha = 0$, we dominate our operator by Bourgain's Circular Maximal Operator in the following way. Recall that

$$\mu_2 = \int_0^1 \sigma_r d\mu(r), \tag{1.11}$$

where σ_r is the rotation invariant measure probability measure on the circle of radius r.

Let $\tau_t g(x) = g(tx)$. Consider the "linearized" maximal operator

$$\mathcal{M}f(x) = A_{t(x)} f(x), \tag{1.12}$$

where $t(x)$ is an arbitrary measurable function of x. It is enough to prove that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^2)$ for $p > 2$. We have

$$|\mathcal{M}f(x)| = |\tau_{t(x)}\mu_d * f(x)| \leq \int_0^1 |d\sigma_{rt(x)} * f(x)| d\mu(r) \leq ||\mu_d|| |\mathcal{T}f(x)|, \tag{1.13}$$

where $\mathcal{T}f(x)$ is Bourgain's Circular Maximal Operator, and $||\mu_d||$ denotes the total mass of μ_d. This completes the proof of Theorem 0.1.

Section II: Proof of Lemma 1.1 and Lemma 1.2

Proof of Lemma 1.1. By Fubini and the definition of the Fourier transform, we see that if $|\xi|$ is large,

$$\int_1^2 |\widehat{\mu}(t\xi)|^2 dt = \int\int\int_1^2 e^{2\pi i(x-y)t\xi} dt d\mu(x) d\mu(y) \tag{2.1}$$

$$= \int\int e^{3\pi i(x-y)\xi} \frac{\sin(\pi(x-y))}{\pi(x-y)\xi} d\mu(x) d\mu(y)$$

$$\leq \int\int \frac{d\mu(x)d\mu(y)}{1+|x-y||\xi|}$$

$$\leq \sum_{j=0}^{\log_2(|\xi|)} \int\int_{\{2^j \leq |x-y||\xi| \leq 2^{j+1}\}} \frac{d\mu(x)d\mu(y)}{1+|x-y||\xi|} + \sum_{j=\log_2(|\xi|)}^{\infty} \int\int_{\{2^j \leq |x-y||\xi| \leq 2^{j+1}\}} \frac{d\mu(x)d\mu(y)}{1+|x-y||\xi|}$$

$$+\int\int_{\{|x-y||\xi|\leq 1\}} d\mu(x)d\mu(y) = I + II + III.$$

It is not hard to see that I is bounded by

$$C_\alpha \sum_{j=1}^{\infty} 2^{-j} \left(\frac{2^j}{|\xi|}\right)^{\alpha} \leq C|\xi|^{-\alpha}, \tag{2.2}$$

since μ is locally uniformly α dimensional. On the other hand, II is bounded by

$$C \sum_{\log_2(|\xi|)}^{\infty} 2^{-j} \leq C|\xi|^{-1}. \tag{2.3}$$

The expression III is bounded by

$$\int_{\{|x-y|\leq \frac{1}{|\xi|}\}} d\mu(x)d\mu(y) \leq C|\xi|^{-\alpha} \tag{2.4}$$

since μ is, by assumption, locally uniformly α-dimensional.

This completes the proof of the first assertion of Lemma 1.1.

To prove the second assertion of Lemma 1.1 we write, with $|\xi| = r$,

$$\widehat{\mu_d}(t\xi) = c\int_E \left(\int_{S^{d-1}} e^{-is\omega\cdot t\xi} d\omega\right) d\mu(s) \tag{2.5}$$

$$= C\int_E J_{\frac{d-2}{2}}(trs)(trs)^{-\frac{d-2}{2}} d\mu(s)$$

$$= \left(C_1 \int_E e^{itrs}(trs)^{-\frac{d-1}{2}} d\mu(s) + C_2 \int_E e^{-itrs}(trs)^{-\frac{d-1}{2}} d\mu(s)\right)$$

$$+O((tr)^{-\frac{d+1}{2}}),$$

using the well known asymptotics of the Bessel function $J_{\frac{d-2}{2}}$ together with our assumption that μ is a measure on the set $E \subset [1,2]$.

At this point, the second assertion of Lemma 1.1 follows from the first assertion applied to the measure $s^{-\frac{d-1}{2}} d\mu(s)$.

Proof of Lemma 1.2. We have

$$\widehat{\phi_k} * \mu_d(x) \le 2^{dk} \int \frac{d\mu_d(y)}{(1+2^k|x-y|)^N} \tag{2.6}$$

$$\le \left(\sum_{-\infty}^{-1} + \sum_{0}^{k} + \sum_{k}^{\infty}\right) \int_{\{2^j \le 2^k|x-y| \le 2^{j+1}\}} 2^{dk} \frac{d\mu_d(y)}{(1+2^k|x-y|)^N} = I + II + III.$$

Using the fact that μ is locally uniformly α dimensional, we see that

$$I = \int_{\{2^k|x-y|\le 1\}} 2^{dk} \frac{d\mu_d(y)}{(1+2^k|x-y|)^N} \le C2^{dk}2^{-k(d-1+\alpha)}\chi_B(x) \le C2^{k(1-\alpha)}\chi_B(x), \tag{2.7}$$

where B is the ball of radius 5 centered at the origin.

Similarly,

$$II \le C \sum_{j=0}^{k} 2^{dk} 2^{(j-k)(d-1+\alpha)} 2^{-jN} \chi_B(x) \le C2^{k(1-\alpha)}\chi_B(x). \tag{2.8}$$

On the other hand,

(2.9)

$$III \le C \sum_{j=k}^{\infty} 2^{dk} 2^{-jN} |B| 2^{(j-k)d} |2^{j-k}B| \frac{\chi_{2^{j-k}B}(x)}{|2^{j-k}B|} = \sum_{j=k}^{\infty} 2^{j(d-N)} |2^{j-k}B||B| \frac{\chi_{2^{j-k}B}(x)}{|2^{j-k}B|}.$$

It follows that $\mathcal{A}^k$ is dominated by $C2^{(1-\alpha)k}$ times the Hardy-Littlewood maximal function. Lemma 1.2 follows.

Section III: Maximal functions over cubes

Let $Q_d(\omega)$ denote the unit cube, rotated so that the normal to one of the faces is $\omega \in S^{d-1}$ (this is of course not unique). Let

$$\mathcal{M}_{d,\omega} f(x) = \sup_{t\in[1,2]} \left| \int_{Q_d(\omega)} f(x'-ty', x_d - t^2 y_d) dy \right|. \tag{3.1}$$

If t^2 is replaced by t, it follows from [Bourg] and [Carb] that $\mathcal{M}^{\alpha}_{d,\omega}$ is bounded on $L^p(\mathbb{R}^d)$ with constants independent of d if $p > \frac{3}{2}$. In fact, this result is established for all convex bodies. In this section we show that the exponent $p > \frac{3}{2}$ can be improved, at least for the cube, if the isotropic dilations are replaced by non-isotropic dilations. The main result of this section is the following.

Theorem 3.1. *Suppose that ω is not one of the coordinate directions, then*

$$||\mathcal{M}_{d,\omega}||_{L^p(\mathbb{R}^d)} \le C_\omega ||f||_{L^p(\mathbb{R}^d)} \tag{3.2}$$

for $p > 1$, and, moreover, C_ω is independent of d if $p > \frac{4}{3}$.

Proof of Theorem 3.1. The fact that this operator is bounded on $L^p(\mathbb{R}^d)$ is classical. The whole point is to establish constants independent of d.

Using Lemma 1.2 and Lemma 1.3, it suffices to prove that if ω is not one of the coordinate directions, then

$$\int_1^2 \left| \int_{Q_d(\omega)} e^{-2\pi i x \cdot T\xi} dx \right|^2 dt \le C_\omega |\xi|^{-3}, \tag{3.3}$$

where C_ω does not depend on d, and $T\xi = (t\xi', t^2\xi_d)$.

Writing the integral in local coordinates and integrating through the final variable, the calculation reduces to showing that

$$\int_1^2 \left| \int e^{-2\pi i (tu\cdot\xi' + t^2\xi_d(c_1u_1 + \cdots + c_{d-1}u_{d-1}))} \psi(u) du \right|^2 dt \le C|\xi|^{-1}, \tag{3.4}$$

with an appropriate constant C, where c_i's depend on ω in the obvious way, and ψ is the characteristic function of the projection of the cube onto the subspace $\{x_d = 0\}$. The left hand side is bounded by

$$\int_1^2 \left| \widehat{\psi}(t\xi_1 + c_2t^2\xi_d, \ldots, t\xi_{d-1} + c_{d-1}t^2\xi_d) \right|^2 dt \tag{3.5}$$

$$\le C' \int_1^2 \left| \widehat{\psi}(\xi_1 + c_2t\xi_d, \ldots, \xi_{d-1} + c_{d-1}t\xi_d) \right|^2 dt. \tag{3.6}$$

Let $j \in \{1, \ldots, d-1\}$, and suppose that $|\xi_j| >> |\xi_k|$, $k \ne j$. Then the expression in (3.6) is bounded by $|\xi_j|^{-2}$ with constants that do not depend on anything interesting. If $|\xi_j| \le C|\xi_d|$ for all $j \in \{1, \ldots, d-1\}$, then we make a change of variables $s = t\xi_d$. We see that (3.6) is bounded by

$$C'|\xi_d|^{-1} \int_{\xi_d}^{2\xi_d} \left| \widehat{\psi}(\xi_1 + c_2\xi_d, \ldots, \xi_{d-1} + c_{d-1}\xi_d) \right|^2 dt \le C''|\xi_d|^{-1}, \tag{3.7}$$

and the proof is complete.

Section IV: Point-wise decay of the Fourier transform in the presence of curvature but very little smoothness

It was recently proved in [BCHIT] that if K is a bounded convex set in $\mathbb{R}^d$, then

$$\left(\int_{S^{d-1}} |\widehat{\chi}_K(r\omega)|^2 d\omega\right)^{\frac{1}{2}} \lesssim r^{-\frac{d+1}{2}}. \tag{4.1}$$

The purpose of this section is to show that the point-wise result is very false even if the Gaussian curvature of ∂K does not vanish. More precisely, we prove the following:

Theorem 4.1. *Given a bounded convex set K such that ∂K has everywhere non-vanishing Gaussian curvature,*

$$|\widehat{\chi}_K(r\omega)| \leq C_1 r^{-2}, \tag{4.2}$$

where C_1 depends on the lower of the Gaussian curvature.

Moreover, there exists an absolute constant $C_2 > 0$ such that given r sufficiently large, there exists a bounded convex set K_r with Gaussian curvature pinched between $\frac{1}{2}$ and 1 and $\omega \in S^{d-1}$ such that

$$|\widehat{\chi}_K(r\omega)| \geq C_2 r^{-2}. \tag{4.3}$$

The first part of Theorem 4.1 is standard. One power of r^{-1} comes from the divergence theorem and the other from integrating by parts once on the boundary. The second part of Theorem 4.1 follows from the following one dimensional estimate.

Theorem 4.2. *For every r sufficiently large, there exists a convex function $\phi_r(t)$ on the interval $[\frac{1}{2}, 4]$, such that the curve $\{(t, \phi_r(t)) : t \in [\frac{1}{2}, 4]\}$ has second derivative pinched between $\frac{1}{2}$ and 1, and*

$$\left| \int e^{ir\phi(t)} \eta(t) dt \right| \geq C r^{-1} \tag{4.4}$$

where η is a smooth non-negative cutoff function supported in $[\frac{1}{2}, 4]$ and identically equal to 1 in $[1, 2]$, and $|\eta(t)|, |\eta'(t)|, |\eta''(t)| \leq 100$.

To reduce Theorem 4.1 to Theorem 4.2, we write $\widehat{\chi}_K$ as a sum of terms of the form $\int_a^b \int_{\Phi_1(x') \leq x_d \leq \Phi_2(x')} e^{-ix' \cdot \xi'} e^{-ix_d \xi_d} \psi(x') dx' dx_d$, where ψ is a smooth cutoff function. Integrating in x_d first and setting $\xi = (0, \dots, 0, r)$, we obtain an integral of the form studied in Theorem 4.2 multiplied by r^{-1}.

Proof of Theorem 4.2. We have for any convex ϕ such that ϕ'' is of bounded variation,

$$\int e^{ir\phi(t)}\eta(t)dt = \int \left[\left(\frac{1}{ir\phi'(t)}\frac{d}{dt}\right)^2 e^{ir\phi(t)}\right]\eta(t)dt \tag{4.5}$$

$$= -\frac{1}{r^2}\int e^{ir\phi(t)}\left(-\frac{\eta(t)\phi'''(t)}{(\phi'(t))^3} + g(t)\right)dt = I + II, \tag{4.6}$$

where

$$g = \frac{4(\phi_r''(t))^2\eta(t) + (\phi_r'(t))^2\eta''(t) - 2\phi_r'(t)\eta'(t)\phi_r''(t)}{(\phi_r'(t))^4} \tag{4.7}$$

is bounded and supported in $[\frac{1}{2}, 4]$. Thus $II = O(r^{-2})$, so we confine our attention to I.

Let ϕ_r and $\{t_n\}$, $t_n \in [1,2]$, $r \le n \le 2r$ be defined so that

$$\phi_r(t_n) = \frac{n\pi}{r}, \tag{4.8}$$

$$1 \le \phi_r'(t) \le 2,\ t \in [1,2], \tag{4.9}$$

$$\phi_r''(t) = \frac{1}{2} \text{ if } t \in I_n,\ n \text{ even}, \tag{4.10}$$

and

$$\phi_r''(t) = 1 \text{ if } t \in I_n,\ n \text{ odd}, \tag{4.11}$$

where $I_n = [t_n, t_{n+1}]$. Since ϕ_r'' is of bounded variation, we have

$$I \ge \frac{1}{r^2}\int \cos(r\phi_r(t))\frac{\eta(t)\phi_r'''(t)}{(\phi_r'(t))^2}dt = \frac{1}{r^2}\sum_n \frac{\eta(t_n)}{(\phi_r'(t_n))^2} \ge \frac{1}{10r}. \tag{4.12}$$

Remark. With some hard work it is possible to construct a single convex function ϕ such that (4.4) holds along an infinite sequence of radii going to infinity.

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