SHARP RATE OF AVERAGE DECAY OF THE FOURIER TRANSFORM OF A BOUNDED SET

L. Brandolini, S. Hofmann, and A. Iosevich

February 10, 2003

Abstract. Estimates for the decay of Fourier transforms of measures have extensive applications in numerous problems in harmonic analysis and convexity including the distribution of lattice points in convex domains, irregularities of distribution, generalized Radon transforms and others. Here we prove that the spherical L^2-average decay rate of the Fourier transform of the Lebesgue measure on an arbitrary bounded convex set in $\mathbb{R}^d$ is

$$\left(\int_{S^{d-1}} |\widehat{\chi}_B(R\omega)|^2 d\omega\right)^{\frac{1}{2}} \lesssim R^{-\frac{d+1}{2}}. \tag{*}$$

This estimate is optimal for any convex body and in particular it agrees with the familiar estimate for the ball. The above estimate was proved in two dimensions by Podkorytov, and in all dimensions by Varchenko under additional smoothness assumptions. The main result of this paper proves $(*)$ in all dimensions under the convexity hypothesis alone. We also prove that the same result holds if the boundary of $\partial\Omega$ is $C^{\frac{3}{2}}$.

Research supported in part by the NSF grants and INdAM

Introduction

Let B be a bounded open set in $\mathbb{R}^d$. If ∂B is sufficiently smooth and has everywhere non-vanishing Gaussian curvature, then

$$|\widehat{\chi}_B(R\omega)| \lesssim R^{-\frac{d+1}{2}}, \tag{0.1}$$

with constants independent of ω, where

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx,$$

denotes the Fourier transform, and $A \lesssim B$ means that there exists a positive constant C such that $|A| \leq C|B|$. The estimate (0.1) is optimal in a very strong sense. One can check that a better rate of decay at infinity is not possible. One can also check that if the Gaussian curvature vanishes at even a single point, then (0.1) does not hold.

In fact, the point-wise estimate may be much worse. For example, if B is convex, one has

$$|\widehat{\chi}_B(R\omega)| \lesssim R^{-1},$$

and the case of a cube $[0,1]^d$ shows that one cannot, in general, do any better. See, for example, [St93], for a nice description of these classical results.

In spite of the fact that the estimate (0.1) does not hold in general, a basic question is whether this estimate holds on average for a large class of domains, for example, bounded open sets with a rectifiable boundary. More precisely, one should like to know for which domains one has the following estimate:

$$\left(\int_{S^{d-1}} |\widehat{\chi}_B(R\omega)|^2 d\omega\right)^{\frac{1}{2}} \lesssim R^{-\frac{d+1}{2}}. \tag{0.2}$$

In some cases, it is equally useful to know whether

$$\left(\int_{S^{d-1}} |\widehat{\sigma}(R\omega)|^2 d\omega\right)^{\frac{1}{2}} \lesssim R^{-\frac{d-1}{2}}, \tag{0.3}$$

where σ is the Lebesgue measure on the boundary of B. Under a variety of assumptions, for example, if ∂B is Lipschitz, (0.2) and (0.3) are linked via the divergence theorem. We use this fact in the proof of our main result below.

An example due to Sjölin ([Sj93]) shows that (0.3) is not purely dimensional. He showed that if σ is an arbitrary $(d-1)$-dimensional compactly supported measure, then the best exponent one can expect on the right hand side of (0.3) is $\frac{d-\frac{3}{2}}{2}$. This means that in order to prove an estimate like (0.2) we must use the fact that ∂B is in some sense a hyper-surface.

Several results of this type have been proved over the years. In [Pod91], Podkorytov proved (0.2) for convex domains in two dimensions using a beautiful geometric argument that relied on the fact that in two dimensions, the Fourier transform of a characteristic function of a convex set in a given direction is bounded by a measure of a certain geometric cap. See, for example, [BrNaWa88] or [BRT98] for more details. Unfortunately, in higher dimensions one cannot bound the Fourier transform of a characteristic function of a convex set by such a geometric quantity. See, for example, [BMVW88]. For the case of average decay on manifolds of co-dimension greater than one, see e.g. [Christ85], [Marshall88], and [IoSa97].

The analytic case has been known for a long time. See, for example, [R66]. In [Var83], Varchenko proved (0.2) under the assumption that ∂B is sufficiently smooth. Smoothness allows one to use the method of stationary phase in a very direct and strong way. In the general case, one must come to grips with the underlying geometry of the problem. In the main result of this paper, we drop the smoothness assumption and prove that (0.2) holds for all bounded open convex sets B in $\mathbb{R}^d$. In addition, we prove the same estimate under an assumption that the boundary is $C^{\frac{3}{2}}$.

The main geometric feature of our approach is a quantitative exploitation of the following simple idea: if $\omega \in S^{d-1}$ is normal to ∂B at x, and y is sufficiently close to x, then $x-y$ cannot be parallel to ω. This allows us to deal with the so-called "stationary" points of the oscillatory integral resulting from (0.3). Unlike the smooth case, where "non-stationary" points are very easy to handle using integration by parts, in the general case one is forced to exploit the smoothness of the sphere along with an appropriate integration by parts argument that exploits either convexity or the $C^{\frac{3}{2}}$ assumption on the boundary.

The estimates (0.2) and (0.3) have numerous applications in various problems of harmonic analysis, analytic number theory and geometric measure theory. Moreover, (0.2) and (0.3) imply immediate generalizations of a number of results in analysis and analytic number theory to higher dimensions. See, for example, [BC00], [BCT97], [BRT98], [CdV77], [Christ85], [Hu96], [IoSa97], [KolWolff02], [Mat87], [Mont94], [R66], [RT01], [Sj93], [Sk98], and [Var83]. We give two simple examples to illustrate the point.

Distribution of lattice points in convex domains. A classical result due to Landau says that if B is convex, and ∂B is smooth and has non-vanishing Gaussian curvature, then

$$\left|\#\{tB \cap \mathbb{Z}^d\} - t^d|B|\right| \leq Ct^{d-2+\frac{2}{d+1}}. \tag{0.4}$$

The proof is based on (0.1). Using (0.2) instead, one can prove the following version of (0.4):

$$\left(\int_{S^{d-1}} \left|\#\{t\rho B \cap \mathbb{Z}^d\} - t^d|B|\right|^2 d\rho\right)^{\frac{1}{2}} \leq Ct^{d-2+\frac{2}{d+1}}, \tag{0.5}$$

where ρB denotes the rotation of B by $\rho \in S^{d-1}$ viewed as an element of $SO(d)$. See, for example, [Ios2001], [BCIPT02], and [BIT01] for a detailed discussion of applications of average decay of the Fourier transform to lattice point problems. Also note that Theorem 1.2 below shows that convexity may be replaced by a $C^{\frac{3}{2}}$ assumption.

Falconer Distance Problem. A result due to Falconer ([Falc86]) says that if the Hausdorff dimension of a set $E \subset [0,1]^d$, $d > 1$, is greater than $\frac{d+1}{2}$, then the distance set $\Delta(E) = \{|x-y| : x, y \in E\}$ has positive Lebesgue measure.

Let $\Delta_B(E) = \{||x-y||_B : x, y \in E\}$, where $||\cdot||_B$ denotes the distance induced by a bounded convex set B. Using (0.2) one can prove that if the Hausdorff dimension of a set $E \subset [0,1]^d$, $d > 1$, is greater than $\frac{d+1}{2}$, then $\Delta_B(\rho B)$ has positive Lebesgue measure for almost every $\rho \in S^{d-1}$ viewed as an element of $SO(d)$. One can also apply this technology to geometric combinatorial results in a discrete setting. See, for example, [HoIo2002] and [IoLa2002] for a detailed discussion of these issues.

Section 1: L^2-Average decay

Our main results are the following two theorems.

Theorem 1.1. *Let B be a bounded convex domain in $\mathbb{R}^d$. Then*

$$\int_{S^{d-1}} |\widehat{\chi}_B(R\omega)|^2 d\omega \lesssim R^{-(d+1)}. \tag{1.1}$$

Theorem 1.2. *Let B be an bounded open set in $\mathbb{R}^d$ satisfying the following assumption. The boundary of B can be decomposed into finitely many neighborhoods such that given any pair of points P, Q in the neighborhood,*

$$|(P-Q)\cdot n(Q)| \lesssim |P-Q|^{\frac{3}{2}}, \tag{1.2}$$

where $n(Q)$ denotes the unit normal to ∂B at Q. Then (1.1) *holds.*

Proof of Theorem 1.1 and Theorem 1.2

We shall give simultaneous proofs of Theorem 1.1 and Theorem 1.2. The argument is based on the fact that both convex surfaces and $C^{\frac{3}{2}}$ surfaces satisfy the following geometric condition:

The boundary of B can be decomposed into finitely many neighborhoods B_j such that on each neighborhood the surface is given as a graph of a Lipschitz function with the Lipschitz constant < 1.

The geometric meaning of this condition is that for x and y belonging to the same neighborhood B_j, the secant vector $x - y$ lies strictly within $\frac{\pi}{4}$ of our local coordinate system's horizon. We shall henceforth refer to this as the "secant property".

This geometric condition is clearly satisfied by C^1 (and hence $C^{\frac{3}{2}}$) surfaces, by taking the neighborhoods to be sufficiently small. For convex domains, we make the following construction. We cover S^{d-1} by a smooth partition of unity η_j, such that $\sum_j \eta_j \equiv 1$ and such that support of each η_j is contained in the intersection of the sphere and a cone of aperture strictly smaller than $\frac{\pi}{2}$. Then if $n(x)$ denotes the Gauss map taking $x \in \partial B$ to the unit normal at x, then $\sum_j \eta_j(n(x))$ induces the desired decomposition on the boundary of B. We note that in the convex case the number of such neighborhoods depends only on dimension.

By the divergence theorem,

$$\widehat{\chi}_B(R\omega) = -\frac{1}{2\pi i R} \int_{\partial B} e^{-ix\cdot R\omega} \left(\omega \cdot n(x)\right) d\sigma(x), \tag{1.3}$$

where $n(x)$ denotes the unit normal to ∂B at x, and $d\sigma$ denotes the surface measure on the boundary. This reduces the problem to the boundary of B.

Decomposition of the boundary. Let ϕ_j denote a smooth partition of unity on ∂B subordinate to the decomposition $\partial B = \cup_{j=1}^{N} B_j$. Moreover, ϕ_js are chosen such that on the support of each ϕ_j, the aforementioned secant property still holds. It follows that the corresponding Lipschitz constant K_j is less than 1. This is the basic building block of our proof.

Let ψ_j be a smooth cutoff function identically equal to 1 on the spherical cap of solid angle $> \frac{\pi}{2}$ and which is supported in a slightly bigger spherical cap which lies at a strict positive distance from all the vectors $\frac{x-y}{|x-y|}$, $x, y \in supp(\phi_j) \subset \partial B$. Notice that our hypothesis make such a decomposition possible and that all the vectors normal to ∂B on the support of ϕ_j lie strictly inside the support of ψ_j.

Singular directions. This part of the proof is identical in the convex and the $C^{\frac{3}{2}}$ cases. In fact, it depends only on the secant property. Let

$$F_j(R\omega) = \int_{\partial B} e^{-ix\cdot R\omega} d\mu_j(x),$$

where

$$d\mu_j = \left(\omega \cdot n(x)\right) \phi_j(x) d\sigma(x).$$

In view of (1.3) and the triangle inequality, it suffices to show that

$$\int_{S^{d-1}} |F_j(R\omega)|^2 d\omega \lesssim R^{-(d-1)}.$$

Now,

$$\int_{S^{d-1}} |F_j(R\omega)|^2 d\omega = \int_{S^{d-1}} |F_j(R\omega)|^2 \psi_j(\omega) d\omega + \int_{S^{d-1}} |F_j(R\omega)|^2 (1-\psi_j(\omega)) d\omega = I + II.$$

We shall refer to the support of ψ_j as "singular" directions, and the other vectors on the sphere as "non-singular" directions. The origin of this notation is the fact that in the smooth case, the singular, or stationary directions are the ones that are normal to the relevant piece of the hyper-surface in question.

We have

$$I = \int_{\partial B}\int_{\partial B}\int_{S^{d-1}} e^{i(x-y)\cdot R\omega}\psi_j(\omega)d\mu_j(x)d\mu_j(y).$$

Using the definition of ψ_j, we integrate by parts N times and obtain

$$I \lesssim \int_{\partial B}\int_{\partial B} \min\{1, (R|x-y|)^{-N}\}d\mu_j(x)d\mu_j(y) \lesssim R^{-(d-1)},$$

since $d\mu_j$ is $d-1$-dimensional and compactly supported.

Non-singular directions. We shall take the following perspective on the spherical coordinates. Let $\omega = \omega(\tau_1, \dots, \tau_{d-2}, \theta)$, where $(\tau_1, \dots, \tau_{d-2})$ denotes the "azimuthal" angles, and θ denotes the remaining angle, i.e $\theta = \tan^{-1}(x_d/x_1)$. Note that for each fixed θ, $(\tau_1, \dots, \tau_{d-2})$ give a coordinate system on the "great circle" tilted at the angle θ from the horizontal.

For each fixed θ, we set up a coordinate system such that

$$II = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int \left|\int e^{iR\omega'\cdot u}\Phi_\theta(u)du\right|^2 J(\tau,\theta)(1-\psi_j(\omega))d\tau d\theta, \tag{2.1}$$

where $\omega = \omega(\tau, \theta)$, $\omega = (\omega', \omega_d)$,

$$\Phi_\theta(u, \omega) = \omega \cdot (\nabla A_\theta(u), -1)\ \phi_j(u, A_\theta(u)),$$

and J is the (smooth) Jacobian corresponding to the spherical coordinates. Here we are viewing this portion of the boundary of B as the graph, of the function A_θ, above the hyperplane determined by the $(d-2)$-dimensional "great circle" obtained by fixing θ. Observe that $\Phi_\theta(u, \omega)$ is linear in ω. This fact and the Minkowski inequality allows to assume in the following $\Phi_\theta(u, \omega)$ independent of ω.

By a further partition of unity, a rotation, and the triangle inequality, we may assume that we are in an arbitrarily small neighborhood of $\omega = (1, 0, \dots, 0)$.

The key object in the remaining part of the proof is the difference operator

$$\Delta_h f(s) = f(s+h) - f(s).$$

We observe that the transpose of this operator

$$\Delta_h^* = \Delta_{-h}.$$

We also note that

$$\Delta_{\frac{1}{R}}(e^{iR\omega_1 u_1}) = (e^{i\omega_1} - 1)e^{iR\omega_1 u_1}.$$

Then by discrete integration by parts, the square root of the portion of (2.1) in the neighborhood of $(1, 0, \ldots, 0)$ equals

$$(2.2) \qquad \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int \left| \frac{1}{e^{i\omega_1} - 1} \int \int e^{iR\omega' \cdot u} \Delta_{-\frac{1}{R}} \Phi_\theta(\cdot, u')(u_1) du_1 du' \right|^2 J(\tau, \theta) \Psi_j(\omega) d\tau d\theta \right)^{\frac{1}{2}},$$

where Ψ_j is an appropriate cut-off function supported in the neighborhood of $(1, 0, \ldots, 0)$, and $u' = (u_2, \ldots, u_{d-1})$.

Applying the Minkowski integral inequality, we see that (2.2) is bounded by

$$(2.3) \qquad \int \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int \left| \int e^{iR\omega'' \cdot u'} \Delta_{-\frac{1}{R}} \Phi_\theta(\cdot, u')(u_1) du' \right|^2 J(\tau, \theta) \Psi_j(\omega) d\tau d\theta \right)^{\frac{1}{2}} du_1,$$

where $\omega'' = (\omega_2, \ldots, \omega_{d-1})$. For a fixed θ, the integration in τ is over the $d-2$-dimensional "great circle". We may parameterize the sphere so that this "great circle" is given by $\omega_1 = \omega_1(\omega'')$.

Expanding (2.3) and rewriting, we get

$$(2.4)$$
$$\int \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int \int e^{iR\omega'' \cdot (u' - v')} \Delta_{-\frac{1}{R}} \Phi_\theta(\cdot, u')(u_1) \Delta_{-\frac{1}{R}} \Phi_\theta(\cdot, v')(u_1) du' dv' J'(\omega'', \theta) \Psi_j(\omega) d\omega'' d\theta \right)^{\frac{1}{2}} du_1,$$

where $J'(\omega'', \theta)$ is smooth in ω''.

Integrating by parts in ω'' we see that (2.4) is bounded by

$$(2.5)$$
$$\int \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int \int \min\{1, R|u' - v'|^{-N}\} |\Delta_{-\frac{1}{R}} \Phi_\theta(\cdot, u')(u_1)| |\Delta_{-\frac{1}{R}} \Phi_\theta(\cdot, v')(u_1)| du' dv' q(\theta) d\theta \right)^{\frac{1}{2}} du_1$$

$$\lesssim R^{-\frac{d-2}{2}} \int \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int |\mathcal{M}' \Delta_{-\frac{1}{R}} \Phi_\theta(\cdot, u')(u_1)|^2 du' q(\theta) d\theta \right)^{\frac{1}{2}} du_1,$$

where $\mathcal{M}'$ is the Hardy-Littlewood maximal function in the u' variable, and q is a smooth cutoff function. The last inequality uses the standard fact that convolution with a radial, integrable and decreasing kernel is dominated by the maximal function. See, for example, [St70], Chapter 3. Since our integrand is compactly supported in the u_1 variable, we may apply Cauchy-Schwarz to obtain that (2.5) is bounded by

$$R^{-\frac{d-2}{2}} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int \int |\Delta_{-\frac{1}{R}} \Phi_\theta(\cdot, u')(u_1)|^2 du' du_1 q(\theta) d\theta \right)^{\frac{1}{2}},$$

since the Hardy-Littlewood maximal function is bounded on L^2.

The conclusion of the theorem now follows from the estimate

$$||\Delta_{\frac{1}{R}}\Phi_\theta||_{L^2(du)} \leq CR^{-\frac{1}{2}}. \tag{2.6}$$

Clearly (2.6) holds if $\partial B \in C^{\frac{3}{2}}$, for in that case $\Phi_\theta \in C^{\frac{1}{2}}$ with compact support. In the convex case we interpolate between the estimates

$$||\Delta_{\frac{1}{R}}\Phi_\theta||_{L^\infty(du)} \leq C, \tag{2.7}$$

and

$$||\Delta_{\frac{1}{R}}\Phi_\theta||_{L^1(du)} \leq CR^{-1}, \tag{2.8}$$

where (2.7) holds because convex surfaces are Lipschitz (hence Φ_θ is bounded), and (2.8) holds by the mean-value theorem, Fubini theorem, and Gauss-Bonnet theorem (for cross-sections) in the u_1 variable.

References

[BMVW88] J. Bak, D. McMichael, J. Vance, and S. Wainger, *Fourier transforms of surface area measure on convex surfaces in* $\mathbb{R}^3$, Amer. J. of Math. **111** (1988), 633-668.

[BC00] L. Brandolini, L. Colzani,, *Decay of Fourier transforms and summability of eigenfunction expansions*, Ann. Scuola Norm. Sup. Pisa **29** (2000), 611-638.

[BCT97] L. Brandolini, L. Colzani, and G. Travaglini, *Average decay of Fourier transforms and integer points in polyhedra*, Ark. Mat. **35** (1997), 253-275.

[BCIPT02] L.Brandolini, L.Colzani, A. Iosevich, A.N.Podkorytov, G.Travaglini, *Geometry of the Gauss map and lattice points in convex domain*, Matematika (to appear).

[BIT01] L. Brandolini, A. Iosevich, and G. Travaglini, *Average decay of the Fourier transform and geometry of the Gauss map*, preprint.

[BRT98] L. Brandolini, M. Rigoli, and G. Travaglini, *Average decay of Fourier transforms and geometry of convex sets*, Revista Mat. Iber. **14** (1998), 519–560.

[BrNaWa88] J. Bruna, A. Nagel and S. Wainger, *Convex hyper-surfaces and Fourier transforms*, Ann. of Math. **127** (1988), 333-365.

[CdV77] Y. Colin de Verdiere, *Nombre de points entiers dans une famille homothtique de domains de R*, Ann. Sci. cole Norm. Sup. **10** (1977), 559–575.

[Christ85] M. Christ, *On the restriction of the Fourier transform to curves: endpoint results and the degenerate case*, Trans. Amer. Math. Soc. **287** (1985), 223-238.

[Falc86] K. J. Falconer, *On the Hausdorff dimensions of distance sets*, Mathematika **32** (1986), 206-212.

[He62] C.S. Herz, *On the number of lattice points in a convex set*, Amer. J. Math. **84** (1962), 126–133.

[Hl50] E. Hlawka, *Über Integrale auf konvexen Körpen, I & II*, Monatsh. Math. **54** (1950), 1–36, 81–99.

[HoIo2002] S. Hofmann and A. Iosevich, *Falconer conjecture in the plane for random metrics*, Israel Journal of Math. (submitted) (2002).

[Hu96] M. N. Huxley, *Area, lattice points, and exponential sums*, Clarendon Press, Oxford (1996).

[Ios2001] A. Iosevich, *Lattice points and generalized diophantine conditions*, Journal of Number Theory **90, no.1** (2001), 19-30.

[IoLo2002] A. Iosevich and I. Laba, *K-distance sets and the Falconer conjecture*, (in preparation) (2002).

[IoSa97] A. Iosevich and E. Sawyer, *Averages over surfaces*, Adv. in Math. **132** (1997), 46-119.

[KolWolff02] M. Kolountzakis and T. Wolff, *On the Steinhaus tiling problem*, Mathematika **46** (2002), 253-280.

[Marshall88] B. Marshall, *Decay rates of Fourier transforms of curves*, Trans. of AMS. **310** (1988).

[Mat87] P. Mattila, *Spherical averages of Fourier transforms of measures with finite energy; dimension of intersections and distance sets* **34** (1987), 207-228.

[Mont94] H. Montgomery, *Ten lessons on the interface between analytic number theory and harmonic analysis*, CBMS Regional Conference Series in Mathematics, American Mathematical Society **84** (1994).

[Pod91] A.N. Podkorytov, *The asymptotic of a Fourier transform on a convex curve*, Vestn. Leningr. Univ. Mat. **24** (1991), 57–65.

[R66] B. Randol, *A lattice-point problem II*, Trans. Amer. Math. Soc. **125** (1966), 101–113.

[RT01] F. Ricci and G. Travaglini, *Convex curves, Radon transforms and convolution operators defined by singular measures*, Proc. Amer. Math. Soc. **129** (2001), 1739-1744.

[Sj93] P. Sjolin, *Estimates of spherical averages of Fourier transforms and dimensions of sets*, Mathematika **40** (1993), 322-330.

[So93] C. Sogge, *Fourier integrals in classical analysis*, Cambridge Univ. Press **105** (1993).

[Sk98] M. M. Skriganov, *Ergodic theory on $SL(n)$, Diophantine approximations and anomalies in the lattice point problem*, Invent. Math. **132** (1998), 1–72.

[St70] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, NJ (1970).

[St93] E. M. Stein, *Harmonic Analysis*, Princeton University Press (1993).

[Str89] R. Strichartz, *Harmonic analysis as spectral theory of laplacians*, J. of Func. Anal. **87** (1989), 51-148.

[Str91] R. Strichartz, *Fourier asymptotics of fractal measures*, J. Funct. Anal. **89** (1990), 154-187.

[Var83] A. Varchenko, *Number of lattice points in families of homothetic domains in $\mathbb{R}^n$*, Funk. An. **17** (1983), 1-6.

[WZ77] R. Wheeden and A. Zygmund, *Measure and Integral*, Pure and Applied Mathematics (1977).

[Wolff00] T. Wolff, *Math 191 lecture notes*, Harmonic analysis lecture notes; CalTech (2000).

L. Brandolini, Dipartimento di Ingegneria, Università di Bergamo, V.le G. Marconi 4, 24044 Dalmine BG, Italy email:brandolini @unibg.it

S. Hofmann, Department of Mathematica, University of Missouri, Columbia Missouri 65211, USA email: hofmann @ math.missouri.edu

A. Iosevich, Department of Mathematica, University of Missouri, Columbia Missouri 65211, USA email: iosevich @ wolff.math.missouri.edu