Freiman's theorem, Fourier transform and additive structure of measures

A. Iosevich* and M. Rudnev†

September 4, 2005

Abstract

We use the Green-Ruzsa generalization of the Freiman theorem to show that if the Fourier transform of certain compactly supported Borel measures satisfies sufficiently bad estimates, then the support of the measure possesses an additive arithmetic structure. This enables one to resolve the Falconer distance problem for such measures. In contrast we show that the Falconer distance problem can also be resolved for a class of measures characterized by total lack of additive structure.

The work was partly supported by the grant DMS02-45369 from the National Science Foundation, the National Science Foundation Focused Research Grant DMS04-56306, and the EPSRC grant GR/S13682/01.

This paper is dedicated to Gregory Freiman, whose beautiful theorem is finding numerous applications many years after its discovery, thus illustrating the power and wisdom of undirected and unrestricted theoretical research.

AMS subject classification 42B, 52C, 97C

*University of Missouri, Columbia, Missouri, 65211 USA, iosevich@math.missouri.edu

†University of Bristol, Bristol BS8 1TW UK, m.rudnev@bris.ac.uk

Contents

1 Introduction

Let γ be a rectifiable curve in $\mathbb{R}^2$ contained in the unit square $[0,1]^2$. Let σ_γ denote the Lebesgue measure on this curve. Let

$$\widehat{\sigma}_\gamma(\xi) = \int_\gamma e^{-2\pi i x\cdot\xi} d\sigma_\gamma(x), \tag{1.1}$$

the Fourier transform of the Lebesgue measure on γ. This object appears in various problems in harmonic analysis, analytic number theory, geometric measure theory and related areas. See for example, [16], [14], [13], and the references contained therein. In order to motivate the main result of this paper we mention the following result due to Podkorytov ([15]) in two dimensions, and to Brandolini, Hofmann and Iosevich ([1]) in higher dimensions.

Theorem 1.1. *Let γ be a convex hypersurface in $\mathbb{R}^d$. Then*

$$\left(\int_{S^{d-1}} |\widehat{\sigma}_\gamma(t\omega)|^2 d\omega\right)^{\frac{1}{2}} \lesssim t^{-\frac{d-1}{2}}. \tag{1.2}$$

Here, and throughout the paper, $X \lesssim Y$ means that there exists $C > 0$ such that $X \leq CY$, $X \gtrsim Y$ means $Y \lesssim X$, and $X \approx Y$ if both $X \lesssim Y$ and $X \gtrsim Y$. Besides, $X \lessapprox Y$ means that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon t^\epsilon Y$, where t is a large controlling parameter.

The decay rate in (1.2) cannot be improved. Suppose that (1.2) holds with $t^{-\frac{d-1}{2}}$ replaced by $t^{-\frac{(d-1+\epsilon)}{2}}$. Then

$$\begin{aligned}
\infty &= \int\int |x-y|^{-(d-1)} d\sigma_\gamma(x) d\sigma_\gamma(y) \\
&\approx \int |\widehat{\sigma}_\gamma(\xi)|^2 |\xi|^{-1} d\xi \\
&\lesssim \int_1^\infty \left(\int |\widehat{\sigma}_\gamma(t\omega)|^2 d\omega \right) t^{d-2} dt \\
&\lesssim \int_1^\infty t^{-1-\epsilon} dt < \infty,
\end{aligned} \tag{1.3}$$

which is absurd.

What's more, if curvature is everywhere non-vanishing on γ, then (1.2) holds point-wise in $\omega \in S^{d-1}$. On the other hand, if γ is a polyhedron, then $\widehat{\sigma}_\gamma(\xi)$ does not decay at all in directions normal to the $(d-1)$-dimensional faces of the polygon. Nevertheless, (1.2) holds, as it does not distinguish between different convex surfaces.

The situation changes if the $L^2(S^{d-1})$ norm in (1.2) is replaced by $L^1(S^{d-1})$. This situation has only been explored, so far, in the two dimensional case. The decay rate in this setting, (see [2]), is determined by the Minkowski dimension of the set of unit vectors normal to γ. Roughly speaking, the lower the dimension of the set of normals, the better the $L^1(S^1)$ decay rate. In particular, the best rate of decay takes place when γ is a polygon, though a precise quantitative version of this statement is yet to be worked out.

On the other end of the $L^p(S^1)$ spectrum, the decay rate is driven by curvature, with the disclaimer that it is difficult to say anything concrete outside of the convex/differentiable category. As we mention above, if γ is convex and has everywhere non-vanishing curvature, then

$$|\widehat{\sigma}_\gamma(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}. \tag{1.4}$$

Conversely, if (1.4) holds under the convexity and sufficient smoothness assumption, then γ has everywhere non-vanishing curvature. This is basically implicit in [8] in much generality, and is worked out explicitly, in a variety of contexts, in [9] and [11]. If the curvature is assumed to vanish to the order at most $m-2$, $m \geq 2$, and $d=2$, then

$$\left(\int_{S^1} |\widehat{\sigma}_\gamma(t\omega)|^p d\omega \right)^{\frac{1}{p}} \lesssim t^{-\frac{1}{2}}, \tag{1.5}$$

for $p < \frac{2(m-1)}{m-2}$. See [3]. If γ is a polyhedron in $\mathbb{R}^d$, it is not difficult to show that in the range $2 \le p \le \infty$,

$$\left(\int_{S^{d-1}} |\widehat{\sigma}_\gamma(t\omega)|^p d\omega \right)^{\frac{1}{p}} \approx t^{-\frac{d-1}{p}}, \tag{1.6}$$

precisely the range obtained by interpolation of the result in the case $p = 2$ given by Theorem 1.1 and complete lack of decay in some directions for $p = \infty$. It would be interesting to show that if (1.6) holds in, say, rectifiable category, then γ must contain a piece of a hyperplane. The purpose of this paper is to investigate a variant of this question, in connection with Λ_p sets and the theory of distance sets.

Recall that a measure μ is called Ahlfors-David regular if there exists some $s \in [0, d]$, such that

$$\mu(B_\delta(x)) \approx \delta^s, \ \forall x \in \operatorname{supp} \mu. \tag{1.7}$$

Recall that μ is a Frostman measure if in the above definition framework,

$$\mu(B_\delta(x)) \lesssim \delta^s, \ \forall x \in \operatorname{supp} \mu. \tag{1.8}$$

Definition 1.2. We say that $\mathbb{A}$ is an *arithmetic progression* in $\mathbb{Z}^d$ of dimension k and size L, if each element of $\mathfrak{g} \in \mathbb{A}$ possesses a representation

$$\mathfrak{g} = \mathfrak{g}_0 + \{r_1\mathfrak{g}_1 + \cdots + r_k\mathfrak{g}_k\}_{1 \le r_j \le L_j}, \tag{1.9}$$

where each r_j is an integer, each $\mathfrak{g}_j$ is a (fixed) element of $\mathbb{Z}^d$ (called a generator), and $L_1 \cdot L_2 \cdot \cdots \cdot L_k = L$. An arithmetic progression is *proper* if the representation (1.9) is unique for each $\mathfrak{g} \in \mathbb{A}$. Proper arithmetic progression is defined similarly for an arbitrary abelian group $\mathfrak{G}$ (to substitute $\mathbb{Z}^d$ in this definition).

Definition 1.3. We say that an Ahlfors-David regular measure μ, supported on a compact set E of Hausdorff dimension α (further E always stands for the support of the measure μ; one always has $\alpha \ge s$, where s is the exponent in (1.61)) is *arithmetic* if there exists $E' \subset E$, of positive α-dimensional Hausdorff measure, such that for each δ sufficiently small, E'_δ, the δ-neighborhood of E' is contained in the δ-neighborhood of a dilate of a proper arithmetic progression $\mathbb{A} = \mathbb{A}(\delta)$ in $\mathbb{Z}^d$. If $\alpha > 0$, the progression has to get longer as $\delta \to 0$.

In this paper we deal with the case $s = \frac{d}{2}$, which is critical from the point of view of the Erdös-Falconer distance problem, discussed in the last section of this paper. Our main result is the following.

Theorem 1.4. *Let μ be a compactly supported Ahlfors-David regular measure, satisfying (1.7) with $s = \frac{d}{2}$. Let $l \geq 2$ be a positive integer. Suppose that*

$$\int_{t \leq |\xi| \leq 2t} |\widehat{\mu}(\xi)|^{2l} d\xi \gtrsim \int_{t \leq |\xi| \leq 2t} |\widehat{\mu}(\xi)|^{2} d\xi \approx t^{\frac{d}{2}}, \tag{1.10}$$

for all sufficiently large t. Then μ is arithmetic.

Moreover, for every $\epsilon > 0$ there there exists $E' \subset E$ is of positive $(1-\epsilon)$-dimensional Minkowski measure, contained in a line segment.

An interesting consequence of our methods is the following average Fourier estimate for Fourier transform of strictly convex curves.

Theorem 1.5. *Let $d = 2$ and μ be a compactly supported Ahlfors-David regular measure, satisfying (1.7) with $s = 1$. Suppose, μ is supported on a one-dimensional strictly convex curve. Then, for $l \geq 2$*

$$\int_{t \leq |\xi| \leq 2t} |\widehat{\mu}(\xi)|^{2l} d\xi \lesssim t^{2^{-l+1}}. \tag{1.11}$$

Remark 1.6. Theorem 1.5 will follow from a recent result by Konyagin, Ten and the authors ([10]), and in view of the conjecture mentioned in that paper, Theorem 1.5 should be true for any $\epsilon > 0$ replacing the exponent 2^{-l+1}.

We stress that (1.11) is much stronger than a result that can be obtained by integrating the estimate (1.2) with respect to the radial variable and interpolating with the trivial L^∞ result. In the special case when γ is the moment curve $\{(s, s^m) : s \in [0,1]\}$, for example, Theorem 1.5 can be verified directly, but in the general case it is difficult to imagine obtaining such a result without the use of arithmetic combinatorics.

In order to illustrate the numerology of Theorem 1.4, consider the case when μ is the Lebesgue measure on the boundary of a square $[-1,1]^2$. Let χ denote the characteristic function of $[-1,1]^2$. It is clear by a direct calculation that

$$|\widehat{\chi}(\xi)| \approx |\xi|^{-1} |\widehat{\mu}(\xi)|.$$

Observe that

$$\widehat{\chi}(\xi) = C \frac{\sin(\pi \xi_1) \sin(\pi \xi_2)}{\xi_1 \xi_2},$$

so we have

$$\int_{t \leq |\xi| \leq 2t} |\widehat{\chi}(\xi)|^4 d\xi$$

$$\approx \int_{t\le|\xi|\le 2t} \frac{\sin^4(\pi\xi_1)\sin^4(\pi\xi_2)}{\xi_1^4\xi_2^4} d\xi_1 d\xi_2$$

$$\gtrsim \int_t^{2t} \frac{dt}{t^4} \;\approx\; t^{-3}.$$

It follows that

$$\int_{t\le|\xi|\le 2t} |\widehat{\mu}(\xi)|^4 d\xi \gtrsim t^{-3}\cdot t^4 = t, \tag{1.12}$$

in accordance with Theorem 1.4.

Corollary 1.7. *Let μ be as in the statement of Theorem 1.4. Let E denote the support of μ. Then the Minkowski dimension of $\Delta(E)$ is 1, where*

$$\Delta(E) = \{|x-y| : x, y \in E\}, \tag{1.13}$$

and $|\cdot|$ denotes the Euclidean distance.

The connection with the theory of distance sets is not accidental. We shall explain the background issues pertaining to this in detail in the final section of this paper. We shall also obtain the following simple positive result which should be contrasted with Theorem 1.4.

Definition 1.8. We say that a compactly supported Borel measure μ is *strictly convex* if the equation

$$x + y = x' + y', \;\; x, y, x', y' \in \operatorname{supp}\mu \tag{1.14}$$

has an at most bounded number of non-trivial solutions, i.e. those when x is not equal x' or y'.

Theorem 1.9. *Let μ be a strictly convex Frostman measure, satisfying (1.8) with the exponent s. Then*

$$\int_{t\le|\xi|\le 2t} |\widehat{\mu}(\xi)|^4 d\xi \lesssim t^{d-2s}. \tag{1.15}$$

Corollary 1.10. *Let $E \subset \mathbb{R}^d$ have Hausdorff dimension α. Suppose that E supports a strictly convex Frostman measure μ, satisfying (1.8) the exponent $s \ge \frac{d}{2}$. Then the Lebesgue measure of the distance set of E is positive. In other words, the Falconer conjecture holds for sets that possess strictly convex measures.*

In summary, our results tell us that sets satisfying *very* poor Fourier estimates have arithmetic structure and thus have large distance sets. Conversely, we know that sets that have almost no arithmetic structure satisfy *very* good Fourier estimates and have large distance sets as well. The question that we are unable to address is, what happens in between?

This paper is structured as follows. In Section 2 we construct a discrete model resulting from the assumptions of Theorem 1.4. The discrete model enables us to prove Theorem 1.5 in that section. In Section 3 we develop a pigeon-holing argument and use the Green-Ruzsa ([5]) variant of Freiman's theorem ([7]) to complete the proof of Theorem 1.4. In the last section we discuss the connection between the problem we are studying and the theory of distance sets. Corollary 1.7 follows immediately. Furthermore, we prove Theorem 1.9 and Corollary 1.10 based on the machinery developed by Mattila ([12]) for the Falconer distance problem ([6]).

1.1 Acknowledgements

We wish to thank Ben Green for several helpful remarks and observations.

2 Discretization

Define $a \approx_\delta b$ if $|a-b| \leq \delta$. Let μ be an arbitrary compactly supported Ahlfors-David regular measure, satisfying (2.7) with $s = \frac{d}{2}$, and such that (1.10) holds. Let $X = (x_1, \dots, x_l) \in \mathbb{R}^{dl}$, $Y = (y_1, \dots, y_l) \in \mathbb{R}^{dl}$ and $\mu^* = \mu_X \times \mu_Y = \mu \times \mu \times \cdots \times \mu$, $2l$ times. Observe that with $\delta \approx \frac{1}{t}$, the condition (1.10) implies that

$$\mu^*\{(x_1, \dots, x_l, y_1, \dots, y_l) : x_1 + \cdots + x_l \approx_\delta y_1 + \cdots + y_l\} \gtrsim \delta^{\frac{d}{2}}, \qquad (2.1)$$

for all sufficiently small values of δ. Indeed, if ψ is a radial cut-off function which is supported in the annulus $\{\xi : .9 \leq |\xi| \leq 2.1\}$, and is identically one for $1 \leq |\xi| \leq 2$, then by the Fubini theorem,

$$\begin{aligned} \int_{t \leq |\xi| \leq 2t} |\widehat{\mu}(\xi)|^{2l} d\xi &\lesssim \int\int \left(\int e^{-2\pi i z\cdot\xi} \psi(\xi/t) d\xi\right) d\mu_X d\mu_Y \\ &= t^d \int\int \hat{\psi}(tz) d\mu_X d\mu_Y, \end{aligned} \qquad (2.2)$$

where $z = x_1 + \ldots + x_l - (y_1 + \ldots + y_l)$. The estimate (2.1) now follows since $\widehat{\psi}$ decays rapidly.

Assume without loss of generality that $E \subset [0,1]^d$. Since μ is Ahlfors-David regular, we can choose $\delta = c_0 t^{-1}$, with some $0 < c_0 < 1$, so that for $N = \delta^{-1}$ there exists $\Gamma_N \subset (NE \cap \mathbb{Z}^d)$ of cardinality $c_1 N^{\frac{d}{2}}$, for some sufficiently small $c_1 \in (0,1)$, such that the left hand side of (2.1) equals

$$\begin{aligned} &\int \mu_X\{(x_1,\dots,x_l) : x_1+\dots+x_l \approx_\delta y_1+\dots+y_l\} d\mu_Y \ \approx \\ &\delta^{dl}\, \#\{(a_1,\dots,a_l,b_1,\dots,b_l) \in \Gamma_N : a_1+\dots+a_l = b_1+\dots+b_l\}. \end{aligned} \tag{2.3}$$

Without loss of generality we may assume that N is an integer, and that $N \approx t$.

Indeed, the support of μ can be expressed as the union of $c_1 N^{\frac{d}{2}}$ δ-separated cubes of the lattice $\delta\mathbb{Z}^d$. It follows that the centers of the cubes satisfy the diophantine equation in (2.3) rather than the approximate relation in (2.1).

We are now in position to give a simple proof of Theorem 1.5 in view of the following result due to Konyagin, Ten and the authors ([10]).

Theorem 2.1. *Let $\{s_j\}_{j=1}^N$ be a strictly convex sequence of real numbers, in the sense that $s_{j+1} - s_j > s_j - s_{j-1}$. Then*

$$\begin{aligned} &\#\{(i_1,\dots,i_l,j_1,\dots,j_l) : s_{i_1}+\dots+s_{i_l} = s_{j_1}+\dots+s_{j_l}\} \\ &\lesssim N^{2l-2+2^{-l+1}}. \end{aligned} \tag{2.4}$$

In particular, it is shown in [10] that without loss of generality, the quantities s_j in Theorem 1.4 can be regarded as integers. Hence, in the special case when μ is supported on a convex planar curve, there is no loss of generality in simply taking $\Gamma_N = \{(i, f(i)),\ i \in \{1,\dots,N\}$, for some strictly convex integer-valued function $f(x)$ in the above described discretization, that is to assume that the grid points $a_1,\dots,b_l$ appearing in (2.3) belong to the support E of μ. In view of (2.2) this proves Theorem 1.5.

We now turn our attention to Theorem 1.4. The discretization procedure in (2.1-2.3) reduces this result to the following combinatorial problem.

Discrete Model Let Γ_N be the aforementioned subset of $\mathbb{Z}^d \cap [0,N]^d$ of cardinality $N^{\frac{d}{2}}$. Suppose that

$$\#\{(a_1,\dots,a_l,b_1,\dots,b_l) \in \Gamma_N : a_1+\dots+a_l = b_1+\dots+b_l\} \gtrsim N^{dl-\frac{d}{2}}. \tag{2.5}$$

The following theorem describes the structure of Γ_N as $N \to \infty$.

Theorem 2.2. *For the discrete model above, the condition (2.5) implies that there exists $c_2 > 0$ such that at least $c_2 N$ points of Γ_N lie on a straight line.*

Theorem 1.4 follows from Theorem 2.2 by definition of Minkowski dimension. As we have shown, the conclusions of Theorem 1.4 imply the assumption (2.5) of Theorem 2.2 quite immediately since μ is assumed to be Ahlfors-David regular, which makes the discretization (2.1-2.3) possible.

To prove Theorem 2.2, define for $u \in l\Gamma_N = \Gamma_N + \Gamma_N + \cdots + \Gamma_N$, l times, the *multiplicity* function

$$n(u) = \#\{(a_1, \ldots, a_l) \in \Gamma_N^l : a_1 + \cdots + a_l = u\}. \tag{2.6}$$

The statement of the theorem can now be rewritten in the form

$$\sum_{u \in l\Gamma_N} n^2(u) \gtrsim N^{dl - \frac{d}{2}}. \tag{2.7}$$

The discrete model above leads to the following combinatorial observation.

Lemma 2.3. *For the discrete model above, there exists a family of subsets $\Gamma_{j,N} \subset \Gamma_N$, $j = 1, \ldots, l$, such that $\#\Gamma_{1,N} \gtrsim \#\Gamma_N$ and*

$$\#(\Gamma_{1,N} + \ldots + \Gamma_{l,N}) \lesssim \#\Gamma_{1,N}. \tag{2.8}$$

The diameter of the set $\Gamma_{1,N}$ is $\gtrsim N$.

Lemma 2.3 will be proved in the next section. In order to take advantage of it, as the final building block of the proof of Theorem 1.4, we shall use a generalization of the following classical result due to G. Freiman ([7]).

Theorem 2.4 (Freiman's theorem)**.** *Let $A_N \subset \mathbb{Z}$ such that $\#A_N = N$, and $\#(A_N + A_N) \leq CN$, where C is independent of N. Then A_N is contained in some k-dimensional arithmetic progression in $\mathbb{Z}$ (see the definition (1.9), where k depends only on C.*

Observe that Freiman's theorem in the above formulation does not extends immediately to $\mathbb{Z}^d$. However, Green and Ruzsa ([5]) proved that Freiman's theorem applies to general abelian groups in the following context.

Definition 2.5. A coset progression in an abelian group $\mathfrak{G}$ is the sum $\mathbb{A}+H$, where $\mathbb{A}$ is a proper arithmetic progression in $\mathfrak{G}$ and H is a subgroup of $\mathfrak{G}$. The sum is direct in the sense that $a+h=a'+h'$ only if $a=a'$ and $h=h'$.

The dimension of the coset progression is the number k in (1.9) above and the size of the coset progression is simply the cardinality of $\mathbb{A}+H$.

The generalization of Freiman's theorem we are going to use is the following result due to Green and Ruzsa ([5]).

Theorem 2.6. *Let $\mathfrak{G}$ be an abelian group and let $A \subset \mathfrak{G}$ such that $\#(A+A) \leq C\#A$. Then A is a subset of a coset progression of dimension $k(C)$ and size $f(C)\#A$.*

In particular, it is immediate from Theorem 2.6 that if $\mathfrak{G}=\mathbb{Z}^d$, then the only possible choice for H is $\{0\}$, the trivial subgroup. In other words, in this case the conclusion of the theorem is that A is contained in a proper arithmetic progression.

Remark 2.7. We note that both Freiman's theorem and the Green/Ruzsa variant (in the case of subsets of $\mathbb{Z}^d$) continue to hold if we assume that $\#A \approx \#B$ and $\#(A+B) \lesssim \#A$. In this case, the conclusion is that at least one of A, B is contained in a generlized arithmetic progression of the designated length and dimension.

One can probably deduce the version of Freiman's theorem that we need in this paper directly from Freiman's original result using projections, without resorting to the fill statement of the Green-Ruzsa generalization. However, we believe that the full strength of the Green-Ruzsa theorem will be necessary in further structure theorems for measures which we hope to explore in subsequent work.

3 Proof of Theorem 1.4

We start out by proving Lemma 2.3 which will put us into the framework of additive number theory. Observe that

$$\begin{aligned} \textstyle\sum_{u\in l\Gamma_N} n^2(u) &\leq \max_{u\in l\Gamma_N} n(u)\cdot \textstyle\sum_{u\in l\Gamma_N} n(u) \\ &\lesssim N^{(l-1)\frac{d}{2}}\cdot N^{l\frac{d}{2}} = N^{dl-\frac{d}{2}}. \end{aligned} \tag{3.1}$$

Comparing this with the condition (2.5) we see that there is a subset $\Upsilon_N \subset l\Gamma_N$ of cardinality at least $c_3 N^{\frac{d}{2}}$, such that for all $u \in \Upsilon_N$ we have

$n(u) \geq c_4 N^{(l-1)\frac{d}{2}}$. Indeed,

$$\#\Upsilon_N \lesssim \frac{N^{l\frac{d}{2}}}{N^{(l-1)\frac{d}{2}}} = N^{\frac{d}{2}} \lesssim \#\Gamma_N. \tag{3.2}$$

Then there exist subsets $\Gamma_{1,N}, \ldots, \Gamma_{l,N}$ such that $\#\Gamma_{j,N} \approx \#\Gamma_N \approx N^{\frac{d}{2}}$, as well as

$$\Gamma_{1,N} + \ldots + \Gamma_{2,N} = \Upsilon_N. \tag{3.3}$$

Finally, all the sets sets $\Gamma_{j,N}$ cannot have a diameter $o(N)$, as this would contradict the Ahlfors-David regularity of the measure μ or the fact that $\dim_H(\operatorname{supp}\mu \geq \frac{d}{2}$. This proves Lemma 2.3.

It follows from Theorem 2.6 that the set $\Gamma_{1,N}$, henceforth denoted by Γ'_N, is contained in some proper arithmetic progression $\mathbb{A}$ in $\mathbb{Z}^d \cap [0,N]^d$. Let $\mathfrak{g}_0, \ldots \mathfrak{g}_k$, where $k = O(1)$ is the generators of the progression, with lengths L_j, $j = 1, \ldots, k$. Note that $L_1 \cdot \ldots \cdot L_k \approx N^{\frac{d}{2}}$. We claim that one of these lengths, L_1, say, is $\gtrsim N$. Assuming the opposite immediately contradicts the statement in Lemma 2.3 about the diameter of the set $\Gamma_{1,N}$. This completes the proof of Theorems 1.2 and Theorem 1.4.

4 Connections with the Falconer distance problem

The Falconer distance conjecture says that if the Hausdorff dimension of a compact set E in $\mathbb{R}^d$ is greater than $\frac{d}{2}$, then the Lebesgue measure of $\Delta(E)$ is positive, where $\Delta(E)$ has been defined by (1.14).

The best known result in this direction is due to Wolff ([17]), in two dimensions, and Erdoğan ([4]), in higher dimensions, who prove that the Lebesgue measure of $\Delta(E)$ is indeed positive if the Hausdorff dimension of E is greater than $\frac{d}{2} + \frac{1}{3}$. These results are partly based on the machinery developed by Mattila ([12]), which can be summarized as follows. Mattila proves that if there exists a Borel measure μ supported on E such that

$$M(\mu) = \int_1^\infty \left(\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega \right)^2 t^{d-1} dt < \infty, \tag{4.1}$$

then the Lebesgue measure of $\Delta(E)$ is positive.

Mattila proves his result by studying the pull-forward ν on $\Delta(E)$ of the measure $\mu \times \mu$ on $E \times E$, under the distance map. If μ is a probability measure, than by Cauchy-Schwartz and Plancherel,

$$1 \lesssim \left(\int d\nu \right)^2 \leq |\Delta(E)| \cdot \int |\widehat{\nu}(t)|^2 dt, \tag{4.2}$$

as long as ν has an L^2 densiity. Mattila then shows that

$$\widehat{\nu}(t) \approx t^{\frac{d-1}{2}} \int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega. \tag{4.3}$$

Both Wolff, Erdoğan and their predecessors (see the reference list in [4]) obtained an upper bound for $M(\mu)$, by first proving a bound of the form

$$\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega \lesssim I_\alpha(\mu) t^{-\beta}, \tag{4.4}$$

(where

$$I_\alpha(\mu) = \int\int \frac{d\mu_x d\mu_y}{|x-y|^\alpha} < \infty, \ \ \forall \alpha < \dim_H(E) \tag{4.5}$$

is the energy integral of μ) for some $\beta > 0$. By using this, polar coordinates, and Plancherel, it follows that

$$\begin{aligned} M(\mu) &\lesssim \int |\widehat{\mu}(\xi)|^2 |\xi|^{-d+(d-\beta)} d\xi \\ &\approx \int\int |x-y|^{-(d-\beta)} d\mu(x) d\mu(y) \\ &\equiv I_{d-\beta}(\mu), \end{aligned} \tag{4.6}$$

the energy integral of μ of order $(d-\beta)$. It is a standard result that the latter integral is bounded whenever $d-\beta$ is smaller than the Hausdorff dimension of E (which is the supremum of the set α such that $I_\alpha(\mu)$ is finite).

Observe that by a trivial application of the Minkowski integral inequality,

$$M(\mu) \leq \int |\widehat{\mu}(\xi)|^4 d\xi, \tag{4.7}$$

so if E carries a Borel measure μ such that $\widehat{\mu} \in L^4(\mathbb{R}^d)$, the Lebesgue measure of the distance set is automatically positive. The main result of this paper, Theorem 1.4, implies that if the Hausdorff dimension of E is $\frac{d}{2}$ and $\widehat{\mu}$ fails to be in $L^4(\mathbb{R}^d)$ sufficiently "badly", then the distance set of E is large. Indeed, all it takes to vindicate Corollary 1.7 is to identify $E' \subset E$, given by Theorem 1.4 with a subset of the distance set $\Delta(E)$.

This opens a possibility of attacking the Falconer conjecture by showing that if the Mattila integral is large, then the underlying set E must have ad additive structure. Theorem 1.5 can be viewed as a small step in that direction.

Finally, to prove Theorem 1.9 we use (4.7) and (3.1) with $l = 2$ that imply

$$M(\mu) \lesssim t^d \, \mu^* \{(x, y, x', y') : x + y =_{t^{-1}} x' + y'\}, \tag{4.8}$$

again with the notation of the proof of Theorem 1.4. Since μ is a Frostman measure that satisfies (1.8), and the equation

$$x + y = x' + y' \tag{4.9}$$

only has trivial solutions by assumption, the right hand side of (4.8) is bounded by

$$\lesssim \; t^j t^{-s} t^{-s} \; = \; t^{d-2s}, \tag{4.10}$$

as desired. This completes the proof of Theorem 1.9.

The proof of Corollary 1.10 follows immediately from Theorem 1.9, the estimate (4.7) of the previous section and Mattila's approach to the distance problem, described in the beginning of this section.

References

[1] L. Brandolini, S. Hofmann, and A. Iosevich. *Sharp rate of average decay of the Fourier transform of a bounded set.* Geom. Funct. Anal. **13** (2003) 671–680.

[2] L. Brandolini, A. Iosevich, and G. Travaglini. *Spherical L^1-averages of the Fourier transform of the characteristic function of a convex set and the geometry of the Gauss map.* Trans. Amer. Math. Soc. **355** (2003) 3513–3535.

[3] L. Brandolini, M. Rigoli, and G. Travaglini. *Average decay of Fourier transforms and geometry of convex sets.* Rev. Mat. Iberoamericana **14** (1998) 519–560.

[4] B. Erdoğan. *A bilinear Fourier extension theorem and applications to the distance set problem.* IMRN (accepted for publication) 2005.

[5] B.J. Green, and I. Ruzsa. *Freiman's theorem in a arbitrary abelian group.* Preprint 2005.

[6] K.J. Falconer. *On the Hausdorff dimensions of distance sets.* Mathematika **32** (1986) 206–212.

[7] G. Freiman. *Foundations of a structural theory of set addition.* Translations of Mathematical Monographs **37**. *Amer. Math. Soc. Providence, RI, USA,* 1973.

[8] L. Hörmander. *The analysis of linear partial differential operators IV: Fourier integral operators.* Gründlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **275**, 1985.

[9] A. Iosevich. *Fourier transform, L^2 restriction theorem, and scaling.* Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. **8** (1999) 383–387.

[10] A. Iosevich, S. Konyagin, M. Rudnev, and V. Ten. *Combinatorial complexity of convex sequences.* Discr. and Comp. Geom. (2005) to appear.

[11] A. Iosevich and G. Lu. *Sharpness results and Knapp's homogeneity argument.* Canad. Math. Bull. **43** (2000) 63–68.

[12] P. Mattila. *Spherical averages of Fourier transforms of measures with finite energy: dimensions of intersections and distance sets.* Mathematika, **34** (1987), 207–228.

[13] P. Mattila. *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability.* Cambridge Studies in Advanced Mathematics **44**, 1995.

[14] H. Montgomery. *Ten lectures on the interface between analytic number theory and harmonic analysis.* CBMS Regional Conference Series in Mathematics **84**, 1994.

[15] A. Podkorytov. *On the asymptotics of the Fourier transform on a convex curve.* Vestnik Leningrad. Univ. Mat. Mekh. Astronom **125** (1991) 50–57.

[16] P. Sjölin. *Estimates of spherical averages of Fourier transforms and dimensions of sets.* Mathematika **40** (1993) 322–330.

[17] T. Wolff. *Decay of circular means of Fourier transforms of measures.* International Mathematics Research Notices **10** (1999) 547–567.