LECTURE #5: BOURGAIN STRIKES AGAIN-
ARITHMETIC KAKEYA ESTIMATES

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Abstract. We present Bourgain's combinatorial approach to the Kakeya conjecture. We prove that there exists $c > \frac{1}{2}$ such that the Minkowski dimension of a Kakeya set is at least $cn + (1-c)$.

In the last lecture, we proved Wolff's result which says that the Hausdroff dimension of a Kakeya set in $\mathbb{R}^n$ is at least $\frac{n+2}{2}$. The purpose of this lecture is to discuss an arithmetic approach to the Kakeya conjecture, introduced by Bourgain. This approach leads to improved lower bounds on the Minkowski and Hausdorff dimension of a Kakeya set when n is relatively large. However, recent work by Nets Katz and Tery Tao shows that this method, combined with earlier ideas due to Wolff and others may well lead to the complete resolution of the conjecture.

In this lecture, we shall concentrate on the Minkowski dimension of a Kakeya set. We shall see that this simplification leads to a purely combinatorial problem, without the added complication of arbitrary scales associated with the Hausdorff dimension. Recall that if E is a set in $\mathbb{R}^n$, $E_\delta = \{x : dist(x, E) \leq \delta\}$, the δ-neighborhood of E, also known as the Minkowski sausage.

Theorem 10.1. *There exists an absolute constant $c > \frac{1}{2}$ such that a Besicovitch set has Minkowski dimension at least $cn + (1-c)$.*

Setup. We will work with the following modified version of a Besicovictch set.

Definition 10.2. We say that E is a modified Besicovitch set if E is a subset of $\mathbb{R}^{n-1} \times [0,1]$ which contains, for each $v \in \mathbb{R}^{n-1}$, $|v| \leq 1$, a line segment of the form $\{(x,0) + t(v,1) : t \in [0,1]\}$.

It is not difficult to show (by scaling and projecting) that if there exists a Besicovitch set of dimension d, then there exists a modified Besicovitch set of dimension at most d. This allows us to work with a modified Besicovitch set in proving Theorem 10.1.

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We proceed by contradiction. Suppose that E is a modified Besicovitch set of Minkowski dimension $cn+(1-c)$. We will see that if c is too close to $\frac{1}{2}$, a contradiction will result. By definition of Minkowski dimension,

$$|E_\delta| \lesssim \delta^{n-(cn+(1-c))} = \delta^{(1-c)(n-1)}. \tag{10.1}$$

For each $t \in [0,1]$, let $E_\delta(t)$ denote the horizontal slice of E_δ with the hyperplane $x_n = t$. By Fubini,

$$\int_0^1 |E_\delta(t)|dt \lesssim \delta^{(1-c)(n-1)}. \tag{10.2}$$

By Chebyshev's inequality, the measure of the set

$$Goodslice = \{t : |E_\delta(t)| \leq C\delta^{(1-c)(n-1)}\} \tag{10.3}$$

is at least .99 if C is chosen to be large enough.

We shall need the fact that $Goodslice$ contains at least one arithmetic progression t_0, t_0+a, t_0+2a. We may assume that $t_0 = 0 \in Goodslice$. Consider the set $A = Goodslice \cap [1/4, 1/2)$ and its double $2A$. By (10.3), $|A| \geq 1/4 - .01$. It follows that the measure of $2A \cap Goodslice$ is at least $2(1/4 - .01) - .01$. It follows that A and $2A \cap Goodslice$ are not empty and we have our arithmetic progression. Observe that this trivial argument fails utterly if we assume that the measure of $Goodslice$ is only, say, $\frac{1}{2}$. However, a deeper line reasoning due to Roth shows that if $Goodslice$ has positive measure, however small, than there exists at least one arithmetic progression.

Discretization. By scaling and chopping we may assume that the arithmetic progression we just found is $0, \frac{1}{2}, 1$. Consider $C^{-1}\delta\mathbb{Z}^{n-1}$ where C is a large constant. Let

$$A(t) = \{i \in C^{-1}\delta\mathbb{Z}^{n-1} : (i,t) \in E_\delta(t)\}. \tag{10.4}$$

It follows easily that

$$\#A(t) \lesssim \delta^{-c(n-1)}, \tag{10.5}$$

with $t = 0, 1/2, 1$.

We are close to the punch-line. Let G consist of those pairs in $A(0) \times A(1)$ that are contained in a single tube. Recall that since E contains many segments, E_δ contains many tubes. Define

$$Sums = \{x+y : (x,y) \in G\}, \tag{10.6}$$

and let

$$Differences = \{x-y : (x,y) \in G\}. \tag{10.7}$$

Since $\frac{x+y}{2}$ is essentially in $A(1/2)$, it follows that

$$\#Sums \lesssim \delta^{-c(n-1)}. \tag{10.8}$$

On the other hand, E has a line segment in every directions. This implies that

$$\#Differences \gtrsim \delta^{-(n-1)}. \tag{10.9}$$

Thus, we have many differences and few sums. We shall see that if c is sufficiently close to $\frac{1}{2}$, (which measures how small the set of differences is), this leads to a contradiction.

Arithmetic combinatorics

The result will follow from the following combinatorial lemma.

Lemma 10.3. *Let $N >> 1$ and let A, B be subsets of a free abelian group with cardinality at most N. Let G be a subset of $A \times B$ such that $\#Sums \leq N$. Then $\#Differences \lesssim N^{2-\epsilon}$ for some absolute $\epsilon > 0$.*

Before embarking upon proving this wonderful result, let's record an amusing fact. If $\epsilon = 0$, the lemma is trivial. Indeed,

$$\#Differences \leq \#G \leq \#A \times \#B \leq N^2. \tag{11.1}$$

It follows that the Minkowski dimension of a Besicovitch set is at least $\frac{n+1}{2}$. On some level, this proof must be the same as the one we gave in Lecture #3, right?

Suppose that we have a counterexample to the lemma. In other words, assume that we have A, B and G such that

$$\#Differences \approx N^2. \tag{11.2}$$

We may assume that the map $(x, y) \to x - y$ is one-to-one when restricted to G since we can clearly refine G to make this the case. It follows that

$$\#G \approx N^2, \tag{11.3}$$

which says that G occupies a significant portion of $A \times B$.

Let G^b denote the $b'th$ row of G,

$$G^b = \{a : (a, b) \in G\}. \tag{11.4}$$

By 11.3 and Fubini we know that

$$\sum_b \#G^b \approx N^2. \tag{11.5}$$

This means that one average, rows of G contain N elements. In fact, we make sure that they all contain about that many elements by throwing out the rows that do not. No harm done...

Time has come to use some advanced mathematics. Observe that

$$a + b = a' + b' \Leftrightarrow a - b' = a' - b. \tag{11.6}$$

It follows that many pairs of elements in $A \times B$ should have the same difference. A precise version of this remarkable fact is the following computation.

Lemma 11.1. *Let A, B be sets of cardinality at most N, and let $G \subset A \times B$ be such that $\#Sums \le N$ and $\#G \approx N^2$. Then there exists a set $I \subset A \times B$ of cardinality $\approx N^2$ such that for all $(a,b) \in I$,*

$$\#\{(a',b') \in A \times B : a - b = a' - b'\} \approx N. \tag{11.7}$$

To prove the lemma, let

$$sumcount(x) = \#\{(a,b) \in A \times B : a + b = x\}. \tag{11.8}$$

By (11.3) we have

$$\sum_{Sums} sumcount(x) \ge \#G \approx N^2. \tag{11.9}$$

On the other hand, since $\#Sums \le N$, Cauchy-Schwartz yields

$$\sum_{Sums} sumcount(x) \le (\#Sums)^{\frac{1}{2}} \left(\sum_{Sums} sumcount^2(x) \right)^{\frac{1}{2}}, \tag{11.10}$$

which implies that

$$\sum_{Sums} sumcount^2(x) \ge N^3. \tag{11.11}$$

A more humane version of this statement is that

$$\#\{(a,b,a',b') \in A \times B \times A' \times B' : a + b = a' + b'\} \gtrsim N^3, \tag{11.12}$$

which implies by (11.6) that

$$\#\{(a,b,a',b') \in A \times B \times A' \times B' : a - b' = a' - b\} \gtrsim N^3. \tag{11.13}$$

Let

$$diffcount(x) = \#\{(a,b) \in A \times B : a - b = x\}. \tag{11.14}$$

It follows by the same reasoning that

$$\sum diffcount^2(x) \gtrsim N^3. \tag{11.15}$$

On the other hand,

$$\sum diffcount(x) \le N^2 \tag{11.16}$$

trivially, which implies that

$$\sum_{diffcount(x)<<N} diffcount^2(x) \lesssim N^3. \tag{11.17}$$

The inevitable conclusion is that

$$\sum_{diffcount(x)\gtrsim N} diffcount^2(x) \gtrsim N^3. \tag{11.18}$$

We are almost done. For every a there is at most one b such that $a - b = x$. It follows that $diffcount(x) \leq N$, and so

$$\sum_{diffcount(x)\gtrsim N} N^2 \gtrsim N^3. \tag{11.19}$$

It follows that

$$\#\{x : diffcount(x) \gtrsim N\} \gtrsim N. \tag{11.20}$$

Let I be the set of pairs (a, b) such that $diffcount(a - b) \gtrsim N$. The condition (11.7) is automatically fulfilled, and the fact that $\#I \approx N^2$ follows from (11.20).

This completes the proof of Lemma 11.1. It is now time for more difficult combinatorics.

Little grey cells

We seem to be on to something, but it is not clear exactly what. We have two subsets of $A \times B$, namely G and I. Both occupy a significant fraction of $A \times B$. However, while differences of G are all distinct, the differences of I are very popular. If G and I have a significant intersection, we are done. Unfortunately, this need not be the case. Bourgain got around this problem in a magnificently clever way. This is why this subsection is called "Little grey cells"...

The basic idea is the following. We will write $a - b$ as

$$a - b = (a_1 - b_1) - (a_2 - b_2) + (a_3 - b_3) \tag{12.1}$$

for a very large number (about N^5) of sextuplets. Since there are only about N^6 such sextuplets, it follows that there are only about N values of $a - b$, which contradicts (11.3).

How could this idea possibly fly? Well, advanced mathematics tells us that

$$a - b = (a - b') - (a' - b') + (a' - b). \tag{12.2}$$

Why does this help? After all, a' and b' range over sets of cardinality about N, so we get only about N^2 representations for $a - b$. Where are we going to find N^5 representations? Well, we can try to make sure that (a, b'), (a', b') and (a', b) are usually in I, so the differences on the right hand side of (12.2) are popular. This will gives us the extra factors we need to build up to N^5 representations.

Everything we just said is fairly imprecise. The devil, as usual, is in the details. The details of this problem were originally worked out by the "little grey cells"...

Let's get to work.

Definition 12.1. If b, b' are elements of B, we say that b and b' communicate, and write $a \sim b$, if

$$\#\{a \in A : (a,b), (a,b') \in I\} \gtrsim N^{1-\epsilon_0}, \tag{12.3}$$

where ϵ_0 is an absolute constant to be chosen later.

We now demonstrate that we can make many rows communicate with each other if we first throw out some bad rows.

Lemma 12.2. *There exists $B' \subset B$ of cardinality $\approx N$ such that*

$$\#\{(b,b') \in B' \times B' : b \nsim b'\} \lesssim N^{2-\epsilon_0}. \tag{12.4}$$

The basic idea of the proof is to let $B' = I_a = \{b : (a,b) \in I\}$ for some $a \in A$ since $\#\{(b,b') \in I_a \times I_a : b \nsim b'\}$ is small on average. Indeed,

$$\sum_{a \in A} \#\{(b,b') \in B' \times B' : b \nsim b'\} = \sum_{a \in A} \sum_{(b,b'):b \nsim b'} \chi_I(a,b)\chi_I(a,b') \tag{12.5}$$

$$= \sum_{(b,b'):b \nsim b'} \sum_{a \in A} \chi_I(a,b)\chi_I(a,b') \tag{12.6}$$

$$\lesssim \sum_{(b,b'):b \nsim b'} N^{1-\epsilon_0} \tag{12.7}$$

$$\lesssim N^{3-\epsilon_0}. \tag{12.8}$$

If we wish to feast on this development, we need a lower bound... Indeed,

$$\sum_{a \in A} \#\{(b,b') \in I_a \times I_a\} = \sum_{a \in A} (\#I_a)^2 \tag{12.9}$$

$$\geq \frac{1}{\#A} \left(\sum_{a \in A} \#I_a \right)^2 \tag{12.10}$$

$$= \frac{(\#I)^2}{\#A} \gtrsim N^3. \tag{12.11}$$

It follows that

$$\sum_{a\in A} \#\{(b,b')\in I_a\times I_a\} - N^{\epsilon_0-\epsilon}\#\{(b,b')\in I_a\times I_a : b\not\sim b'\} \gtrsim N^3 \tag{12.12}$$

for arbitrarily small ϵ. We deduce that there exists $a\in A$ such that

$$\#\{(b,b')\in I_a\times I_a\} \gtrsim N^{\epsilon_0}\#\{(b,b')\in I_a\times I_a : b\not\sim b'\} + N^2. \tag{12.13}$$

This completes the proof of the lemma. We now have a subset $A\times B'$ of $A\times B$ with very communicative rows. Moreover, it contains many elements of G because (11.5) and the discussion that follows implies that

$$\#(G\cap(A\times B')) = \sum_{b\in B'} \#G^b \approx N^2. \tag{12.14}$$

Having dealt with the rows, we now turn to weeding the columns. Let

$$G_a = \{b\in B' : (a,b)\in G\} \tag{12.15}$$

and let

$$A' = \{a\in A : \#G_a\approx N\}. \tag{12.16}$$

It is not hard to see that $\#A'\approx N$ and hence

$$\#(G\cap(A'\times B'))\approx N^2, \tag{12.17}$$

so we are ready to cruise to the finish. Apply Lemma 11.1 to $G\cap(A'\times B')$ with (A',B') in place of (A,B). We get a set $I'\subset A'\times B'$ such that for each $(a,b)\in I'$, $a-b$ can be written in $\approx N$ different ways as $a'-b'$ with $a'\in A'$, $b'\in B'$.

What is the cardinality of the set of triplets

$$\{(a,b,b')\in A'\times B'\times B' : (a,b)\in G, (a,b')\in I'\}? \tag{12.18}$$

There are $\approx N^2$ ways to chose (a,b'), and by the construction of A', there are then $\approx N$ ways to chose b. It follows that we have at least $\approx N^3$ triplets in our set. Let us also count the set

$$\{(a,b,b')\in A'\times B'\times B' : (a,b)\in G, (a,b')\in I', b\not\sim b'\}. \tag{12.19}$$

By Lemma 12.2 there are $\lesssim N^{2-\epsilon_0}$ ways to chose (b,b'), and then $\lesssim N$ ways to chose a. It follows that there are $\lesssim N^{3-\epsilon_0}$ triplets in this set. It follows that if ϵ_0 is small,

$$\#\{(a,b,b')\in A'\times B'\times B' : (a,b)\in G, (a,b')\in I', b\sim b'\} \gtrsim N^3. \tag{12.20}$$

In other words,

$$\sum_{(a,b)\in G\cap(A'\times B')} \#\{b' : b \sim b', (a,b') \in I'\} \gtrsim N^3. \tag{12.21}$$

There are at most N^2 pairs (a,b) being summed over and the summand is at most N. It follows that there exist $\approx N^2$ pairs (a,b) such that

$$\#\{b' : b \sim b', (a,b') \in I'\} \gtrsim N. \tag{12.22}$$

By definition 12.1 this means that

$$\#\{(a',b') : (a,b') \in I', (a',b), (a',b') \in I\} \gtrsim N^{2-\epsilon_0}. \tag{12.23}$$

for these pairs (a,b).

Since I and I' are popular, we have

$$\#\{(a_1,a_2,a_3,b_1,b_2,b_3) : a-b = (a_1-b_1)-(a_2-b_2)+(a_3-b_3)\} \gtrsim N^{5-\epsilon_0}. \tag{12.24}$$

What do we have? There are $\approx N^2$ pairs (a,b) in G, whose differences are each representable in $\approx N^{5-\epsilon_0}$ different ways. However, there are only $\approx N^6$ ways to write something in the form $(a_1-b_1)-(a_2-b_2)+(a_3-b_3)$. This contradicts (11.2) if ϵ_0 is small enough. The proof of Theorem 10.1 is complete.