LECTURE #1: KAKEYA PROBLEM, KAKEYA MAXIMAL OPERATOR, AND THE RESTRICTION PHENOMENON: CONNECTIONS AND RELATIONSHIPS

Alex Iosevich

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Abstract. The purpose of this lecture is to introduce the Kakeya set problem, the Kakeya maximal operator, and the restriction problem, and to discuss the connections between those three.

Section 0: Introduction

A Kakeya set is a compact set $E \subset \mathbb{R}^n$ containing a unit line segment in every direction. In other words, for every $e \in S^{n-1}$, there exists $x \in \mathbb{R}^n$, such that $x + te \in E$, $t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. A construction due to A. S. Besicovitch shows that such sets can be of measure 0. However, this turns out to be the beginning rather than the end of the story, and our **first question** is, what is the Hausdorff dimension of E? To show that the Hausdorff dimension of E is d, it is enough to show that if $\{B(x_i, r_i)\}$ is a cover of E (by balls of radius r_i centered at x_i), then

$$\sum_i r_i^s \geq C, \tag{0.1}$$

for any $s < d$, where C is a uniform constant.

Define a tube centered at a of width δ in the direction e by

$$T_e^\delta(a) = \left\{ x \in \mathbb{R}^n : |(x-a) \cdot e| \leq \frac{1}{2},\ |(x-a)^\perp \cdot e| \leq \delta \right\} \tag{0.2}$$

with $e \in S^{n-1}$, $\delta > 0$, and $a \in \mathbb{R}^n$. Define the Kakeya maximal function f_δ^* by

$$f_\delta^*(e) = \sup_{a \in \mathbb{R}^n} \frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f|. \tag{0.3}$$

This definition is due to Bourgain. Our **second question** is, given $\epsilon > 0$, does there exists C_ϵ such that

$$||f_\delta^*||_{L^n(S^{n-1})} \leq C_\epsilon \delta^{-\epsilon} ||f||_{L^n(\mathbb{R}^n)}? \tag{0.4}$$

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This estimate is related to our first question in the same way as the Hardy-Littlewood maximal theorem is related to Lebesgue's theorem on the points of density. We shall see later that an affirmative answer to (0.4) implies that the Hausdorff dimension of a Kakeya set in $\mathbb{R}^n$ is n.

Let σ denote the Lebesgue measure on the $(n-1)$-dimensional sphere. Our **third question** is, does

$$||\widehat{fd\sigma}||_{L^p(\mathbb{R}^n)} \lesssim ||f||_{L^p(d\sigma)} \tag{0.5}$$

for all $p > \frac{2n}{n-1}$, where $A \lesssim B$ means that there exists a uniform constant C such that $A \leq CB$. It is not difficult to see that this inequality does not hold for $p \leq \frac{2n}{n-1}$ since $\widehat{d\sigma}(\xi) = \cos\left(\xi - \frac{\pi n}{4}\right) |\xi|^{-\frac{n-1}{2}} +$ small error. Inequality (0.5) is an example of the so-called *restriction phenomenon* which has numerous applications in harmonic analysis and partial differential equations. We shall discuss this topic in more detail later in these lecture notes.

Again, we shall see that the affirmative answer to our third question implies that the dimension of a Kakeya set in $\mathbb{R}^n$ is n. Moreover, we shall see that weaker versions of (0.4) and (0.5) still say something about the dimension of a Kakeya set.

This lecture is organized as follows. In Section I: Kakeya maximal operator, we shall see that bounds for the Kakeya maximal operator imply the coresponding lower bounds on the Hausdorff dimension of a Kakeya set. In Section II: The restriction phenomenon, we shall analyze a variant of (0.5) from the same point of view. Finally, in Section III: Technical appendix, we shall briefly discuss some of the analytic tools used in this lecture.

Acknowledgements. Most of the material in these lecture notes comes from the following sources:

[TerryTao1] T. Tao, *Dyadic pigeonholes in harmonic analysis (expository note)*, http://www.math.ucla.edu/ tao (1999).

[TerryTao2] T. Tao, *Lecture notes for Math 254B-UCLA*, http://www.math.ucla.edu/ tao (1999).

[TomWolff1] T. Wolff, *Lecture notes for Math 191-Caltech* (2000).

[TomWolff2] T. Wolff, *Recent work connected with the Kakeya problem*, (preprint) (1998).

Section 1: Kakeya maximal operator

The purpose of this section is to establish a link between the first and the second question in the introduction.

Suppose that we have the following *restricted weak type* version of (0.4):

$$||f_\delta^*||_{q,\infty} \lesssim \delta^{-\alpha}||f||_{p,1}, \tag{1.1}$$

by which we mean that if A is any measurable set, and $f = \chi_A$, then $|\{e \in S^{n-1} : f_\delta^*(e) > \lambda\}| \lesssim (\lambda^{-1}\delta^{-\alpha}|A|^{\frac{1}{p}})^q$, $\lambda \in (0,1]$. A systematic treatment of the restricted weak type can be found in, for example, "Introduction to Fourier analysis on Euclidean spaces", by E. M. Stein and G. Weiss.

Theorem 1.1. *The estimate* (1.1) *implies that the Hausdorff dimension of a Kakeya set is at least* $n - p\alpha$.

To prove the lemma, fix $s < n - p\alpha$. Let E be a Kakeya set and for each $e \in S^{n-1}$, fix $x_e \in \mathbb{R}^n$ such that $x_e + te \in E$, $t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Let $\{B(x_j, r_j)\}$ denote the cover of E by balls of radius r_j centered at x_j's. We must prove that (0.1) holds with $s < n - p\alpha$. Note that all the r_j's may be chosen to be < 1, since otherwise the estimate obviously holds...

One inconvenience we are facing is that r_i's may be very different in size. In an attempt to correct for this annoyance, we define

$$\Sigma_k = \{j : 2^{-k} \leq r_j \leq 2^{-k+1}\}. \tag{1.2}$$

Let $E_k = E \cap \{\cup\{B(x_j, r_j) : j \in \Sigma_k\}\}$. Let $B'_j = B(x_j, 2r_j)$, and $E'_k = \cup\{B'_j : j \in \Sigma_k\}$. Note that in the definition of E_k we are intersecting with E, whereas in the definition of E'_k, we are not...

Since $\cup_k E_k = E$, the pigeonhole principle implies that given $e \in S^{n-1}$,

$$\left|\left\{t \in \left[-\frac{1}{2}, \frac{1}{2}\right] : x_e + te \in E_k\right\}\right| \geq \frac{6}{\pi^2}\frac{1}{k^2} \tag{1.3}$$

for some $k = k_e$, since $\sum_k \frac{6}{\pi^2}\frac{1}{k^2} = 1$.

Let $\Omega_k = \{e \in S^{n-1} : k = k_e\}$. Clearly, $\sum |\Omega_k| = |S^{n-1}|$. It follows by the pigeonhole principle that we can find a fixed k and a set $\Omega = \Omega_k$, so that $k = k_e$ when $e \in \Omega$, and

$$|\Omega| \geq \frac{6}{\pi^2}\frac{1}{k^2}, \tag{1.4}$$

where we have normalized things so that $|S^{n-1}| = 1$.

The point. Since E_k contains many line segments, E'_k contains many tubes of width 2^{-k}. We are now ready to use the tube technology, i.e the estimate (1.1) for f^*_δ with $\delta = 2^{-k}$.

We have

$$\frac{1}{k^2}\left|T_e^{2^{-k}}(x_e)\right| \lesssim |T_e^{2^{-k}}(x_e) \cap E'_k| \tag{1.5}$$

for any $e \in \Omega$. Recalling the definition of $f^*_{2^{-k}}$, with $f = \chi_{E'_k}$, we have

$$\frac{1}{k^2} \lesssim \left|\left\{e \in \Omega : \frac{1}{k^2} \lesssim f^*_{2^{-k}}(e)\right\}\right| \leq \left|\left\{e \in S^{n-1} : \frac{1}{k^2} \lesssim f^*_{2^{-k}}(e)\right\}\right|. \tag{1.6}$$

On the other hand, (1.1) says that

$$\left|\left\{e : \frac{1}{k^2} \lesssim f^*_{2^{-k}}(e)\right\}\right| \lesssim \left(k^2 2^{k\alpha}|E'_k|^{\frac{1}{p}}\right)^q. \tag{1.7}$$

It follows, up to logarithms, that

$$2^{-k\alpha p} \lesssim |E'_k|. \tag{1.8}$$

However, $|E'_k| \lesssim |\Sigma_k| 2^{-kn}$, so

$$2^{k(n-\alpha p)} \lesssim |\Sigma_k|. \tag{1.9}$$

We are now ready to complete the proof. We have

$$\sum_{\Sigma_k} r_j^s \geq 2^{-ks}|\Sigma_k| \geq 2^{-ks}2^{k(n-\alpha p)} \tag{1.10}$$

and we win because $s < n - \alpha p$.

Section 2: The restriction phenomenon

The purpose of this section is to link the second and the third (and consequently the first and the third) question in the introduction.

Applying Holder's inequality and interpolation along with (0.5), we get

$$||\widehat{fd\sigma}||_q \lesssim ||f||_{L^p(d\sigma)}, \; p < \frac{2n}{n-1}, \; q > \frac{n+1}{n-1}p'. \tag{2.1}$$

Theorem 2.1. *Suppose that* (2.1) *holds for a given* $p \geq 2$ *and* $q \geq 2$. *Then, with* $r = \left(\frac{q}{2}\right)'$ *and* $s = \left(\frac{p}{2}\right)'$, *the restricted weak type* (r, s) *norm of the Kakeya maximal operator is* $\lesssim \delta^{-2(\frac{n}{r}-1)}$.

Applying Theorem 1.1 with $p = r, q = s$, and $\alpha = 2\left(\frac{n}{r} - 1\right)$, we see that under the assumptions of Theorem 2.1, the Hausdorff dimension of a Kakeya set is at least $\frac{2q}{q-2} - n$. Those familiar with classical restriction theorems may find the following observation amusing. The classical Stein-Tomas restriction theorem says that (2.1) holds with $p = 2$ and $q_n = \frac{2(n+1)}{n-1}$. Since $\frac{2q_n}{q_n-2} - n = 1$, we see that Stein-Tomas restriction theorem does not provide us with meaningful information about the dimension of a Kakeya set via Theorem 2.1.

To prove Theorem 2.1, let $\{T_j\}_{j=1}^N$, $T_j = T_{e_j}^\delta(a_j)$ be any collection of δ tubes with δ separated directions e_j. Let $T_j' = \{x \in \mathbb{R}^n : \delta^2 x \in T_j\}$ denote the dilation of T_j by the factor δ^{-2}. Let χ_j and χ_j' denote the characteristic functions of T_j and T_j' respectively. Let C_j denote a spherical cap of radius $\approx \delta$, i.e

$$C_j = \{e \in S^{n-1} : e \cdot e_j \geq 1 - C^{-1}\delta^2\}, \tag{2.2}$$

where C is a suitable constant. Construct a bump function $\phi_j \in C_0^\infty(C_j)$ with

$$||\phi_j||_\infty = 1, \; \phi_j \geq 0, \text{ and } ||\phi_j||_1 \approx \delta^{n-1}, \tag{2.3}$$

and let

$$\psi_j(\xi) = exp(\xi \cdot \delta^{-2} a_j)\phi_j(\xi), \tag{2.4}$$

where $exp(t) = e^{2\pi i t}$.

We want to argue that

$$|\widehat{\psi_j d\sigma}| \geq \delta^{n-1}\chi_j', \tag{2.5}$$

which would follow if the expression for $\widehat{\psi_j d\sigma}$ involved no cancellations. Indeed,

$$\widehat{\psi_j d\sigma} = \int_{S^{n-1}} \psi_j(\xi) exp(-x \cdot \xi) d\xi \tag{2.6}$$

$$= exp(-(x-\delta^{-2}a_j)\cdot e_j)\int_{C_j}\phi_j(\xi)exp(-(\xi-e_j)\cdot(x-\delta^{-2}a_j))d\xi, \tag{2.7}$$

and the estimate (2.5) follows by the uncertainty principle type reasoning.

Now the fun begins. Consider $f=\sum_j \pm\psi_j$, where the signs are chosen randomly. Since the supports of ψ_j's are disjoint, we have

$$||f||_{L^p(d\sigma)} \lesssim (N\delta^{n-1})^{\frac{1}{p}}, \tag{2.8}$$

which implies, by the assumption of the lemma, that

$$||\widehat{fd\sigma}||_q \lesssim (N\delta^{n-1})^{\frac{1}{p}}. \tag{2.9}$$

It is time to get some mileage out of the $\pm$ business. If we combine (2.5) and Khinchin's inequality, we get

$$\mathbb{E}(|\widehat{fd\sigma}|^q) \gtrsim \delta^{q(n-1)}\left(\sum_j \chi'_j\right)^{\frac{q}{2}} \tag{2.10}$$

pointwise, where $\mathbb{E}()$ denotes the expected value of (). It follows that

$$\delta^{2(n-1)-\frac{4n}{q}}\left|\left|\sum_j \chi_j\right|\right|_{\frac{q}{2}} \lesssim (N\delta^{n-1})^{\frac{2}{p}}. \tag{2.11}$$

O.K., I am being intentionally malicious here. To get (2.11), first establish the appropriate inequality with χ'_j in place of χ_j, then use the fact that χ_j and χ'_j just differ by the scaling factor δ^{-2}, and, finally, take $\frac{q}{2}$ of both sides.

Punchline. Let A be a set, $f=\chi_A$, and

$$\Omega=\{e: f^*_\delta(e)\geq\lambda\}. \tag{2.12}$$

Let $\{e_j\}_{j=1}^N$ be a maximal δ-separated subset of Ω and for each j chose a δ-tube T_j as above with

$$|A\cap T_j|\geq\lambda|T_j|. \tag{2.13}$$

Then

$$N\lambda\delta^{n-1}\leq\sum_j |T_j\cap A| \tag{2.14}$$

$$\leq |A|^{1-\frac{2}{q}} \left\| \sum_j \chi_j \right\|_{\frac{q}{2}} \tag{2.15}$$

$$\leq |A|^{1-\frac{2}{q}} (N\delta^{n-1})^{\frac{2}{p}} \delta^{-2(n-1)+\frac{4n}{q}},$$

where the first line follows from (2.13), the second line by Holder, and the third line by (2.11). Now, this is nice and all, but we need to relate it to an estimate for $|\Omega|$. Let $\mathcal{N}_\delta(\Omega)$ denote the maximum possible cardinality of $\{e_j\}_{j=1}^N$. It is not hard to see that

$$\mathcal{N}_\delta(\Omega) \gtrsim \frac{|\Omega|}{\delta^{n-1}} \tag{2.16}$$

since e_j's are δ-separated. Plugging this into (2.15) we see that

$$|\Omega|^{1-\frac{2}{p}} \lesssim \lambda^{-1} |A|^{1-\frac{2}{q}} \delta^{-2(n-1)+\frac{4n}{q}}, \tag{2.17}$$

which means that

$$|\Omega|^{\frac{1}{s}} \lesssim \lambda^{-1} |A|^{\frac{1}{r}} \delta^{-2(\frac{n}{r}-1)} \tag{2.18}$$

as promised. This completes the proof of Theorem 2.1.

In the next lecture we shall prove the (0.4) and (0.5) indeed hold in the plane. From there, we shall venture forth into the wild jungle of higher dimensions.

Section 3: Technical appendix

In Section II we made use of the following inequality due to Khinchin's:

Lemma 3.1. *Let $\{\omega_i\}_{i=1}^N$ denote a family of independent random variables taking on values ± 1 with equal probability. Let $\{a_i\}_{i=1}^N$ denote a sequence of complex numbers. Then*

$$\mathbb{E}\left(\left| \sum_{i=1}^N a_i \omega_i \right|^p \right) \approx \left(\sum_{i=1}^N |a_i|^2 \right)^{\frac{p}{2}}, \tag{3.1}$$

where $\mathbb{E}()$ denotes the expected value of ().

The proof can be found in many books. I personally recommend the proof given by Tom Wolff in [TomWolff1]. There is a large number of beautiful applications of Khinchin's inequality. Among them is the Littlewood-Payley square function inequality and the proof that the classical Hausdorff-Young inequality cannot be reversed. The former can be found in many introductory harmonic analysis texts, whereas the latter is in the aforementioned notes of Tom Wolff.