

**MAXIMAL OPERATORS ASSOCIATED TO  
FAMILIES OF FLAT CURVES IN THE PLANE**

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**Introduction:** Let  $C$  denote a smooth curve in the plane. Let  $M_t f(x) = \int_C f(x - ty)d\sigma(y)$ , where  $d\sigma(y)$  denotes a cutoff function times the Lebesgue measure on  $C$ . Let  $\mathcal{M}f(x) = \sup_{t>0} M_t f(x)$ . A question we ask is, for what range of the exponents  $p$ , is the following a priori inequality satisfied:

$$(1.1) \quad \|\mathcal{M}f\|_p \leq C_p \|f\|_p, \quad f \in \mathcal{S}.$$

Bourgain showed that if  $C$  has non-vanishing curvature, the inequality (1.1) holds for  $p > 2$  (see [B]). In this paper, we shall consider a situation when the curvature is allowed to vanish of finite order on a finite set of isolated points. We shall need the following definition:

**Definition 1.1.** Let  $C : I \rightarrow \mathbb{R}^2$ , where  $I$  is a compact interval in  $\mathbb{R}$ , and  $C$  is smooth. We say that  $C$  is finite type if  $\langle (C(x) - C(x_0)), \mu \rangle$  does not vanish of infinite order for any  $x_0 \in I$ , and any unit vector  $\mu$ .

We shall also need a more precise definition which would specify the order of vanishing at each point. Let  $a_0$  denote a point in the compact interval  $I$ . We can always find a smooth function  $\gamma$ , such that in a small neighborhood of  $a_0$ ,  $C(s) = (s, \gamma(s))$ , where  $s \in I$ .

**Definition 1.2.** Let  $C$  be defined as before. Let  $C(s) = (s, \gamma(s))$  in a small neighborhood of  $a_0$ . We say that  $C$  is finite type  $m$  at  $a_0$  if  $\gamma^{(k)}(a_0) = 0$  for  $1 \leq k < m$ , and  $\gamma^{(m)}(a_0) \neq 0$ .

Our main result is the following:

**Theorem 1.1.** Let  $C$  be a finite type curve which is finite type  $m$  at  $a_0$ . Let  $M_t f(x) = \int_C f(x - ty)d\sigma(y)$ , where  $d\sigma$  is the Lebesgue measure on  $C$  multiplied by a smooth cutoff function supported in a sufficiently small neighborhood of  $a_0$ . Let  $\mathcal{M}f(x) = \sup_{t>0} M_t f(x)$ . Then,

$$(1.2) \quad \|\mathcal{M}f\|_p \leq C_p \|f\|_p \quad \text{for } p > m.$$

Furthermore, the result is sharp. Let  $h_p(x) = |x_2|^{-\frac{1}{p}} \log\left(\frac{1}{|x_2|}\right)^{-1} \phi(x)$ , where  $\phi(x)$  is a non-negative  $C_0^\infty$  function supported in the unit ball, such that  $\phi \equiv 1$  near the origin. It

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is not hard to check that  $h_p \in L^p(\mathbb{R}^2)$ , if  $p > 1$ . Locally,  $C(s) = (s, \gamma(s))$ . Without loss of generality, let  $a_0 = 0$ , and  $\gamma(0) = -1$ . Then,  $M_t h_p(x) = \infty$ , for  $p \leq m$ , if  $|x_2| = t$ , and  $x_2 < 0$ .

We shall also consider a variable coefficient version of Theorem 1. The averaging operator  $M_t$  that we have considered so far is called translation invariant, or the constant coefficient operator because it averages a function over the translates and dilates of a fixed curve. We are now going to consider an operator which averages a function over a more general distribution of curves in the plane. We are also going to consider a more general time dependence. As before, we are going to define a maximal operator by taking the supremum over the time dependence. In order to formulate the conditions for  $L^p$  boundedness of such operators, a more sophisticated geometric setup is required.

Let's consider a curve distribution in  $\mathbb{R}^2$  such that through every point  $x \in \mathbb{R}^2$  we have a  $C^\infty$  curve. If we assume that this curve distribution is a smooth submanifold for each fixed  $t$ , then we can locally express this curve distribution as

$$\mathcal{D}_t = \{(x, y) : y_2 = x_2 + A(x_1, y_1, x_2, t)\} \text{ for some smooth } A(x_1, y_1, x_2, t).$$

Let  $S_{x,t}$  denote  $\{y : (x, y) \in \mathcal{D}_t\}$  and consider a family of operators

$$(1.3) \quad M_t(f)(x) = \int_{S_{x,t}} f(x-y) d\sigma_{x,t}(y),$$

where  $d\sigma_{x,t}$  is the smooth cutoff function times the Lebesgue measure on  $S_{x,t}$ .

After taking a partial Fourier transform, we can rewrite  $M_t(f)(x)$  as

$$(1.4) \quad (2\pi)^{-1} \int \int e^{i\theta(A(x_1, y_1, x_2, t) + x_2 - y_2)} \psi(x, y) f(y) d\theta dy,$$

where  $\psi(x, y)$  is a cutoff function.

After setting  $M_t(f)(x) = (Ff)(x, t)$  and using a standard theorem about orders of Fourier integral operators (See [So1]), we see that the  $M_t$ 's are Fourier integral operators of the order given by

$$\text{order(symbol)} + \left(\frac{1}{2} \# \text{oscillating vars}\right) - \frac{1}{4}(\#x \text{ vars} + \#y \text{ vars}) = 0 + \frac{1}{2} - 1 = -\frac{1}{2}.$$

Let  $F_t(x, y)$  denote the kernel of this Fourier integral operator. The distribution defined by  $(x, y) \rightarrow F(x, y, t)$  is a Lagrangian distribution of order  $-\frac{1}{2}$ , the wave front set of which is contained in a subset of the cotangent bundle of  $\mathbb{R}^2 \times \mathbb{R}^2$  with the zero section removed (See e.g. [So1]). This subset is called a canonical relation. The canonical relation is a Lagrangian submanifold of the cotangent bundle with respect to the canonical symplectic form  $dx \wedge d\xi - dy \wedge d\mu$ .

Since the kernel  $F_t(x, y)$  is supported in  $\mathcal{D}_t$ , it is not hard to see that if we consider a "twisted" canonical relation  $C$  where  $(x, \xi, y, \mu)$  is replaced by  $(x, \xi, y, -\mu)$ ,  $C$  is in fact a

conormal bundle of  $\mathcal{D}_t$  denoted by  $N^*(D_t)$  (See [Trvs]). This is related to the fact that the Fourier transform of the smooth density on a hypersurface decays rapidly in every direction except the normal directions to the hypersurface (See [Hor1]).

Let  $\mathcal{C}_t$  denote the canonical relation for a fixed  $t$ . If we let  $X, Y$  denote the support of  $x \rightarrow F_t(x, y)$  and  $y \rightarrow F_t(x, y)$  respectively,  $\mathcal{C}_t$  can be naturally viewed as a Lagrangian submanifold of the cotangent space of  $X \times Y$  with the zero section removed. Let  $\pi_l : \mathcal{C}_t \rightarrow T^*(X) \setminus 0$  and  $\pi_r : \mathcal{C}_t \rightarrow T^*(Y) \setminus 0$  denote the natural projections of  $\mathcal{C}_t$ .

We say that  $\mathcal{C}_t$  is a local canonical graph if

$$\mathcal{C}_t = \{(x, \xi, y, \mu) : (y, \mu) = \Xi_t(x, \xi)\},$$

where  $\Xi_t$  is a symplectomorphism for each  $t$ . The distribution of curves is then said to satisfy the rotational curvature condition (See [PhSt1]). It is not hard to see that this condition is equivalent to the condition that both  $\pi_l$  and  $\pi_r$  are local diffeomorphisms.

As we have noted earlier, the canonical relation  $\mathcal{C}_t$  is the conormal bundle of the curve distribution  $\mathcal{D}_t$ . In other words, the canonical relation is parameterized by the phase function of the operator (1.4),  $\theta\Phi(x, y, t)$ , where  $\Phi(x, y, t) = y_2 - x_2 - A(x, y_1, t)$ .

The condition that the projection  $\pi_l$  is a local diffeomorphism means that given  $(x_0, \xi_0) \in T^*(X) \setminus 0$  we can find  $(y, \theta) \in Y \times \mathbb{R}$  such that  $\Phi(x_0, y) = 0$  and  $\theta d_x \Phi(x_0, y, t) = \xi_0$ . This means that the Jacobian of the map  $(\theta, y) \rightarrow (\Phi(x, y, t), \theta d_x \Phi(x, y, t))$  is non-zero. The resulting Jacobian is called the Monge-Ampere determinant:

$$(1.5) \quad J_t(x, y) = Det \begin{pmatrix} 0 & \Phi_{x_1} & \Phi_{x_2} \\ \Phi_{y_1} & \Phi_{x_1 y_1} & \Phi_{x_2 y_1} \\ \Phi_{y_2} & \Phi_{x_1 y_2} & \Phi_{x_2 y_2} \end{pmatrix}.$$

We are interested in the defining function of the form  $y_2 - x_2 - A(x_1, y_1, x_2, t)$ . Hence, the Monge-Ampere determinant becomes

$$(1.6) \quad (1 - A_{x_2})A_{x_1 y_1} + A_{x_1} A_{x_2 y_1}.$$

Since the Monge-Ampere determinant is symmetric in the  $x$  and  $y$  variables, we see that  $\pi_l$  is a local diffeomorphism if and only if  $\pi_r$  is also.

If each  $\mathcal{C}_t$  is a local canonical graph, we can express the full canonical relation in the form:

$$(1.7) \quad \mathcal{C} = \{(x, \xi, y, \mu, t, \tau) : (y, \mu) = \Xi_t(x, \xi), \tau = q^*(x, t, \xi)\},$$

where  $q^*(x, t, \xi)$  is homogeneous of degree 1 in  $\xi$ .

Sogge showed that the rotational curvature condition is not sufficient in two dimensions (see [So1]). He showed that the following extra assumption is necessary.

**Cone condition.** We say that the canonical relation  $\mathcal{C}$  as in (1.7) satisfies the cone condition if the cone given by the equation  $\tau = q^*(x, t, \xi)$ , has exactly one non-vanishing principal curvature.

We are going to generalize Sogge's result using the following definition due to Phong and Stein:

**Definition 1.3.** Let  $\Sigma_t = \{a \in \mathcal{C}_t : \pi_l \text{ is not locally 1-1}\}$  where  $\pi_l, \pi_r : \mathcal{C}_t \rightarrow T^*\mathbb{R}^2 \setminus 0$  are natural projections. We say that  $\mathcal{C}_t$  is folding of order  $m - 2$  if the following conditions hold

- (1)  $\Sigma_t$  is a submanifold of  $\mathcal{C}_t$  of codimension 1.
- (2)  $\det(d\pi_l)$  and  $\det(d\pi_r)$  vanish of order  $m - 2$  along  $\Sigma_t$
- (3)  $T_a(\Sigma_t) \oplus \text{Ker}(d\pi_l)_a = T_a(\mathcal{C}_t)$
- (4)  $T_a(\Sigma_t) \oplus \text{Ker}(d\pi_r)_a = T_a(\mathcal{C}_t)$

Conditions (3) and (4) are called the transversality conditions.

We note that when  $m = 3$ , the above condition is equivalent to the condition that both  $\pi_l$  and  $\pi_r$  are Whitney folds. (See e.g [MlTr], [Hor1]).

Let  $\pi_{X \times Y}$  denote a projection of the canonical relation onto the  $x$  and  $y$  variables. Let  $V_t = \pi_{X \times Y}(\Sigma_t)$ . It follows from the definition that locally  $V_t$  can be expressed in the form  $\{(x, y) : y_1 = \chi_t(x_1, x_2), y_2 = x_2 + A(x, y_1, t)\}$  where  $\chi_t$  is a local diffeomorphism for each fixed  $(x_2, t)$ . In other words, we can find a smooth function  $\chi_t^*(y_1, x_2)$  such that  $V_t = \{(x, y) : x_1 = \chi_t^*(y_1, x_2), y_2 = x_2 + A(x, y_1, t)\}$  where  $\chi_t^*(y_1, x_2)$  is a local diffeomorphism for each fixed  $(x_2, t)$ .

$V_t$  can be viewed as a parameterization of the zeroes of the Monge-Ampere determinant intersected with the curve distribution  $\mathcal{D}_t$ . In the translation invariant case, we have  $y_2 = x_2 + A(x, y_1, t)$  with  $A(x, y_1, t) = \gamma(\frac{y_1 - x_1}{t})$ , where  $\gamma : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function satisfying the finite type condition described earlier. The Monge-Ampere determinant in this case is just  $\gamma^{(2)}(\frac{y_1 - x_1}{t})$ . Hence, the Monge-Ampere determinant vanishes of order  $m - 2$  along the diagonal  $x = y$ . Consequently,  $V_t$  is just the intersection of  $\mathcal{D}_t$  with the diagonal.

In general, the situation is more complicated. However, locally a curve distribution whose canonical relation has a two-sided fold behaves very much like a translation invariant family. This idea is contained in the following result:

**Lemma 1.1.** Suppose that the canonical relation associated to the curve distribution  $\mathcal{D}_t$  has a two-sided fold of order  $m - 2$ . Then, for each fixed  $(x', t')$  the curve given by the equation  $y_2 = x'_2 + A(x', y_1, t')$  is a curve of finite type  $m$  with a flat point at  $y_1 = \chi_t(x', t')$ .

*Proof.* Without loss of generality let  $\chi'_t(x') = 0$ . We must show that the second partial derivative of  $A(x', y_1, t')$  with respect to  $y_1$  vanishes of order  $m - 2$  at  $y_1 = 0$ .

By our observation in (1.6) the Monge-Ampere determinant is given by  $A_{x_1 y_1} - A_{x_2} A_{x_1 y_1} + A_{x_1} A_{x_2 y_1}$ . By assumption and our observations, the Monge-Ampere determinant vanishes of order  $m - 2$  along  $V_t$  which is given by  $\{(x, y) : y_1 = \chi_t(x_1, x_2), y_2 = x_2 + A(x, y_1, t)\}$ . Hence, along  $V_t$ ,  $A_{x_1} = A_{y_1} \frac{\partial \chi_t}{\partial x_1}$ . Since  $\chi_t$  is a local diffeomorphism for

each fixed  $(x, t)$ , we conclude that under our assumptions,  $A_{y_1 y_1}(x', y_1, t')$  vanishes of order  $m - 2$ . Hence, the curve obtained by fixing  $(x, t)$  is finite type  $m$ .

We can actually prove a little more. Fix  $(x', t')$  as before. We can again assume that  $\chi_{t'}(x') = 0$ . Then, by Lemma 1.1 and the Malgrange Preparation Theorem (See [Hor1]), there exist  $\delta_1, \delta_2, \delta_3 > 0$ , such that if  $|t - t'| < \delta_1$ ,  $|y_1| < \delta_2$ , and  $|x - x'| < \delta_3$ , then

$$(1.8) \quad A(x, y_1, t) = g(x, y_1, t)(y_1^m + a_{m-1}(x, t)y_1^{m-1} + \dots + a_0(x, t)),$$

where  $g(x, y_1, t)$  is smooth,  $g(x', 0, t') \neq 0$ ,  $a_j$ 's are smooth, and  $a_j(x', t') = 0$ .

However, we must conclude that  $a_j = 0$  for  $j > 0$ . If not, let  $t''$  denote a point in  $\{t : |t - t'| < \delta_1\}$  such that  $a_j(t'') \neq 0$ . Then the proof of Lemma 1.1 would show that the Monge-Ampere determinant at  $(x', 0, t'')$  vanishes of order  $j - 2$ . This is a contradiction, since by the assumption of Lemma 1.1, the Monge-Ampere determinant can only vanish of order  $m - 2$ . Hence, we have shown that in a small neighborhood of  $(x', t')$ , the defining function  $A(x, y_1, t)$  is of the form

$$(1.9) \quad A(x, y_1, t) = g(x, y_1, t)(y_1^m + a_0(x, t)),$$

where  $g(x, y_1, t)$  and  $a_0(x, t)$  have the properties described above.

We can now state the variable coefficient version of Theorem 1.1.

**Theorem 1.2.** *Let  $M_t$  be as in (1.3). Let  $\mathcal{M}(f)(x) = \sup_{t>0} M_t(f)(x)$ . Suppose that for each  $t$  the canonical relation is folding of order  $m - 2$  and the cone condition (see [So1], p.352) is satisfied away from  $\Sigma$ . Then,*

$$(1.10) \quad \mathcal{M} : L^p(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2) \text{ for } p > m.$$

### Proof of Theorem 1.1:

Our proof will consist of three main steps. First, we will decompose each  $M_t$  away from the flat point. Then, we will use the method of stationary phase to express each dyadic operator in terms of the Fourier transform of the surface measure on each dyadic piece. We will then use a stretching argument to expose the behavior of our operator near the flat point.

To complete the proof, we will use a scaling argument and a technical lemma in order to reduce our problem to the local smoothing estimates of Mockenhaupt, Seeger, and Sogge ([MSSo2]). In the process, we shall take advantage of the fact that the local smoothing arguments in question are valid under small smooth perturbations.

We now turn to the details. We wish to use the following stationary phase result (See [So2]) which we shall only use for curves in  $\mathbb{R}^2$ .

**Lemma 1.2.** *Let  $S$  be a smooth hypersurface in  $\mathbb{R}^n$  with non-vanishing Gaussian curvature and  $d\mu$  a  $C_0^\infty$  measure on  $S$ . Then,*

$$(1.11) \quad |\widehat{d\mu}(\xi)| \leq \text{const.}(1 + |\xi|)^{-\frac{n-1}{2}}.$$

Moreover, suppose that  $\Gamma \subset \mathbb{R}^n \setminus 0$  is the cone consisting of all  $\xi$  which are normal to some point  $x \in S$  belonging to a fixed relatively compact neighborhood  $\mathcal{N}$  of  $\text{supp } d\mu$ . Then,

$$\left(\frac{\partial}{\partial \xi}\right)^\alpha \widehat{d\mu}(\xi) = O(1 + |\xi|)^{-N} \quad \forall N, \text{ if } \xi \notin \Gamma,$$

$$(1.12) \quad \widehat{d\mu}(\xi) = \sum e^{-i\langle x_j, \xi \rangle} a_j(\xi) \text{ if } \xi \in \Gamma,$$

where the finite sum is taken over all  $x_j \in \mathcal{N}$  having  $\xi$  as the normal and

$$(1.13) \quad \left| \left(\frac{\partial}{\partial \xi}\right)^\alpha a_j(\xi) \right| \leq C_\alpha (1 + |\xi|)^{-\frac{n-1}{2} - |\alpha|}.$$

As we have noted earlier, if  $C$  is finite type  $m$  at  $a_0$ , the curvature at  $a_0$  vanishes of order  $m-2$ . Hence, in order to take advantage of the above lemma, we must first decompose each  $M_t$  away from the flat point. Without loss of generality,  $a_0 = (0, c)$ .

Since our curve is finite type  $m$ , locally we can write  $C(s) = (s, g(s)s^m + c)$ . With that in mind we define  $\rho \in C_0^\infty(\mathbb{R})$  such that  $\text{supp}(\rho) \subset (\frac{B}{2}, 2B) \cup (-2B, -\frac{B}{2})$ ,  $B > 0$ , and  $\sum \rho(2^k y_1) \equiv 1$ , where  $B$  is chosen to be small enough so that the interval  $(-2B, 2B)$  does not contain any other flat points.

Let  $M_t^k(f)(x) = \int_C f(x - ty)\rho(2^k y_1)d\sigma(y)$ . Then,  $M_t(f) = \sum_{k=0}^\infty M_t^k(f)$ . Let  $\mathcal{M}^k(f)(x) = \sup_{t>0} M_t^k(f)(x)$ . Hence, it would suffice to show that

$$(1.14) \quad \mathcal{M}^k : L^p(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2) \text{ with norm } 2^{-k\epsilon(p)} \text{ for some } \epsilon(p) > 0.$$

We can now apply Lemma 1.2 to each  $M_t^k$  which is defined over a dyadic piece of our curve. Simultaneously, we perform a stretching transformation

$$(1.15) \quad y_1 \rightarrow 2^k y_1 \quad y_2 \rightarrow 2^{mk} y_2$$

which sends each dyadic piece to the curve of unit length  $2^{mk}c$  units up the  $y_2$ -axis.

We now apply the lemma to the "stretched" operator  $M_t$ . Keeping in mind the Jacobian of the stretching transformation we get an operator of the form

$$(1.16) \quad \mathcal{G}_k(f)(x, t) = (2\pi)^{-2} 2^{-k} \int_\Gamma e^{i\langle x, \xi \rangle} e^{itq_k(\xi)} e^{t2^{mk}c\xi_2} \frac{a_k(t\xi)}{(1 + t|\xi|)^{\frac{1}{2}}} \hat{f}(\xi) d\xi,$$

where  $\Gamma$  is a fixed cone away from coordinate axes,  $q_k(\xi)$  is homogeneous of degree 1, and  $a_k(t\xi)$  is a symbol of order 0.

It suffices to show that

$$(1.17) \quad \left\| \sup_{t>0} \mathcal{G}_k(f)(x, t) \right\|_p \leq 2^{-k\epsilon(p)} C_p \|f\|_p \text{ for some } \epsilon(p) > 0.$$

We complete the microlocalization of our operator by introducing  $\beta \in C_0^\infty(\mathbb{R})$  satisfying

$$\text{supp}(\beta) \subset [1/2, 2], \quad \sum_{-\infty}^{\infty} \beta(2^{-j}s) = 1, \quad s > 0.$$

Let

$$(1.18) \quad \mathcal{G}_{k,j}(f)(x, t) = (2\pi)^{-2} 2^{-k} \int_{\Gamma} e^{i\langle x, \xi \rangle} e^{itq_k(\xi)} e^{it2^{m_k} c\xi_2} \frac{a_k(t\xi)}{(1+t|\xi|)^{\frac{1}{2}}} \beta(|\xi|/2^j) \hat{f}(\xi) d\xi.$$

If we take the supremum over  $t$  of the absolute value of the difference between  $\mathcal{G}_k(f)(x, t)$  and  $\sum_{j=1}^{\infty} \mathcal{G}_{k,j}(f)(x, t)$ , we see that it is dominated by the Hardy–Littlewood Maximal function of  $f$ . Hence, it suffices to show that

$$(1.19) \quad \left\| \sup_{t>0} \mathcal{G}_{k,j}(f)(x, t) \right\|_p \leq C_p 2^{-k\epsilon(p)} 2^{-j\epsilon'(p)} \|f\|_p$$

for  $m < p < \infty$  and some  $\epsilon(p), \epsilon'(p) > 0$ .

Since we are dealing with dyadic operators, we can use Littlewood–Paley theory (see [So3]) to see that the inequality holds iff

$$(1.20) \quad \left\| \sup_{t \in [1, 2]} |\mathcal{G}_{k,j}(f)(x, t)| \right\|_p \leq C_p 2^{-k\epsilon(p)} 2^{-j\epsilon'(p)} \|f\|_p, \quad m < p < \infty.$$

In order to complete our argument, we need another technical lemma.

**Lemma 1.3.** *Suppose that  $F$  is  $C^1(\mathbb{R})$ . Then if  $p > 1$  and  $1/p + 1/p' = 1$ ,*

$$\sup_{\lambda} |F(\lambda)|^p \leq |F(0)|^p + p \left( \int |F(\lambda)|^p d\lambda \right)^{1/p'} \times \left( \int |F'(\lambda)|^p d\lambda \right)^{1/p}.$$

To prove the lemma, we just express  $F(\lambda)$  as an integral of its derivative using the fundamental theorem of calculus. Then, if we use Hoelder's inequality, we get the desired result (see [So2]).

If we let  $\rho$  be the cutoff function defined previously and apply the lemma, we see that  $\|\sup_{t \in [1,2]} |\rho(t)\mathcal{G}_{k,j}(f)(x,t)|\|_p^p$  is dominated by

$$(1.21) \quad \left\| 2^{-k} \int_{\Gamma} e^{i\langle x, \xi \rangle} e^{itq_k(\xi)} e^{it2^{mk}c\xi_2} \frac{a_k(t\xi)}{(1+t|\xi|)^{\frac{1}{2}}} \beta(|\xi|/2^j) \hat{f}(\xi) d\xi \right\|_p^{p-1} \times \\ \left\| 2^{-k} \int_{\Gamma} e^{i\langle x, \xi \rangle} e^{itq_k(\xi)} e^{it2^{mk}c\xi_2} \beta(|\xi|/2^j) A_{m,k}(\xi,t) \hat{f}(\xi) d\xi \right\|_p,$$

where

$$A_{m,k}(\xi,t) = i(q_k(\xi) + 2^{mk}c\xi_2) \frac{a_k(t\xi)}{(1+t|\xi|)^{\frac{1}{2}}} + \frac{d}{dt} \frac{a_k(t\xi)}{(1+t|\xi|)^{\frac{1}{2}}},$$

where we are taking the  $L^p$  norm with respect to  $\mathbb{R}^2 \times [1,2]$ .

Since  $q_k(\xi) \approx |\xi| \approx 2^j$  on the support of  $\beta$ , we can count the orders of the symbols to see that the expression in (1.21) is dominated by

$$(1.22) \quad C 2^{-k(1-\frac{m}{p})} 2^{-j(\frac{1}{2}-\frac{1}{p})} \|\mathcal{F}_{k,j}f\|_{L^p(\mathbb{R}^3)},$$

where

$$(1.23) \quad \mathcal{F}_{k,j}f(x,t) = \rho(t) \int_{\Gamma} e^{i\langle x, \xi \rangle} e^{itq_k(\xi)} e^{it2^{mk}c\xi_2} \beta(|\xi|/2^j) a_k(t,\xi) \hat{f}(\xi) d\xi,$$

and where  $a_k(t,\xi)$  is a symbol of order 0 in  $\xi$ .

Since  $-(1-\frac{m}{p}) < 0$  if  $p > m$ , we can take  $\epsilon = -1 + \frac{m}{p}$ . Hence, it suffices to show that

$$(1.24) \quad \|\mathcal{F}_{k,j}f\|_{L^p(\mathbb{R}^3)} \leq C_{\epsilon'} 2^{j(\frac{1}{2}-\frac{1}{p}-\epsilon'(p))} \|f\|_{L^p(\mathbb{R}^2)}, \quad m < p < \infty,$$

for some  $\epsilon' > 0$ .

In [MSSo1] and [MSSo2], Mockenhaupt, Seeger, and Sogge proved  $L^p(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^3)$  estimates for the operators of the form

$$(1.25) \quad P_j f(x,t) = \rho(t) \int e^{i\langle x, \xi \rangle} e^{itq(\xi)} a(t,\xi) \hat{f}(\xi) \beta(|\xi|/2^j) d\xi,$$

where  $a(t,\xi)$  is a symbol of order 0 in  $\xi$ , and the Hessian matrix of  $q$  always has rank one. They showed that

$$(1.26) \quad \|P_j f\|_{L^p(\mathbb{R}^3)} \leq C_{\delta} 2^{j(\frac{1}{2}-\frac{1}{p}-\frac{1}{2p}+\delta(p))} \|f\|_{L^p(\mathbb{R}^2)}.$$

Moreover, the proof of [MSSo2] shows that this estimate is valid under small smooth perturbations. The constants in those estimates depend on only a finite number of derivatives



of the phase function and the symbol. The same estimates are still valid for small smooth perturbations of this operator with perhaps larger constants.

In order to take advantage of this fact, we need to make several observations about the operator  $\mathcal{F}_{k,j}$ .

The phase function  $q_k(\xi) + 2^{mk}c\xi_2$  is the phase function of the Fourier transform of the Lebesgue measure on the stretched dyadic piece of the curve  $C$ . As we have noted, without loss of generality  $C(s) = (s, g(s)s^m + c)$ , where  $g \in C^\infty(\mathbb{R})$ ,  $g(0) \neq 0$ , and  $c$  is a constant.

If we take a dyadic piece of this curve, and then dilate and stretch, we get  $(2^kts, 2^{mk}t\{g(s)s^m + c\})$ , and setting  $u = 2^ks$ , we get  $(ut, tg(u/2^k)u^m + 2^{mk}ct)$ . We notice that as  $k \rightarrow \infty$  this family of curves smoothly approaches a fixed family of curves  $C'(s) = (s, g(0)s^m + c)$ . Lemma 1.2 tells us that the phase function of the Fourier transform of the curve carried measure is given by  $e^{\langle x_j, \xi \rangle}$  where  $\xi$  is normal to  $x_j$ . If we note that the Gauss map taking the point on the curve to the normal at that point is smooth as long as the Gaussian curvature does not vanish, we conclude that inside  $\Gamma$ ,  $q_k(\xi) + 2^{mk}c\xi_2$  smoothly converges to the phase function  $q(\xi) + 2^{mk}c\xi_2$ . In particular,

$$|D_\xi^\alpha q_k(\xi)| \leq C(1 + |\xi|)^{1-|\alpha|},$$

where  $C$  does not depend on  $k$ . Moreover,  $q(\xi) + c\xi_2$  is the phase function of the Fourier transform of the Lebesgue measure on the unit length piece of the curve  $C'(s) = (s, g(0)s^m + c)$ .

It is important to note that the Hessian of  $q$  always has rank one. In order to see this, we explicitly compute  $q(\xi)$  up to a multiplicative constant using Lemma 1.2. After computing the unit normal and taking the dot product, we get

$$(1.27) \quad \frac{(m-1)s^m}{(m^2s^{2m-2} + 1)^{\frac{1}{2}}}.$$

If we note that the normal  $\vec{N} = (ms^{m-1}, -1)$ , then we see that

$$(1.28) \quad q(\xi) + c\xi_2 = \text{const.} \frac{\xi_1^{\frac{m}{m-1}}}{\xi_2^{\frac{1}{m-1}}} + c\xi_2.$$

At this point, a direct computation shows that the Hessian of  $q$  has rank one for  $\xi \in \Gamma$ , since  $\Gamma$  is a cone away from the coordinate axes.

We also observe that a similar argument shows that  $\{a_k(t, \xi)\}$  is contained in a bounded subset of symbols of order 0. More precisely, for  $t \in [1, 2]$ ,  $|D_\xi^\alpha a_k(t, \xi)| \leq C_\alpha(1 + |\xi|)^{-\alpha}$ , where  $C_\alpha$  does not depend on  $k$ .

Let  $a(t, \xi) = a_N(t, \xi)$  for  $N$  very large, and let  $q(\xi) + c\xi_2$  be the limiting phase function discussed above. Let

$$(1.29) \quad \mathcal{F}_{k,j}^* f(x, t) = \rho(t) \int_\Gamma e^{i\langle x, \xi \rangle} e^{itq(\xi)} e^{it2^{mk}c\xi_2} a(t, \xi) \beta(|\xi|/2^j) \hat{f}(\xi) d\xi.$$

It is not hard to check that

$$(1.30) \quad \mathcal{F}_{0,j}^* f(x, t) = \mathcal{F}_{k,j}^* f(x_1, x_2 + t2^{mk}c, t).$$

Since the Lebesgue measure is invariant under translations, (1.30) implies that

$$(1.31) \quad \|\mathcal{F}_{k,j}^* f\|_{L^p(\mathbb{R}^3)} = \|\mathcal{F}_{0,j}^* f\|_{L^p(\mathbb{R}^3)}.$$

The operator  $\mathcal{F}_{0,j}^*$  satisfies the aforementioned local smoothing estimates of Mockenhaupt, Seeger and Sogge. More precisely,

$$(1.32) \quad \|\mathcal{F}_{0,j}^* f\|_{L^p(\mathbb{R}^3)} \leq C_\delta 2^{j(\frac{1}{2} - \frac{1}{p} - \frac{1}{2p} + \delta(p))} \|f\|_{L^p(\mathbb{R}^2)}, \quad \delta(p) > 0.$$

The argument above shows that  $\mathcal{F}_{k,j}$  is a smooth family of Fourier integral operators which belong to a bounded subset of Fourier integral operators of order 0 (see e.g [So2], Ch.6). The proof of the local smoothing estimates (see [MSSo2]) shows that the estimates are valid under small perturbations. More precisely, the proof of the local smoothing estimates combined with the statement (1.31) and our observations about the  $\mathcal{F}_{k,j}$ 's, imply that the operator  $\mathcal{F}_{k,j}$ , for a large enough  $k$ , satisfies the estimate (1.32) with perhaps a larger constant.

Hence, if we let  $\epsilon' = \frac{1}{2p} - \delta(p)$ , we see that the estimate (1.24) is satisfied. This completes the proof.

### Proof of Theorem 1.2:

Our argument will be based on a scaling argument, Lemma 1.1, the proof of Theorem 1, and most importantly the local smoothing estimates of Mockenhaupt, Seeger and Sogge ([MSSo2]).

Fix  $(x', t')$ . Consider a curve given by  $y_2 = x_2' + A(x', y_1, t)$ . The remarks following the proof of Lemma 1.1 show that there exists a  $\delta_1 > 0$ , such that for  $|t - t'| < \delta_1$ , this curve is finite type  $m$  with a flat point at  $y_1 = \chi_{t'}(x')$ .

We extend this curve to a translation invariant family by defining  $y_2 = x_2 + A(x', y_1 - x_1 + x_1', t)$ . Again using the proof of Lemma 1.1 and the discussion that followed, we see that for  $|t - t'| < \delta_1$ ,

$$(1.33) \quad A(x', y_1, t) = g(x', y_1, t)((y_1 - \chi_{t'}(x'))^m + a_0(x', t)),$$

where  $a_0(x', t) = 0$ ,  $g(x', y_1, t) \neq 0$  when  $y_1 = \chi_{t'}(x')$ , and  $t \in \{t : |t - t'| < \delta_1\}$ .

We shall first argue that the maximal operator associated to this translation invariant family satisfies the conclusions of Theorem 1.2. As we did in the proof of Theorem 1.1, we localize our operator near the flat point by introducing a cutoff function  $\rho$  with the same properties as before. We perform a stretching transformation

$$(1.34) \quad (y_1 - \chi_{t'}(x')) \rightarrow 2^k(y_1 - \chi_{t'}(x')) \quad y_2 \rightarrow 2^{mk}y_2.$$

The limiting operator under this stretching transformation corresponds to the family of curves given by

$$(1.35) \quad y_2 = x_2 + g(x', \chi_{t'}(x'), t)((y_1 - \chi_{t'}(x'))^m + a_0(x', t)).$$

The only difference between this family of curves and the ones handled in Theorem 1.1 is the  $t$  dependence. However, using a Littlewood-Paley argument as we did in the proof of Theorem 1.1, we see that it suffices to take a supremum over  $t \in [1, 2]$ . Since  $g(x', \chi_{t'}(x'), t)$  does not vanish in intersection of this interval with the set  $\{t : |t - t'| < \delta_1\}$ , we can treat  $g(x', \chi_{t'}(x'), t)$  as our time parameter, and the same argument goes through.

We again use the fact that the variable coefficient estimates of Mockenhaupt, Seeger, and Sogge (See [MSSo2]) are valid under small smooth perturbations. The estimates only depend on the finite number of derivatives of the phase function and the symbol of the corresponding Fourier integral operator. Hence, the estimates which are valid for the limiting operator are also valid for the sufficiently small perturbation of that operator with perhaps a larger constant. This is directly related to the fact that the cinematic curvature condition is stable under smooth changes of coordinates. In other words, if we consider a family of curves satisfying the cinematic curvature condition, this family will still satisfy that condition in a new smooth coordinate system (See [So3]).

In order to complete the proof, we localize the operator corresponding to the general family of functions. As before, we use the cutoff function  $\rho$  and we define

$$(1.36) \quad M_t^k f(x) = \int e^{i\theta(y_2 - x_2 - A(x, y_1, t))} \rho(y_1 - \chi_{t'}(x')) \psi(x, y) f(y) dy.$$

If we perform a stretching transformation sending

$$\begin{aligned} (x_1 - x_1') &\rightarrow 2^k(x_1 - x_1') & (x_2 - x_2') &\rightarrow 2^{mk}(x_2 - x_2') \\ (y_1 - \chi_{t'}(x')) &\rightarrow 2^k(y_1 - \chi_{t'}(x')) & y_2 &\rightarrow 2^{mk}y_2, \end{aligned}$$

we can use Lemma 1.1 to see that our family of curves smoothly converges to the family of translation invariant curves defined above. Moreover, each  $M_t^k$  satisfies the cinematic curvature condition. If we again use the fact that local smoothing estimates used to analyze the translation invariant family are valid under small perturbations, we see that our localized operators satisfy the right estimates.

More precisely, we can show that if we localize near the point  $x'$  in the plane and take the supremum over  $t$  in a sufficiently small neighborhood of some  $t'$ , the resulting maximal operator maps  $L^p(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2)$  for  $p > m$ . As we have noted earlier, we only need to consider  $t \in [1, 2]$  and  $x$  in some compact subset of the plane. Hence, using partitions of unity and the triangle inequality, we complete the proof.

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