DISPERSIVE EFFECTS IN A MODIFIED

KURAMOTO-SIVASHINSKY EQUATION

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Abstract.

We study the limiting behavior of the Kuramoto-Sivashinsky/Korteweg-de Vries equation

$$u_t = -\beta_1 u_{xx} - \beta_2 u_{xxxx} - \delta u_{xxx} - u u_x.$$

We show that in the appropriate sense, the solutions of $(*)$ tend to the solutions of the standard Korteweg-de Vries equation

$$u_t = -\delta v_{xxx} - v v_x,$$

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as $\delta \to \infty$. The proof relies, to a large extent, on precise estimates for oscillatory integrals that yield pointwise bounds on Green's functions.

1. INTRODUCTION

The Kuramoto-Sivashinsky/Korteweg-de Vries (KS/KdV) equation

$$u_t = -\beta_1 u_{xx} - \beta_2 u_{xxxx} - \delta u_{xxx} - u u_x, \tag{1.1}$$

arises in the modelling of flow on an inclined plane [15], and has been studied by a number of authors [2], [4], [5], [6], [9]. Kuramoto-Sivashinsky (KS)-type equations with more general dispersive terms have been proposed as models for a variety of phenomena in fluid dynamics. For example, a hybrid of KS and the fifth-order KdV equation,

$$u_t = -u_{xx} - u_{xxxx} - u_{xxx} - \eta u_{xxxxx} - u u_x,$$

has been studied as a model for long waves in films on an inclined plane [1], [3], [7], and KS with a nonlocal dispersive term given in terms of the modified Bessel functions I_0 and I_1,

$$\begin{aligned} u_t &= -\epsilon u_{xx} - \delta \mathcal{L}(-i\partial_x) u - u u_x, \\ \mathcal{L}(k) &= \frac{k^2 I_1^2(k)}{k I_1^2(k) - k I_0^2(k) + 2 I_0(k) I_1(k)}, \end{aligned} \tag{1.2}$$

has been suggested as a model for core-annular flow in the absence of gravity [10], [13]. As it is the simplest equation combining KS with dispersion, analysis of KS/KdV can be expected to shed light on this larger family of systems.

The well-posedness of (1.1) was shown in [2], using energy arguments analogous to those used in [14] to establish well-posedness for the Kuramoto-Sivashinsky equation ((1.1) with $\beta_1 = \beta_2 = 0$) (also see [6] for work on well-posedness for KS). Additionally, a number of authors have studied (1.1) as a perturbation of the Korteweg-de Vries (KdV) equation, letting $\delta = 1$ and $\beta_1 = \beta_2 = \epsilon \ll 1$, both in the whole-line ($x \in (-\infty, \infty)$) and periodic cases [2], [4], [5], [9]. In particular, for the whole-line case it was shown in [9] that (1.1) possesses localized traveling wave solutions that converge to KdV solitons as $\beta_1 = \beta_2 \to 0$. Furthermore, it was shown in [2] that reasonably smooth solutions of (1.1) tend toward solutions of the KdV equation

$$v_t = -\delta v_{xxx} - v v_x \tag{1.3}$$

as $\beta_1 = \beta_2 \to 0$, uniformly for fixed δ and for bounded time intervals. In [13], numerical evidence was presented suggesting that solutions of (1.2) are well approximated by solutions of (1.1) when the coefficient of the dispersive term is large.

In the present work, we are concerned with a regime where dispersive effects are important. Specifically, we consider (1.1) with $\beta_1 = 1$, $\beta_2 = \delta^a$:

$$u_t = -u_{xx} - \delta^a u_{xxxx} - \delta u_{xxx} - u u_x \tag{1.4}$$

for large δ, and compare its solutions with those of the KdV equation (1.3). We show that in an appropriate sense, solutions of (1.1) on the whole line approach solutions of (1.3) as $\delta \to \infty$, except for a domain of rapid oscillations at very large (but bounded) values of x. Furthermore, the convergence is uniform for times of order δ^ϵ for some $\epsilon > 0$, even for a range of $a < 1$. This long time scale and uniformity for a range of a extend the results of [2].

In what follows, $\|f\|_{L^p(S)}$ denotes the L^p norm of the function f restricted to the domain $x \in S$. If no S is specified, we assume $S = (-\infty, \infty)$.

H^1 refers to a nonhomogeneous Sobolev space; thus,

$$\|f\|_{H^1} = \|f\|_{L^2} + \|f_x\|_{L^2}.$$

The symbol $\hat{}$ denotes the (spatial) Fourier transform, and $\mathcal{X}_I$ denotes the characteristic function of the interval I.

Our main result is

Theorem 1.1. *Let $u(x,t)$ and $v(x,t)$ be solutions of (1.1) and (1.3) respectively, with identical initial data*

$$u(\cdot,0) = v(\cdot,0) = u_0(\cdot,0) \in H^1 \cap W^{1,1},$$

and let

$$w = u - v.$$

Suppose that $\frac{2}{1501} < a < 1$. For any $\epsilon_1 > 0$ sufficiently small and for any $h > 0$, let $r < \frac{1}{2}(1-a) + \epsilon_1$ and define

$$\begin{aligned} I_{\text{bad}} &= \left(-\delta^r - \delta^{\frac{3}{2}(1-a)+\epsilon_1}, \delta^r - \delta^{\frac{1}{2}(1-a)+\epsilon_1}\right), \\ I_{\text{good}} &= \mathbf{R} \backslash I_{\text{bad}}. \end{aligned} \tag{1.5}$$

Then for any $\epsilon_2 > 0$, there exist $\epsilon_3 > 0$, $\epsilon_4 > 0$, and $C > 0$ such that

$$\begin{aligned} \|w(\cdot,t)\|_{L^2(I_{\text{good}})} &\leq C\delta^{-\epsilon_4} \\ \|w(\cdot,t)\|_{L^2(I_{\text{bad}})} &\leq C\delta^{\epsilon_2} \end{aligned}$$

uniformly for $\delta > 0$ and $0 < t < \delta^{\epsilon_3}$, for all v such that $\|v(\cdot,t)\|_{W^{1,1}} + \|v(\cdot,t)\|_{L^2}$ is bounded independent of δ and

$$\|v(\cdot,t)\|_{W^{1,1}(\mathbf{R}\backslash(-\delta^r,\delta^r))} \leq C\delta^{-h} \tag{1.6}$$

for some C independent of $\delta > 0$.

In other words, we show that the difference w between solutions of (1.1) and (1.3) with identical initial data is bounded by a negative power of δ for

δ large, except for a region of rapid oscillations far from the origin. (If more restrictive hypotheses were placed on v, one should be able to remove this region of growing oscillations and, in effect, derive a different asymptotic expansion for the error $w = u - v$ as $\delta \to \infty$. From the details of the proof of the current theorem, it appears that the necessary hypotheses would include decay of the derivatives of the solutions v as functions of δ; in the present work we have chosen to focus instead on the details of the linear analysis and present only the above.)

Theorem 1.1 provides precise information on the spatial structure of the error $w = u - v$; this information is obtained through a careful accounting of dispersive effects, in the form of pointwise bounds on Green's functions defined by oscillatory integrals. In contrast, the results of [2] were obtained using energy estimates and Hausdorff-Young inequalities, in ways similar to those used in [14] to show well-posedness for the Kuramoto-Sivashinsky equation with no dispersive term. These techniques exploited the dissipative nature of the linearized KS/KdV problem. However, they were insensitive to dispersion. The methods used here yield a fine analysis of the local behavior of the linearized kernel, and therefore reveal how dissipation and dispersion interact to produce local decay or growth of the KS/KdV solution.

The proof of Theorem 1.1 is technical, involving different estimates for the kernel in different physical (x) space, Fourier and time domains. The various necessary lemmas are presented in Sections 3 (pointwise kernel estimates) and 4 (L^p kernel estimates). The proof of Theorem 1.1, which uses the technical linear estimates to obtain nonlinear bounds on the error w, is presented earlier, in Section 2.

An announcement of these results appeared in [8]; however, [8] contained an error in that the theorem hypotheses stated there were not strong enough to eliminate the region of growing oscillations as claimed.

2. THE LINEAR KERNEL, AND PROOF OF THEOREM 1.1

To begin the proof of Theorem 1.1, let $u(x,t)$ be a solution of KS/KdV (1.4) and $v(x,t)$ be a solution of KdV (1.3), with initial data

$$v(\cdot,0) = u(\cdot,0) = u_0.$$

Then the error $w = u - v$ satisfies

$$w_t = -w_{xx} - \delta^a w_{xxxx} - \delta w_{xxx} - v_{xx} - \delta^a v_{xxxx} - uu_x - vv_x.$$

Since $w(\cdot,0) \equiv 0$, the Duhamel form of the equation for w is (integrating by parts one time and writing $u = w + v$)

$$w(\cdot,t) = \int_{s=0}^{t} K_1(\cdot,t-s;\delta)\star\left[(v_x + \delta^a v_{xx})_x + \tfrac{1}{2}w(w+2v)\right](\cdot,s)\,ds, \tag{2.1}$$

where K_1 is the first derivative of the linearized KS/KdV Green's function. In general, we will denote the m-th derivative of the Green's function by K_m; thus, up to a constant factor,

$$K_m(x,t;\delta) = \int_{k\in\mathbf{R}} e^{i\Phi(k;x,t,\delta)} k^m e^{(k^2-\delta^a k^4)t}\,dk, \tag{2.2}$$

where

$$\Phi(k;x,t,\delta) = kx - \delta k^3 t$$

is the relevant phase function. We note that points of stationary phase in (2.2) occur when $x \geq 0$ and are given by $\pm k_0$, where

$$k_0 = \sqrt{\frac{x}{3\delta t}}; \tag{2.3}$$

the stationary phase points are nondegenerate except when $x = 0$.

The proof of Theorem 1.1 is based on L^p estimates for the derivative kernels K_m, partitioned in a way that shows where the sole non-decaying part of the kernel is localized in both x and t. The required kernel estimates for $t \geq 1$ are as follows.

Proposition 2.1. *Let m, p, a_{min} and a_{max} be as in any row of Table 1. Then for any $a \in (a_{min}, a_{max})$, there exist positive constants c_1 and ϵ_3 such that*

$$\|K_m(\cdot, \delta, t)\|_{L^p} \leq Ce^{\delta^{-a}t/2}\delta^{-c_1}$$

uniformly for $t \in [1, \delta^{\epsilon_3})$. Furthermore, if $m = 4$ then $c_1 > a$ strictly.

m	p	a_{min}	a_{max}
4	2	$\frac{2}{1501}$	$\frac{257}{256}$
1	2	$\frac{2}{2501}$	$\frac{1281}{1280}$
2	1	$\frac{4}{7}$	$\frac{11903}{12800}$

Table 1. Key parameter values for Proposition 2.1.

The required kernel estimates for $t \in (0, 1)$ are somewhat more complicated. In particular, the interaction between stationary phase and dissipation results in rapid oscillations, at a large distance from the origin, when $t = \mathcal{O}(\delta^{-a})$. To obtain estimates valid for $t < 1$, it is therefore helpful to first isolate the stationary phase points by breaking the kernel K_m into three pieces with a cutoff function $\Psi(k)$. Thus,

$$K_m = K_m^A + K_m^B + K_m^C,$$

where

$$\begin{aligned} K_m^A(x, t; \delta) &= \int e^{i[kx - \delta k^3 t]}\Psi(k) k^m e^{(k^2 - \delta^a k^4)t}\, dk, \\ K_m^B(x, t; \delta) &= \int_{|k| \leq k_0} e^{i[kx - \delta k^3 t]}(1 - \Psi(k)) k^m e^{(k^2 - \delta^a k^4)t}\, dk, \\ K_m^C(x, t; \delta) &= \int_{|k| \geq k_0} e^{i[kx - \delta k^3 t]}(1 - \Psi(k)) k^m e^{(k^2 - \delta^a k^4)t}\, dk. \end{aligned} \tag{2.4}$$

Here Ψ is a C^∞ function such that

$$\operatorname{supp} \Psi \subset \{k : ||k| - k_0| \leq k_0/2\},$$

$$\operatorname{supp} \Psi' \subset \{k : k_0/4 \leq ||k| - k_0| \leq k_0/2\},$$

and for all $m \geq 0$ there exists $C = C(m) > 0$ such that

$$\|\partial_k^m \Psi(k)\|_{L^\infty} \leq C|k_0|^{-m};$$

if $x < 0$ we define

$$k_0 = \sqrt{|x|/3\delta t}$$

analogously to (2.3), although k_0 is a stationary phase point only for $x \geq 0$.

Proposition 2.2. *Let m, p, a_{min} and a_{max} be as in any row of Table 2. Then for any $a \in (a_{min}, a_{max})$, there exists $c > 0$ such that*

$$\|K_m^B(\cdot, \delta, t) + K_m^C(\cdot, \delta, t)\|_{L^p} \leq Ce^{\delta^{-a}t/2}\delta^{-c}$$

uniformly for $\delta > 1$ and $t \in (0, 1)$. Furthermore, if $m = 4$ then $c > a$ strictly.

m	p	a_{min}	a_{max}
4	2	0	1
1	2	0	$\frac{7}{3}$
2	1	$\frac{1000}{1499}$	$\frac{39}{37}$

Table 2. Key parameter values for Proposition 2.2 .

Proposition 2.3. *Let $a \in (0, 2)$. Then for any $\epsilon_5 > 0$ small, there exist $C > 0$ and $\epsilon_6 > 0$ such that for all $\delta \geq 1$,*

$$\begin{aligned} &\int_0^{\delta^{-a-\epsilon_5}} \|K_m(\cdot; \delta, t)\|_{L^2}\, dt \leq C\delta^{-a-\epsilon_6} \quad \text{if } m < 7/2 \\ &\int_{\delta^{-a+\epsilon_5}}^1 \|K_m(\cdot; \delta, t)\|_{L^2}\, dt \leq C\delta^{-a-\epsilon_6} \quad \text{if } m > 7/2. \end{aligned} \tag{2.5}$$

Proposition 2.4. *Let $a \in (0, 2)$. Then for any $\epsilon_5 > 0$ and $\epsilon_1 > 0$ sufficiently small, there exist $C > 0$, $\epsilon_2 > 0$ and $\epsilon_6 > 0$ such that for all $\delta \geq 1$, the following holds:*

a) *If $m \geq 0$, then*

$$\int_{\delta^{-a-\epsilon_5}}^{\delta^{-a+\epsilon_5}} \|K_m^A(\cdot; \delta, t)\|_{L^2(\delta^{\frac{1}{2}(1-a)-\epsilon_1}, \delta^{\frac{3}{2}(1-a)-\epsilon_1})}\, dt \leq C\delta^{-a+\epsilon_2}. \tag{2.6}$$

b) If $m > 3 - 2/p$, then

$$\int_{\delta^{-a-\epsilon_5}}^{\delta^{-a+\epsilon_5}} \|K_m^A(\cdot;\delta,t)\|_{L^2(0,\delta^{\frac{1}{2}(1-a)-\epsilon_1})}\,dt \le C\delta^{-a-\epsilon_6}, \tag{2.7}$$

c) If $m \ge 0$, then

$$\int_{\delta^{-a-\epsilon_5}}^{\delta^{-a+\epsilon_5}} \|K_m^A(\cdot;\delta,t)\|_{L^2(\delta^{\frac{3}{2}(1-a)-\epsilon_1},\infty)}\,dt \le C\delta^{-a-\epsilon_6}. \tag{2.8}$$

Furthermore, ϵ_2 can be made arbitrarily small by choosing ϵ_5 sufficiently small.

Note. In Propositions 2.3 and 2.4, the various ϵ_i $(i = 5, 4, 3)$ have been named to correspond with their roles in the proof of the theorem below.

We now proceed to use the above estimates on the linear kernel and its derivatives to obtain the nonlinear estimates necessary to prove Theorem 1.1. (The proofs of Propositions 2.1–2.4 will be given at the end of this section.)

Proof of Theorem 1.1. We begin by rewriting the Duhamel equation (2.1) as

$$\begin{aligned} w(\cdot,t) = &\int_{s=0}^{t} K_1(\cdot,t-s;\delta)\star v_x(\cdot,s)\,ds - \int_{s=0}^{t} \delta^a K_4(\cdot,t-s;\delta)\star v(\cdot,s)\,ds \\ &+ \tfrac{1}{2}\int_{s=0}^{t} K_1(\cdot,t-s;\delta)\star(w^2+2vw)(\cdot,s)\,ds \end{aligned} \tag{2.9}$$

where $w = u - v$ as in (2.1). The assumption that $\|v\|_{W^{1,1}}$ is uniformly bounded, together with Propositions 2.1, 2.2 and 2.4, implies a bound on the first integral on the right-hand side of the expression for w (2.9):

$$\left\| \int_{s=0}^{t} K_1(\cdot,t-s;\delta)\star v_x(\cdot,s)\,ds \right\|_{L^2} \le C\delta^{-\epsilon_6} \tag{2.10}$$

for some $\epsilon_6 > 0$, uniformly for $0 < t < \delta^{\epsilon_3}$ for some $\epsilon_3 > 0$ as in Proposition 2.1.

To handle the second integral in (2.9) requires several steps. First, reasoning as for (2.10) we have (by Proposition 2.2)

$$\left\| \int_{s=0}^{t} \delta^a \left(K_4^B + K_4^C\right)(\cdot, t-s; \delta) \star v(\cdot, s)\, ds \right\|_{L^2} \leq C\delta^{-\epsilon_6} \tag{2.11}$$

for some $C > 0$ independent of $\delta > 1$ and some $\epsilon_6 > 0$ which we take to be the same as in (2.10). Next, Proposition 2.3 shows that the bound (2.11) holds for the stationary-phase kernel K_4^A as well if we omit t near δ^{-a}. Specifically, we have (redefining ϵ_6 if necessary)

$$\left\| \int_{\substack{s=0 \\ s \notin (\delta^{-a-\epsilon_5}, \delta^{-a+\epsilon_5})}}^{t} \delta^a K_4^A(\cdot, t-s; \delta) \star v(\cdot, s)\, ds \right\|_{L^2} \leq C\delta^{-\epsilon_6} \tag{2.12}$$

for arbitrary $\epsilon_5 > 0$.

It remains only to consider $\int_{s=\delta^{-a-\epsilon_5}}^{\delta^{-a+\epsilon_5}} \delta^a K_4^A(\cdot, t-s; \delta) \star v(\cdot, s)\, ds$. (For convenience we now assume $t > \delta^{-a+\epsilon_5}$; the case of smaller t differs only trivially from this case.) Writing the convolution explicitly, we first split the integral into two parts:

(2.13)

$$\begin{aligned}
&\int_{s=\delta^{-a-\epsilon_5}}^{\delta^{-a+\epsilon_5}} \delta^a K_4^A(\cdot, t-s; \delta) \star v(\cdot, s)\, ds = \\
&= \delta^a \int_{s=\delta^{-a-\epsilon_5}}^{\delta^{-a+\epsilon_5}} \int_{y=x-\delta^{1/2(1-a)+\epsilon_1}}^{x-\delta^{\frac{3}{2}(1-a)-\epsilon_1}} K_4^A(x-y, t-s; \delta) v(y, s)\, dy\, ds \\
&+ \delta^a \int_{s=\delta^{-a-\epsilon_5}}^{\delta^{-a+\epsilon_5}} \left[\int_{y=-\infty}^{x-\delta^{1/2(1-a)+\epsilon_1}} + \int_{y=x-\delta^{\frac{3}{2}(1-a)-\epsilon_1}}^{\infty} \right] K_4^A(x-y, t-s; \delta) \cdot \\
&\cdot v(y, s)\, dy\, ds \\
&= I_{\text{Kbad}} + I_{\text{Kgood}}
\end{aligned}$$

where $\epsilon_1 > 0$ is an arbitrary small constant. (The designations I_{Kbad} and I_{Kgood} reflect the fact that the only growing term will prove to be isolated within I_{Kbad}.)

The hypothesis (1.6) on the localization of v, together with the bounds (2.7) and (2.8) on K_4^A, implies that

$$\|I_{\text{Kgood}}\|_{L^2} \leq C\delta^{-\epsilon_6}$$

for some $C > 0$ independent of $\delta > 1$ and some $\epsilon_6 > 0$ which we take to be the same as in (2.12). To handle I_{Kbad}, we let $\mathcal{X}$ denote the characteristic function of $\{y : |y| < \delta^r\}$, where r is as in the hypothesis of the theorem, and further split I_{Kbad} into two parts:

$$\begin{aligned}
I_{\mathrm{Kbad}} =& \delta^a \int_{s=\delta^{-a-\epsilon_5}}^{\delta^{-a+\epsilon_5}} \int_{y=x-\delta^{1/2(1-a)+\epsilon_1}}^{x-\delta^{\frac{3}{2}(1-a)-\epsilon_1}} K_4^A(x-y,t-s;\delta) v(y,s) \mathcal{X}(y)\, dy\, ds \\
&+ \delta^a \int_{s=\delta^{-a-\epsilon_5}}^{\delta^{-a+\epsilon_5}} \int_{y=x-\delta^{1/2(1-a)+\epsilon_1}}^{x-\delta^{\frac{3}{2}(1-a)-\epsilon_1}} K_4^A(x-y,t-s;\delta) \cdot \\
&\cdot v(y,s)\,(1-\mathcal{X}(y))\, dy\, ds \\
=& I_{\mathrm{Kbadvin}} + I_{\mathrm{Kbadvout}}
\end{aligned}$$

(here ϵ_1 is as in (2.13)). Note that if

$$x < -\delta^r - \delta^{\frac{3}{2}(1-a)+\epsilon_1}$$

or

$$x > \delta^r - \delta^{\frac{1}{2}(1-a)+\epsilon_1},$$

i.e. if $x \in I_{\mathrm{good}}$ (as defined in (1.5)), then

$$I_{\mathrm{Kbadvin}} = 0.$$

On the other hand, if $x \in I_{\mathrm{bad}} = \mathbf{R} \backslash I_{\mathrm{good}}$, then using the bound (2.6) on K_4^A and the hypothesis that $\|v\|_{L^1}$ is uniformly bounded, we obtain

$$\|I_{\mathrm{Kbadvin}}(\cdot,t)\|_{L^2} = \|I_{\mathrm{Kbadvin}}(\cdot,t)\|_{L^2(I_{\mathrm{bad}})} \leq C\delta^{\epsilon_2} \tag{2.14}$$

for some $C > 0$ independent of $\delta > 1$ and some $\epsilon_2 > 0$ which can be made arbitrarily small for a given ϵ_1, depending on the choice of ϵ_5. Furthermore, the bound (2.6) combined with the hypothesis (1.6) on the localization of v implies that

$$\|I_{\mathrm{Kbadvout}}\|_{L^2} \leq C\delta^{\epsilon_2 - h}.$$

By assuming $\epsilon_2 < h$, therefore, we find that the only non-decaying contribution to the second integral in (2.9) is I_{Kbadvin}, for $x \in I_{\text{bad}}$.

We now return to the Duhamel equation for w (2.9). Using the bounds obtained above on the first two integrals and letting

$$A = I_{\text{Kbadvin}}, \qquad B = w - A,$$

we obtain from (2.9)

$$\|B(\cdot,t)\|_{L^2} \leq C\delta^{-\epsilon_6} + \tfrac{1}{2}\left\| \int_{s=0}^{t} K_1(\cdot, t-s; \delta) \star \left[B^2 + 2(A+v)B + A(A+2v)\right](\cdot, s)\, ds \right\|_{L^2}$$

where $C > 0$ is independent of $\delta > 1$ (here $\epsilon_6 > 0$ is as in the bounds above, and we may assume $\epsilon_2 - h < -\epsilon_6$ by our choice of ϵ_6). We define

$$M(t) = \sup_{s \in [0,t]} \|B(\cdot, s)\|_{L^2}.$$

Using the bounds from Propositions 2.1–2.4 and (2.14) on K and I_{Kbadvin}, along with (2.16) and the theorem hypotheses bounding v, then yields the bound

$$M(t) \leq C\delta^{-\epsilon_6} + Ce^{\delta^{-a}t/2} t\delta^{2\epsilon_2 - c_1}\left[M(t)^2 + M(t) + 1\right] \tag{2.15}$$

where c_1 as as in Proposition 2.1 (note that we may assume $\epsilon_3 < c_2$, where c_2 is as in Proposition 2.1).

We note that for $\epsilon_3 > 0$ sufficiently small, there is some $\epsilon_4 > \epsilon_6$ (redefining ϵ_6 if necessary) such that $e^{\delta^{-a}t/2} t\delta^{2\epsilon_2 - c_1} < \delta^{-\epsilon_4}$ for all $t \in (0, \delta^{\epsilon_3})$. Furthermore, $M(t)$ is a continuous function of t with $M(0) = 0$. Therefore (taking the roots of the quadratic in M), the inequality (2.15) implies that (for some $C > 0$ independent of $\delta > 1$)

$$M(t) \leq \tfrac{1}{2}\left[C\delta^{\epsilon_4} - 1 - C\delta^{\epsilon_4 - \epsilon_6} - \sqrt{\left(C\delta^{\epsilon_4} - 1 - C\delta^{\epsilon_4-\epsilon_6}\right)^2 - 4}\right]$$

and thus (letting δ go to infinity) $M(t) \le C\delta^{-\epsilon_4}$ for all $t < \epsilon_3$.

To complete the proof of the theorem, it remains only to prove Propositions 2.1–2.4.

Proof of Proposition 2.1. The desired estimates are obtained by patching together the estimates in Lemmas 4.1–4.7, using the parameters specified in a given row of Table 3.

m	p	η	$n_{C,\infty}$	r_C	ρ_C	$n_{C,0}$	r_B	ρ_B	n_B	a_{min}	a_{max}
4	2	$\frac{5}{16}$	8	$\frac{17}{64}$	0	1	$\frac{29}{64}$	$\frac{1}{8}$	100	$\frac{2}{1501}$	$\frac{257}{256}$
1	2	$\frac{41}{256}$	2	$\frac{2}{5}$	0	1	$\frac{1}{4}$	$\frac{1}{4}$	20	$\frac{2}{2501}$	$\frac{1281}{1280}$
2	1	$\frac{31}{200}$	4	$\frac{119}{256}$	0	2	$\frac{1}{4}$	$\frac{1}{4}$	20	$\frac{4}{7}$	$\frac{11903}{12800}$

Table 3. Parameter values for the proof of Proposition 2.1.

Specifically, we use the parameters subscripted "C" or "C,∞" in Lemma 4.1 (which provides estimates on the operator $K_m^{C,\infty}$), "C" or "$C,0$" in Lemmas 4.2 and 4.3 (which estimate $K_m^{C,0}$), and "B" in Lemmas 4.6 and 4.7 (which estimate K_m^B). Combining the resulting estimates with Lemmas 4.5 and 4.6, we obtain

$$\|K_m(\cdot,\delta,t)\|_{L^p(\mathbf{R})} \le Ce^{\delta^{-a}t}\delta^{-\alpha}t^{\beta}$$

for some $C > 0$ independent of $\delta > 1$, for some α, $\beta > 0$. The bounds stated in the proposition therefore hold for $1 < t < \delta^{\alpha/\beta-\epsilon}$, for any $\epsilon > 0$.

The proof of Proposition 2.2 is similar to that of Proposition 2.1. Advantageous combinations of parameters are given in the Table 4 (where "large" means "arbitrarily large").

m	p	η	$n_{C,\infty}$	r_C	ρ_C	$n_{C,0}$	r_B	ρ_B	n_B	a_{min}	a_{max}
4	2	$\frac{3}{16}$	large	$\frac{1}{8}$	$\frac{1}{16}$	1	$\frac{1}{2}$	$\frac{1}{2}$	2	0	1
1	2	$\frac{5}{16}$	10	$\frac{1}{8}$	$\frac{1}{16}$	1	$\frac{1}{2}$	$\frac{1}{2}$	2	0	$\frac{21}{23}$
1	2	$\frac{3}{16}$	1	$\frac{1}{8}$	$\frac{1}{16}$	1	$\frac{1}{2}$	$\frac{1}{2}$	2	$\frac{1}{2}$	$\frac{7}{3}$
2	1	$\frac{3}{16}$	10	$\frac{1}{8}$	$\frac{1}{16}$	1	$\frac{1}{2}$	$\frac{1}{2}$	2	$\frac{1000}{1499}$	$\frac{39}{37}$

Table 4. Parameter values for the proof of Proposition 2.2 .

Proof of Proposition 2.3. Integrating the simple Hausdorff-Young estimate

$$\|K_m(\cdot,\delta,t)\|_{L^2} \leq (\delta^a t)^{-(2m+1)/8} \tag{2.16}$$

from $t = 0$ to $t = \delta^{-a-\epsilon_5}$ with $m < 7/2$, or from $t = \delta^{-a+\epsilon_5}$ to $t = 1$ with $m > 7/2$, establishes (2.5).

Proof of Proposition 2.4. In the following, ϵ is used generically and may change when this is unlikely to cause confusion.

Any of a variety of methods gives the pointwise bound

$$|K_m^A(x,\delta,t)| \leq Cx^{(m-3)/2}(\delta t)^{-(m-3)/2}(\delta^a t)^{-1}e^{-x^2/(\delta^{2-a}t)},$$

for some $C > 0$ independent of x and t, uniformly for $\delta > 1$. For $t \in (\delta^{-a-\epsilon}, \delta^{-a+\epsilon})$, then, we have

$$|K_m^A(x,\delta,t)| \leq Cx^{(m-3)/2}\delta^{-(1-a)(m-3)/2}\delta^{\epsilon}e^{-x^2/\delta^{2-2a+\epsilon}}.$$

Note that

$$\left[\int_0^{\delta^r} x^{p(m-3)/2}\,dx\right]^{1/p} \delta^{-(1-a)(m-3)/2} \leq \delta^{-\epsilon}$$

for some $\epsilon > 0$ if and only if $m > 3 - 2/p$ and

$$r\left(\frac{m-3}{2} + \frac{1}{p}\right) < \frac{m-3}{2}(1-a).$$

Letting $m = 4$, $p = 2$ and integrating with respect to t establishes (2.7). Analogously, letting $m = 0$, $p = 2$ establishes (2.8). Finally, replacing K_m by K_m^A in (2.16) and integrating with respect to t establishes (2.6).

3. POINTWISE BOUNDS ON GREEN'S FUNCTIONS

Propositions 2.1–2.4, the L^p bounds on Green's functions used to prove Theorem 1.1, are proved by first obtaining pointwise bounds through a detailed analysis of Fourier integrals, and then integrating the pointwise bounds to obtain L^p estimates. In this section we present a number of lemmas giving the necessary pointwise bounds on the kernel components K_m^A, K_m^B and K_m^C (2.4), and give representative proofs. The corresponding L^p estimates are presented in Section IV.

Lemma 3.1. *Suppose that $x > 0$ and $t > 0$. Then for any $\epsilon > 0$ sufficiently small, there exists $C > 0$ independent of $\delta > 1$, x, and t such that*

(3.1)
$$|K_m^A(x,t;\delta)| \leq Ce^{\delta^{-a}t/2}e^{-\delta^{a-2}x^2/2t} \cdot \left[\begin{array}{c} x^{(m-\epsilon)/2-1/4}\delta^{-(m-\epsilon)/2-1/4}t^{-(m-\epsilon)/2-3/4} \\ +x^{m/2+\epsilon/2+1/4}\delta^{-(2m+3)/4-\epsilon/2}t^{-(2m+5)/4-\epsilon/2} \\ +x^{(m+3\epsilon)/2+3/4}\delta^{[-5+(1+2\epsilon)a]/4-(m+3\epsilon)/2}t^{-(m+3\epsilon)/2-7/4} \end{array} \right].$$

Furthermore, if in addition there exists $\gamma > 0$ such that

$$x > \delta^{1-(a-\gamma)/2}t^{(1-\gamma)/2},$$

then

$$|K_m^A(x,t;\delta)| \leq Ce^{\delta^{-a}t/2}e^{-\delta^{\gamma}t^{-\gamma}/2}\delta^{-(a-\gamma)(m+1)/4}t^{-(1+\gamma)(m+1)/4}. \tag{3.2}$$

Proof. The proof of Lemma 3.1 is a stationary-phase argument. We assume $k > 0$; the case when $k < 0$ is similar. To begin, note that Taylor's theorem implies that

$$\Phi(k) = \Phi(k_0) + \tfrac{1}{2}\Phi''(k_0)(z)^2,$$

where

$$(3.3) \qquad z = Z(k) = \sqrt{\int_{k_0}^{k} \frac{\Phi''(s)}{\Phi''(k_0)}(k-s)\,ds}.$$

Letting

$$(3.4) \qquad \psi(k) = k^m e^{(k^2-\delta^a k^4)t}\Psi(k)$$

and

$$v(z) = \frac{\psi(Z^{-1}(z))}{Z'(Z^{-1}(z))},$$

therefore, we have (taking a Fourier transform)

$$\begin{aligned} K_m^A(x,t;\delta) &= e^{i\Phi(k_0)t}\int e^{\frac{1}{2}iz^2\Phi''(k_0)t}\,v(z)\,dz \\ &= \frac{Ce^{i\Phi(k_0)t}}{\sqrt{\Phi''(k_0)t}}\int e^{-\frac{1}{2}i\xi^2/(\Phi''(k_0)t)}\,\hat{v}(\xi)\,d\xi, \end{aligned}$$

where the constant C (both here and in the remainder of the proof) may be chosen to be independent of $m \in [0,4]$ and of δ and a for δ^a bounded away from 0.

Proceeding with the Plancherel lemma as in the standard stationary phase argument [11], [12] then gives

$$(3.5) \qquad |K_m^A(x,t;\delta)| \le \frac{C}{\sqrt{\Phi''(k_0)t}}\|v\|_{H^{1/2+\epsilon}}$$

for any $\epsilon > 0$. Reversing the change of variables $Z : k \to z$, we obtain

$$\begin{aligned} \|v\|_{L^2}^2 &\le \int \left(\frac{|\psi|^2}{|Z'|}\right)\,dk, \\ \|v\|_{H^1}^2 &\le \int \left(\frac{|\psi|^2}{|Z'|} + \frac{2|\psi'|^2}{|Z'|^3} + \frac{2|\psi|^2|Z''|^2}{|Z'|^5}\right)\,dk. \end{aligned}$$

Since $\Phi''(k) = -6\delta tk$, furthermore, we have from (3.3)

$$(3.6) \qquad Z(k) = \frac{1}{\sqrt{k_0}}\sqrt{\int_{k_0}^{k} s(k-s)\,ds} = \frac{1}{\sqrt{k_0}}\sqrt{\tfrac{1}{2}k(k^2-k_0^2) - \tfrac{1}{3}(k^3-k_0^3)}.$$

It follows that

$$Z'(k) = C\frac{1+\frac{k}{k_0}}{\sqrt{2+\frac{k}{k_0}}} \tag{3.7}$$

(here C is independent of k/k_0 and k_0), so that $1/Z'(k)$ is a bounded function of $k > 0$, uniformly for $k_0 > 0$. In addition, we have

$$Z''(k) = C\frac{3+\frac{k}{k_0}}{k_0(2+\frac{k}{k_0})^{3/2}}, \tag{3.8}$$

which is $1/k_0$ times a bounded function of $k > 0$, uniformly for $k_0 > 0$. Plugging (3.7) and (3.8) into (3.6), we obtain

$$\begin{aligned} \|v\|_{L^2}^2 &\leq C\int |\psi|^2\,dk, \\ \|v\|_{H^1}^2 &\leq \left(1+\frac{C}{k_0^2}\right)\int |\psi|^2\,dk + C\int |\psi'|^2\,dk. \end{aligned} \tag{3.9}$$

Using the inequality

$$\|v\|_{H^{1/2+\epsilon}} \leq \|v\|_{H^1}^{\frac{1}{2}+\epsilon}\|v\|_{L^2}^{\frac{1}{2}-\epsilon},$$

it follows from (3.5) that

$$\begin{aligned} &|K_m^A(x,t;\delta)| \leq \\ &\frac{C}{\sqrt{\Phi''(k_0)t}}\left(\begin{array}{c} \left[1+\frac{1}{k_0^2}\right]^{\frac{1}{2}(\frac{1}{2}+\epsilon)}\left[\int |\psi(k)|^2\,dk\right]^{\frac{1}{2}} \\ +\left[\int |\psi(k)|^2\,dk\right]^{\frac{1}{2}(\frac{1}{2}-\epsilon)}\left[\int |\psi'(k)|^2\,dk\right]^{\frac{1}{2}(\frac{1}{2}+\epsilon)} \end{array}\right). \end{aligned} \tag{3.10}$$

We now must estimate ψ and its derivatives. From the definition (3.4) we find that

$$\begin{aligned} &\int |\psi(k)|^2\,dk \leq Ck_0^{2m+1}e^{2(k_0^2-\delta^a k_0^4)t}, \\ &\int |\psi'(k)|^2\,dk \leq C\left(k_0^{2m-1}+k_0^{2m+3}+\delta^a k_0^{2m+7}\right)e^{2(k_0^2-\delta^a k_0^4)t}. \end{aligned}$$

Furthermore, for any k we have

$$e^{(k^2-\delta^a k^4)t} \leq e^{\delta^{-a}t/2}e^{-\delta^a k^4 t/2}. \tag{3.11}$$

The inequality (3.10) thus implies (via Minkowski's inequality) that

$$|K_m^A(x,t;\delta)| \leq C(\Phi''(k_0)t)^{-\frac{1}{2}} \begin{pmatrix} k_0^{m+\frac{1}{2}} + k_0^{m-\epsilon} \\ +k_0^{m+1+\epsilon} + \delta^{(\frac{1}{2}+\epsilon)a/2} k_0^{m+2+3\epsilon} \end{pmatrix} \cdot e^{\frac{1}{2}\delta^{-a}t} e^{-\frac{1}{2}\delta^a k_0^4 t}.$$

For $\epsilon > 0$, $k_0^{m+1/2}$ is bounded above and below by constant multiples of $k_0^{m-\epsilon}+k_0^{m+1+\epsilon}$. Recalling that $k_0 = \sqrt{x/3\delta t}$ (2.3), and thus $\Phi''(k_0) = 6\sqrt{x\delta t}$, then gives the bounds stated in (3.1).

To complete the proof of the lemma, note that if $x > \delta^{1-(a-\gamma)/2} t^{(1-\gamma)/2}$ (for some $\gamma > 0$), then $\delta^a k_0^4 t \geq \delta^\gamma t^{-\gamma}$. The bounds stated in (3.2) then follow from (3.11) and the simple bound

$$|K_m^A(x,t;\delta)| \leq \int k^m e^{-\delta^a k^4 t} \Psi(k)\, dk.$$

(Note that van der Corput lemma cannot be applied directly to prove the lemma, though techniques closely related to the proof of this lemma are very much in play. The point is that the presence of non-oscillatory terms requires a more careful analysis.)

We now turn to estimating the nonstationary-phase integrals K_m^B and K_m^C defined in (2.4). The following definitions will prove useful:

$$\begin{aligned}
\alpha_1(n) &= \frac{n + (n \bmod 2)}{2}, \\
\alpha_2(n) &= n \bmod 2, \\
\alpha_3(n) &= \frac{n - 1 + (n - 1 \bmod 2)}{2}, \\
\alpha_4(n) &= \frac{3 + n - (n - 1 \bmod 4)}{4}, \\
\alpha_5(n) &= 3 - (n - 1 \bmod 4), \\
\alpha_6(n) &= \frac{3(n-1) + (n - 1 \bmod 4)}{4}, \\
\alpha_7(n) &= (n-1) \bmod 2.
\end{aligned} \tag{3.12}$$

Lemma 3.2. *Suppose that $\delta > 0$, $t > 0$ and $m \geq 0$. Then for any $n \geq 1$, there is a constant C independent of x, δ and t such that*

(3.13)
$$|K_m^B(x,t;\delta)| \leq Ce^{\delta^{-a}t}\cdot \\ \cdot\left(\begin{array}{c} x^{(m-n-1)/2}\delta^{-(m+n-1)/2}t^{-(m+n-1)/2} \\ +x^{(m+\alpha_2(n)-2n+1)/2}\delta^{-(m+\alpha_2(n)+1)/2}t^{\alpha_1(n)-(m+\alpha_2(n)+1)/2} \\ +x^{(m-n+1)/2}\delta^{-(m+n+1)/2}t^{-(m-n+1)/2} \\ +x^{(m+\alpha_5(n)-2n+1)/2}\delta^{a\alpha_4(n)-(m+\alpha_5(n)+1)/2}t^{\alpha_4(n)-(m+\alpha_5(n)+1)/2} \\ +x^{(m+n+1)/2}\delta^{an-(m+3n+1)/2}t^{-(m+n+1)/2} \\ +x^{(m-3n+1)/2}\delta^{-(m-n+1)/2}t^{-(m-n+1)/2} \end{array}\right)$$

and

$$|K_m^B(x,t;\delta)| \leq Ce^{\delta^{-a}t}\cdot \tag{3.14}$$
$$\cdot\left(\begin{array}{c} x^{-(n+1)}\delta^{-a(m+n+1)/4+1}t^{-(m+n+1)/4+1} \\ +\left(\begin{array}{c} x^{-(n+3+[(n-1)\bmod 2])/2}\cdot \\ \cdot\delta^{-a(m+(n\bmod 2)+1)/4-(n-3+[(n-1)\bmod 2])/2}\cdot \\ \cdot t^{-(m+2n+[(n-1)\bmod 2]-4)/4} \end{array}\right) \\ +x^{-n}\delta^{-a(m+\alpha_2(n)+1)/4}t^{\alpha_1(n)-(m+\alpha_2(n)+1)/4} \\ +x^{-n}\delta^{-a(m+n+1)/4}t^{-(m-3n+1)/4} \\ +x^{-n}\delta^{a(\alpha_4(n)-(m+\alpha_5(n)+1)/4)}t^{\alpha_4(n)-(m+\alpha_5(n)+1)/4} \\ +x^{-n}\delta^{-a(m-n+1)/4}t^{-(m-n+1)/4} \end{array}\right)$$

Proof of Lemma 3.2. We begin by establishing preliminary bounds on the integrand in the definition (2.4) of K_m^B. Since $\Phi \neq 0$ on $\mathrm{supp}(1-\Psi)$, integrating (2.4) by parts gives (up to a constant factor)

$$\begin{aligned} &K_m^B(x,t;\delta) = \\ &= \int_{|k|\leq 3k_0/4} e^{i\Phi(k)}\left(-\partial_k\circ\frac{1}{\Phi'(k)}\right)^n\left[k^m e^{(k^2-\delta^a k^4)t}(1-\Psi(k))\right]\,dk \\ &= \int_{|k|\leq 3k_0/4} e^{i\Phi(k)}\left(-\partial_k\circ\frac{1}{3\delta t(k_0^2-k^2)}\right)^n\left[k^m e^{(k^2-\delta^a k^4)t}(1-\Psi(k))\right]\,dk \end{aligned}$$

for any $n \geq 0$. To obtain the desired bounds, we first note that

$$\left|\partial_k^n\left(\frac{1}{k_0^2-k^2}\right)\right| \leq C\left(\frac{1}{(k_0^2-k^2)^{n+1}}\right)\begin{cases} k^n + kk_0^{n-1} & (n \text{ odd}) \\ k^n + k_0^n & (n \text{ even}), \end{cases}$$
$$\left|\partial_k^n e^{k^2 t}\right| \leq Ct^{\alpha_1}k^{\alpha_2}(1+(k^2t)^{\alpha_3})e^{k^2t},$$
$$\left|\partial_k^n e^{-\delta^a k^4 t}\right| \leq C(\delta^a t)^{\alpha_4}k^{\alpha_5}\left[1+(k^4\delta^a t)^{\alpha_6}\right]e^{-\delta^a k^4 t}$$
$$|\partial_k^n(1-\Psi(k))| \leq \begin{cases} Ck_0^{-n} & (k \in \mathrm{supp}(\Psi')) \\ 0 & (k \notin \mathrm{supp}(\Psi')) \end{cases}$$

where $C > 0$ is independent of δ, t, k and k_0 (but not of n), and the α_i are as in (3.12). It follows (using (3.11) as well) that

$$|K_m^B(x,t;\delta)| \le C(\delta t)^{-n} e^{\delta^{-a}t/2} \cdot \int_{|k|\le 3k_0/4} \left| \left(\frac{1}{k_0^2-k^2}\right)^n e^{-\frac{1}{2}\delta^a k^4 t} \begin{pmatrix} \frac{k^m(k^n + k_0^{n-1+(n-1)\bmod 2} k^{n \bmod 2})}{k_0^2-k^2} \\ +t^{\alpha_1} k^{m+\alpha_2}\left[1+(k^2t)^{\alpha_3}\right] \\ +(\delta^a t)^{\alpha_4} k^{m+\alpha_5}\left[1+(k^4\delta^a t)^{\alpha_6}\right] \\ +Q(k;m,n)k^{m-n} \end{pmatrix} \right| dk$$

for some $C > 0$ independent of x, δ, and t, where

$$Q(k;m,n) = \begin{cases} 1-\Psi(k) & \text{if } m \ge n, \\ e^{-x^2/(2\delta^{2-a}t)} \mathcal{X}(k_0/2 \le |k| \le 3k_0/4) & \text{if } m < n. \end{cases}$$

Manipulation of the various α_i and the observation that $|k| \le 3k_0/4$ implies $(k_0^2-k^2)^{-1} \le Ck_0^{-2}$ then yield (for any $n \ge 1$)

(3.15)
$$|K_m^B(x,t;\delta)| \le Ce^{\delta^{-a}t}x^{-n} \cdot \int_{|k|\le 3k_0/4} e^{-\frac{1}{2}\delta^a k^4 t} \begin{pmatrix} k^{m+n}k_0^{-2} \\ +k^{m+(n \bmod 2)} k_0^{n-3+((n-1)\bmod 2)} \\ +t^{\alpha_1(n)}k^{m+\alpha_2(n)} \\ +t^n k^{m+n} \\ +(\delta^a t)^{\alpha_4(n)} k^{m+\alpha_5(n)} \\ +(\delta^a t)^n k^{m+3n} \\ +Q(k;m,n)k^{m-n} \end{pmatrix} dk.$$

We may now obtain (3.13) from the integral in (3.15), by using the definition $k_0 = (|x|/3\delta t)^{1/2}$ and the facts that $|k| \le 3k_0/4$ for k in the domain of integration and $k = \mathcal{O}(k_0)$ for $k \in \operatorname{supp}(Q)$. Similarly, we obtain (3.14) from (3.15) by integration using the substitution

$$k = (\delta^a t)^{-1/4}\kappa.$$

This completes the proof of Lemma 3.2.

We now state lemmas giving pointwise estimates on K_m^C; the proofs are similar to the proof of Lemma 3.2 and will be omitted.

For these estimates, it turns out to be useful to break K_m^C into two pieces, using a cutoff function $\Psi_2(k)$ with the following properties:

$$
\begin{aligned}
&\operatorname{supp}\Psi_2 \subset \left(\frac{5k_0}{4} + (\delta t)^{-\eta}, \infty\right) \\
&\operatorname{supp}\Psi_2' \subset \left(\frac{5k_0}{4} + (\delta t)^{-\eta}, 2k_0 + 2(\delta t)^{-\eta}\right) \\
&|\Psi_2^{(n)}(k)| \le C\left(k_0 + (\delta t)^{-\eta}\right)^{-n} \quad \text{for all } n \ge 0
\end{aligned}
$$

for some $C > 0$ independent of δ and k_0 (here $\eta > 0$ is a constant to be determined later). Applying the cutoff Ψ_2 to the integrand of K_m^C (2.4) gives

$$K_m^C = K_m^{C,0} + K_m^{C,\infty},$$

where

$$
\begin{aligned}
K_m^{C,0}(x,t;\delta) &= \int_{|k|\ge 5k_0/4} e^{i[kx-\delta k^3 t]}(1-\Psi_2(k))(1-\Psi(k))k^m e^{(k^2-\delta^a k^4)t}\,dk, \\
K_m^{C,\infty}(x,t;\delta) &= \int_{|k|\ge 5k_0/4} e^{i[kx-\delta k^3 t]}\Psi_2(k)(1-\Psi(k))k^m e^{(k^2-\delta^a k^4)t}\,dk.
\end{aligned}
$$

We now obtain estimates for the operators $K_m^{C,0}$ and $K_m^{C,\infty}$ separately.

Lemma 3.3 Suppose that $\delta > 0$, $t > 0$, and $m \ge 0$, and let $\eta \in (0,1)$. *Then for any integer $n \ge 0$, there exists a constant $C > 0$ independent of x, δ, and t such that*

$$
|K_m^{C,\infty}(x,t;\delta)| \le C e^{\delta^{-a}t/2} e^{-x^2/(2\delta^{2-a}t)} (\delta t)^{(2\eta-1)n} \cdot
\left(\begin{array}{c}
\delta^{-2\eta-(m+n+1)a/4} t^{-2\eta-(m+n+1)/4} \\
+(\delta t)^{-2\eta-\alpha_3(n)} (\delta^a t)^{-(m+n \bmod 2+1)/4} |x|^{\alpha_3(n)} \\
+\delta^{-(m+\alpha_2(n)+1)a/4} t^{\alpha_1(n)-(m+\alpha_2(n)+1)/4} \\
+\delta^{-(m+n+1)a/4} t^{-(m-3n+1)/4} \\
+\delta^{[\alpha_4(n)-(m+\alpha_5(n)+1)/4]a} t^{\alpha_4(n)-(m+\alpha_5(n)+1)/4} \\
+\delta^{-(m-n+1)a/4} t^{-(m-n+1)/4} \\
+\delta^{r_\delta(\eta,n;m,a)} t^{r_t(\eta,n;m,a)}
\end{array}\right)
$$

where

$$
r_\delta(\eta,n;m,a) = \begin{cases} -(m-n+1)a/4 & \text{if } n \le m, \\ -a/4 - \eta(m-n) & \text{if } n > m, \end{cases}
$$

and

$$r_t(\eta, n; m, a) = r_\delta(\eta, n; m, 1).$$

Lemma 3.4 Suppose that $\delta > 0$, $t > 0$ and $m \geq 0$, and that

$$|x| \leq (\delta t)^{1-2\eta}$$

for some $\eta \in (0, 1/4)$. *Then for any $n \geq 1$, there exists a constant $C > 0$ independent of x, δ, and t such that*

$$|K_m^{C,0}(x,t;\delta)| \leq Ce^{\delta^{-a}t/2}e^{-x^2/(2\delta^{2-a}t)}.$$
$$\cdot \left(\begin{array}{c} |x|^{-(n+1)}(\delta t)^{1-(m+n+1)\eta} \\ +|x|^{-(3n+3-\alpha_7(n))/2}(\delta t)^{-(n-3+\alpha_7(n))/2-(m+[n \bmod 2]+1)\eta} \\ +|x|^{-n}t^{\alpha_1(n)}(\delta t)^{-(m+\alpha_2(n)+1)\eta} \\ +|x|^{-n}t^n(\delta t)^{-(m+n+1)\eta} \\ +|x|^{-n}(\delta^a t)^{\alpha_4(n)}(\delta t)^{-(m+\alpha_5(n)+1)\eta} \\ +|x|^{-n}(\delta^a t)^n(\delta t)^{-(m+3n+1)\eta} \\ +|x|^{-n+\min(m-n,0)/2}(\delta t)^{Q_1+1}(\delta^a t)^{Q_2+1} \end{array} \right)$$

where

$$Q_1 = -\eta \max(m-n, 0),$$

$$Q_2 = -\min(m-n+1, 0)/4.$$

Lemma 3.5 Suppose that $\delta > 0$, $t > 0$ and $m \geq 0$, and that $|x| > (\delta t)^{1-2\eta}$ for some $\eta < \min(1/2 - a/8, 1/3)$. *Then*

$$|K_m^{C,0}| = e^{\delta^{-a}t}e^{-x^2/(4\delta^{2-a}t)}f(x,\delta,t;\eta),$$

where f decays arbitrarily (polynomially) rapidly as $\delta \to \infty$, uniformly for $|x| > (\delta t)^{1-2\eta}$ and t bounded away from 0.

4. L^p BOUNDS ON GREEN'S FUNCTIONS

In this section we present L^p estimates for derivatives of K_m^A, K_m^B and K_m^C for various x-domains. These are obtaining by integrating the pointwise bounds in Section 3. We will first state all the bounds as a series of lemmas, and then sketch the proofs.

We begin with L^p bounds on $K_m^{C,\infty}$ that are valid for all $x \in \mathbf{R}$.

Lemma 4.1. *Suppose that $\delta > 0$, $t > 0$, and $m \geq 1$, and let $\eta \in (0, 1/4)$. Then for any $p \geq 1$, there exists $C > 0$ independent of δ and t such that*

$$\|K_m^{C,\infty}(\cdot, t; \delta)\|_{L^p(\mathbf{R})} \leq Ce^{\delta^{-a}t/2}(\delta^{2-a}t)^{1/(2p)}\,(\delta t)^{(2\eta-1)n} \cdot$$
$$\cdot \left(\begin{array}{c} \delta^{2\eta-(m+n+1)a/4}t^{2\eta-(m+n+1)/4} \\ + \left(\begin{array}{c} \delta^{2\eta-\alpha_3(n)-(m+n \bmod 2+1)a/4}t^{-2\eta-\alpha_3(n)-(m+n \bmod 2+1)/4} \cdot \\ \cdot (\delta^{2-a}t)^{(n-1+\alpha_7(n))/4} \end{array} \right) \\ +\delta^{-(m+\alpha_2(n)+1)a/4}t^{\alpha_1(n)-(m+\alpha_2(n)+1)/4} \\ +\delta^{-(m+n+1)a/4}t^{-(m-3n+1)/4} \\ +\delta^{[\alpha_4(n)-(m+\alpha_5(n)+1)/4]a}t^{\alpha_4(n)-(m+\alpha_5(n)+1)/4} \\ +\delta^{-(m-n+1)a/4}t^{-(m-n+1)/4} \\ +\delta^{r_\delta(\eta,n;m,a)}t^{r_t(\eta,n;m,a)} \end{array} \right),$$

where the α_i, r_δ and r_t are defined as in Lemma 3.3.

To estimate the remaining parts of the kernel, we divide the x-axis into four intervals:

$$\begin{array}{rc} S_1 = & (-\delta^r t^\rho, \delta^r t^\rho), \\ S_2 = & \left(-(\delta t)^{1-2\eta}, -\delta^r t^\rho\right] \cup \left[\delta^r t^\rho, (\delta t)^{1-2\eta}\right), \\ S_3 = & \left(-\infty, (\delta t)^{1-2\eta}\right] \cup [(\delta t)^{1-2\eta}, \infty), \end{array} \tag{4.1}$$

where r, ρ and η are positive constants to be chosen later. Different pointwise estimates must be used to obtain the needed L^p estimates on each interval, as given in the following lemmas.

Lemma 4.2 Let $a \in [0, 2]$, $\delta > 0$, $t > 0$, and $\eta > 0$, and let r and ρ be positive constants such that $\delta^r t^\rho < (\delta t)^{1-2\eta}$. *Then there exists $C > 0$ independent of δ and t such that*

$$\|K_m^{C,0}(\cdot, t; \delta)\|_{L^p(S_1)} \leq Ce^{\delta^{-a}t/2}(\delta t)^{-(m+1)\eta} \min\left(\delta^{r/p}t^{\rho/p}, \delta^{(1-a/2)/p}t^{1/(2p)}\right),$$

where S_1 is as in (4.1).

Lemma 4.3. *Let $m \geq 1$ and $p \geq 1$ be integers, and let n be an integer such that $np > 1$. Let $\eta \in (0, 1/4)$, $a \in [0, 2]$, $\delta > 0$, $t > 0$, and let r and ρ be positive constants such that $\delta^r t^\rho < (\delta t)^{1-2\eta}$. Then there exists $C > 0$*

independent of δ *and* t *such that*

$$\|K_m^{C,0}(\cdot,t;\delta)\|_{L^p(S_2)} \le Ce^{\delta^{-a}t/2}e^{-\delta^{2(r-1)+a}t^{2\rho-1}}\cdot$$
$$\cdot\left(\begin{array}{c} (\delta^r t^\rho)^{1/p-(n+1)}(\delta t)^{1-(m+n+1)\eta} \\ +(\delta^r t^\rho)^{1/p-(3n+3-\alpha_7(n))/2}(\delta t)^{-(n-3+\alpha_7(n))/2-(m+[n \bmod 2]+1)\eta} \\ +(\delta^r t^\rho)^{1/p-n}t^{\alpha_1(n)}(\delta t)^{-(m+1+\alpha_2(n))\eta} \\ +(\delta^r t^\rho)^{1/p-n}t^{n}(\delta t)^{-(m+n+1)\eta} \\ +(\delta^r t^\rho)^{1/p-n}(\delta^a t)^{\alpha_4(n)}(\delta t)^{-(m+\alpha_5(n)+1)\eta} \\ +(\delta^r t^\rho)^{1/p-n}(\delta^a t)^{n}(\delta t)^{-(m+3n+1)\eta} \\ +(\delta^r t^\rho)^{1/p-n+\min(m-n,0)/2}(\delta t)^{-\eta(m-n)}(\delta^a t)^{-\min(m-n+1,0)/4} \end{array}\right),$$

where S_2 *is as in (4.1).*

Lemma 4.4. *Suppose that* $m \ge 1$, $p \ge 1$, $a \in [0,2]$, $\delta > 0$, *and* $t > 0$. *Let* $0 < \eta < \min(1/2 - a/8, 1/3)$, *and let* S_3 *be as in (4.1). Then* $\|K_m^{C,0}(\cdot,t;\delta)\|_{L^p(S_3)}$ *decays arbitrarily (polynomially) rapidly as* $\delta \to \infty$, *uniformly for* t *bounded away from* 0.

Lemma 4.5. *Suppose that* $\gamma > 0$, $\delta > 0$ *and* $t > 0$. *Then for any* $p \ge 1$ *and* $\epsilon > 0$ *small, there exists* $C > 0$ *independent of* δ *and* t *such that*

$$\|K_m^{A}(\cdot,t;\delta)\|_{L^p} \le Ce^{\delta^{-a}t/2}\cdot$$
$$\cdot\left[\begin{array}{c} (\delta^{1-a/2}t^{1/2})^{(m-\epsilon)/2-1/4+1/p}\delta^{-(m-\epsilon)/2-1/4}t^{-(m-\epsilon)/2-3/4} \\ +(\delta^{1-a/2}t^{1/2})^{(m+\epsilon)/2+1/4+1/p}\delta^{-(2m+3)/4-\epsilon/2}t^{-(2m+5)/4-\epsilon/2} \\ +\left(\begin{array}{c} (\delta^{1-a/2}t^{1/2})^{(m+3\epsilon)/2+3/4+1/p}\cdot \\ \cdot\delta^{[-5+(1+2\epsilon)a]/4-(m+3\epsilon)/2}\cdot \\ \cdot t^{-(m+3\epsilon)/2-7/4} \end{array}\right) \end{array}\right].$$

Lemma 4.6. *Suppose that* $m \ge 0$, $\delta > 0$ *and* $t > 0$. *Then for any* $p \ge 1$, $r \ge 0$ *and* $\rho \ge 0$, *there exists* $C > 0$ *independent of* δ *and* t *such that*

$$\begin{aligned} &\|K_m^B\|_{L^p(S_1)} \le Ce^{\delta^{-a}t/2}\delta^{(m+1)(r-1)/2+r/p}t^{(m+1)(\rho-1)/2+\rho/p}, \\ &\|K_m^B\|_{L^p(S_1)} \le Ce^{\delta^{-a}t/2}\delta^{r/p-a(m+1)/4}t^{\rho/p-(m+1)/4}. \end{aligned} \tag{4.2}$$

where S_1 *is as in (4.1).*

Furthermore, if $p \ge 2$ *then*

$$\|K_m^B\|_{L^p(S_1)} \le Ce^{\delta^{-a}t/2}(\delta^a t)^{-(m+1-1/p)/4}. \tag{4.3}$$

Lemma 4.7. *Suppose that $m \geq 0$, $\delta > 0$ and $t > 0$, and let $p \geq 1$, $r \geq 0$ and $\rho \geq 0$. Also suppose that $n > m + 1 + 2/p$. Then there exists $C > 0$ independent of δ and t such that*

$$\|K_m^B\|_{L^p(S_2 \cup S_3)} \leq Ce^{\delta^{-a}t} \cdot \left(\begin{array}{c} (\delta t)^{(-n+1-m)/2}(\delta^r t^\rho)^{(m-n-1)/2+1/p} \\ +(\delta t)^{-(n+m+1)/2}(\delta^r t^\rho)^{(-n+m+1)/2+1/p} t^n \\ +(\delta t)^{-(m+1)/2}(\delta^r t^\rho)^{-n+m/2+1/p+(1+[n \bmod 2])/2}\delta^{-[n \bmod 2]/2} t^{n/2} \\ +(\delta^r t^\rho)^{-n+1/p}\delta^{-a(m+n+1)/4} t^{-(m-3n+1)/4} \\ +(\delta^r t^\rho)^{-n+1/p}(\delta^a t)^{\alpha_4(n)-(m+\alpha_5(n)+1)/4} \\ +(\delta^r t^\rho)^{-n+1/p}(\delta^a t)^{-(m-n+1)/4} \end{array} \right)$$

where S_2 and S_3 are as in (4.1).

Sketch of proofs. In what follows, constants C are always independent of δ, t, and (where relevant) x.

To prove Lemma 4.2, note that if $x \in S_1$, then the assumption that $\delta^r t^\rho < (\delta t)^{1-2\eta}$ implies that

$$k_0 = (x/(\delta t))^{1/2} \leq C(\delta t)^{-\eta}.$$

Thus we have the crude estimate

$$\begin{aligned} |K_m^{C,0}(x,t;\delta)| &\leq Ce^{\delta^{-a}t/2}e^{-x^2/(2\delta^{2-a}t)} \int_{k=5k_0/4+(\delta t)^{-\eta}}^{2k_0+2(\delta t)^{-\eta}} k^m \, dk \\ &\leq Ce^{\delta^{-a}t/2}e^{-x^2/(2\delta^{2-a}t)}(\delta t)^{-(m+1)\eta}. \end{aligned}$$

Ignoring the Gaussian term and taking the norm of the resulting (constant) estimate in $L^p(S_1)$ shows that

$$\|K_m^{C,0}(\cdot,t;\delta)\|_{L^p(S_1)}^p \leq Ce^{\delta^{-a}t/2}(\delta t)^{-(m+1)\eta}(\delta^r t^\rho)^{1/p}.$$

On the other hand, using the Gaussian term in taking the L^p norm yields

$$\|K_m^{C,0}(\cdot,t;\delta)\|_{L^p(S_1)}^p \leq Ce^{\delta^{-a}t/2}(\delta t)^{-(m+1)\eta}(\delta^{(1-a/2)}t^{1/2})^{1/p}.$$

This establishes Lemma 4.2. Lemmas 4.1, 4.3–4.5 and 4.7 are similarly proved by integrating pointwise bounds. We note that the proof of Lemma 4.5 uses the additional fact that for any $r \geq 0$,

$$\|x^r e^{-\frac{1}{2}\delta^{a-2}x^2/t}\|_{L^p} = C\left(\delta^{1-a/2}t^{1/2}\right)^{r+1/p}.$$

Also, to prove Lemma 4.7, one can choose the pointwise bound for each of the seven terms in the relevant integral from (3.13) or (3.14), independently of the choices for the other six terms. It turns out to be advantageous to use the estimates in (3.13) for the first three terms, and the estimates in (3.14) for the other four terms.

The proof of Lemma 4.6 is as follows. The first inequality in (4.2) is obtained by integrating (with respect to x) the naive estimate

$$|K_m^B(x,t;\delta)| \leq Ce^{\delta^{-a}t/2}|k_0^{(m+1)}| \leq Ce^{\delta^{-a}t/2}(|x|/(\delta t))^{-(m+1)/2}.$$

Similarly, the second inequality in (4.2) is obtained from the pointwise bound

$$|K_m^B(x,t;\delta)| \leq Ce^{\delta^{-a}t/2}\int_{|k|\leq k_0} k^m e^{-\delta^a k^4 t/2}\, dk.$$

Finally, (4.3) is a consequence of the Hausdorff-Young inequality, if one views the integral defining K_m^B (2.4) as a Fourier transform from k to x and determines the L^q norm of the integrand (with $1/p + 1/q = 1$) in k-space by integrating with the change of variables $k = (\delta^a t)^{-1/4}\kappa$.

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