Basic skills: geometric series and summation by parts

Alex Iosevich

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- Statistics, physics, computer science, chemistry and engineering majors may find these lectures helpful as well.
- Most of these lectures will be accessible to advanced high school students.

A bit more motivation

- Calculus is not a prerequisite for watching this lecture. However, the ideas we will go over will be quite helpful when you take calculus.

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- If you have already taken calculus, you know to calculate integrals like

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- Since calculus is often taught as a collection of mechanical tricks, many calculus students are not exposed to the analogous sum

$$\sum_{k=a}^{b} k \cdot 2^k,$$

and this is the type of an issue we are going to address in the lecture.

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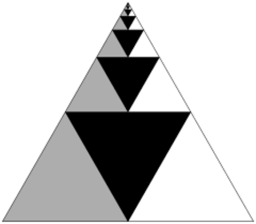
- The geometric series need not start at 1, so
- $$A^k + A^{k+1} + \cdots + A^n$$

  is also a geometric series, where k is a positive integer $< n$.

Geometric series-simple diagrams from wikipedia



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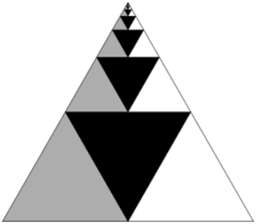

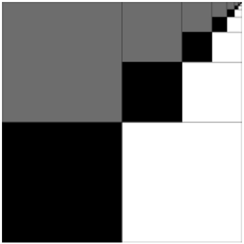

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- Then
$$A \cdot \square = A^{k+1} + A^{k+2} + \cdots + A^n + A^{n+1}.$$

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- Here is a simple example to give ourselves a sanity check. According to our formula,

- $$1 + 2 + \cdots + 2^4 = 2^5 - 1 = 31,$$

  which is, indeed, true!

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- However, themes tend to recur, so it is useful to understand what happened.
- The key observation behind what we did is that multiplying a geometric series $A^k + A^{k+1} + \cdots + A^n$ by A
- yields another geometric series

$$A^{k+1} + A^{k+2} + \cdots + A^n + A^{n+1}$$

which differs from the original geometric series in only two entries.

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- Finally, if $n+1$ is odd and k is even, we get -1.

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- But we persist and try to correct by adding

$$A^2 + A^3 + \cdots + A^n.$$

Just how spicy is it? (continued)

- The correction term we added helped a bit. We now have one factor of A, which is correct, and two factors of A^2, which is again correct, but we only have two factors of A^3 and we need three, and so on.

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- Let us fully write out the case $n = 3$.

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- It is a very good time to recall that we have shown above that

$$\square_k = \frac{A^{n+1} - A^k}{A - 1}.$$

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  which is true.
- In order to built up these skills further, we need to go back and redo all these calculations using the summation notation.

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- We want to subtract $\sum_{j=k}^{n} A^j$ from

$$A \cdot \sum_{j=k}^{n} A^j = \sum_{j=k}^{n} A^{j+1}.$$

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- We can fix the problem as follows. Let $m = j + 1$. Then since j ranges from k to n, m ranges from $k + 1$ to $n + 1$.
- It follows that

$$\sum_{j=k}^{n} A^{j+1} = \sum_{m=k+1}^{n+1} A^{m}.$$

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$$A \cdot \sum_{j=k}^{n} A^j - \sum_{j=k}^{n} A^j = \sum_{j=k+1}^{n+1} A^j - \sum_{j=k}^{n} A^j.$$

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- and we conclude that
$$\sum_{j=k}^{n} A^j = \frac{A^{n+1} - A^k}{A-1},$$
as before.

Double summation (continued)

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- This is where the fundamental idea behind summation by parts comes into play.

Telescoping series

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- This is a special case of a general simple formula

$$\sum_{j=1}^{k} a_j - a_{j-1} = (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_k - a_{k-1})$$

$$= a_k - a_0.$$

Telescope

- It follows that

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- $$= \frac{2A^{n+1}}{A-1} \sum_{j=1}^{n} j - \frac{2}{A-1} \sum_{j=1}^{k} jA^j$$

- $$- \frac{A^{n+1}}{A-1} \sum_{j=1}^{k} 1 + \frac{1}{A-1} \sum_{j=1}^{k} A^j = I + II + III + IV.$$

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$$\sum_{j=1}^{k} j.$$

Sum of consecutive integers

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- It follows that

$$Apple = \frac{k(k+1)}{2}.$$

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- and the question that immediately arises is how to expand the expression

$$j^a - (j-1)^a?$$

(Slightly) advanced "FOIL" method

- In order to make sense of this expression, we need to figure out how to expand expressions of the form

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- Multiplying out this expression amounts to selecting either x or y from each set of parentheses and multiplying them together.

Let's count!

- It follows that this expression is equal to

$$C(a,0)x^a + C(a,1)x^{a-1}y^1 + C(a,2)x^{a-2}y^2 + \cdots + C(a,a)y^a,$$

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- You may already know that

$$C(a,j) = \frac{a!}{j!(a-j)!},$$

where

$$k! = 1 \cdot 2 \cdot \cdots \cdot k.$$

Pascal's triangle

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

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$$= j^a + \sum_{m=1}^{a} (-1)^m j^{a-m} C(a,m),$$

- which allows us to explicitly express

$$j^a - (j-1)^a$$

as a polynomial in j of degree $a-1$.

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- and

$$\sum_{k=1}^{n} k^b, \text{ also with } b < a.$$