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Chapter 4

V = vector space over $\mathbb{R}$

$V^k = V \times V \times \dots \times V$ k-times

$T : V^k \longrightarrow \mathbb{R}$

multi-linear if

$T(v_1, \dots, v_i + v_i', \dots, v_k) = T(v_1, \dots, v_i, \dots, v_k) + T(v_1, \dots, v_i', \dots, v_k)$

and $T(v_1, \dots, a v_i, \dots, v_k) = a\, T(v_1, \dots, v_i, \dots, v_k)$

Multi-linear $T : V^k \rightarrow \mathbb{R}$ is called a k-tensor, and the collection of all of them, $\mathcal{T}^k(V)$ is a vector space w/

$(S+T)(v_1, \dots, v_k) = S(v_1, \dots, v_k) + T(v_1, \dots, v_k)$

$(aS)(v_1, \dots, v_k) = a\, S(v_1, \dots, v_k)$

If $S \in \mathcal{T}^k(V)$ & $T \in \mathcal{T}^\ell(V)$,

$S \otimes T \in \mathcal{T}^{k+\ell}(V)$

tensor product

$S \otimes T(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+\ell}) = S(v_1, \dots, v_k) \cdot T(v_{k+1}, \dots, v_{k+\ell})$

(2)

$V = \mathbb{R} \quad S : V^2 \longrightarrow \mathbb{R}$

$S(v_1, v_2) = v_1 v_2$

$T : V^2 \longrightarrow \mathbb{R}$

$T(v_3, v_4) = v_3$

$S \otimes T(v_1, v_2, v_3, v_4) = S(v_1, v_2)\, T(v_3, v_4) =$

$v_1 v_2 \cdot v_3$

$T \otimes S(v_1, v_2, v_3, v_4) = v_1 \cdot v_2 \cdot v_3$, so

$S \otimes T = T \otimes S$, but...

suppose that $T(v_3, v_4) = v_3 \cdot A v_4$

for some matrix A,

where now $v_3 \in \mathbb{R}^2$, $v_4 \in \mathbb{R}^2$

and $S(v_1, v_2) = v_1 \cdot \underbrace{B}_{\text{another matrix}} v_2$

So $S \otimes T(v_1, v_2, v_3, v_4) =$

$S(v_1, v_2)\, T(v_3, v_4) = (v_1 \cdot B v_2)(v_3 \cdot A v_4)$

$T \otimes S(v_1, v_2, v_3, v_4) = (v_1 \cdot A v_2)(v_3 \cdot B v_4)$

(3)

Let's say $B =$ identity, so we have

$(v_1 \cdot v_2)(v_3 \cdot A v_4)$ vs $(v_1 \cdot A v_2)(v_3 \cdot v_4)$

Choose $v_1 = v_2 = (1,0)$ & $v_3 = (1,0)$ $v_4 = (0,1)$

We have

$$1 \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_4^1 \\ v_4^2 \end{pmatrix} \cdot \begin{pmatrix} v_3^1 \\ v_3^2 \end{pmatrix}$$

$$= \left(a_{11} v_4^1 + a_{12} v_4^2\right) \cdot v_3^1 = a_{12}$$

$$+ \left(a_{21} v_4^1 + a_{22} v_4^2\right) \cdot v_3^2 = 0$$

$$a_{12} + 0 = a_{12}$$

So as long as $a_{12} \neq 0$, $T \otimes S \neq S \otimes T$

Observation: $\mathcal{T}^1(V) = V^*$ ~ dual space

Theorem: $v_1, \dots, v_n$ basis of V

$\varphi_1, \dots, \varphi_n$ the dual basis

$\varphi_i(v_j) = \delta_{ij}$

Then $\{\varphi_{i_1} \otimes \dots \otimes \varphi_{i_k},\ 1 \leq i_1, \dots, i_k \leq n$ is a basis for $\mathcal{T}^k(V)$, which therefore has dimension $\underline{n^k}$.

Proof:

$\varphi_{i_1} \otimes \dots \times \varphi_{i_K}(v_{j_1}, \dots, v_{j_K}) =$

$$\delta_{i_1 j_1} \dots \delta_{i_K j_K} = \begin{cases} 1, & \text{if } j_1 = i_1, \dots \\ 0, & \text{otherwise} \end{cases}$$

If $w_1, \dots, w_K$ are K-vectors w/ $w_i = \sum_{j=1}^{n} a_{ij} v_j$

and T is in $\mathcal{J}^K(V)$, then

$$T(w_1, w_2, \dots, w_K) = \sum_{j_1, \dots, j_K = 1}^{n} a_{1, j_1} \dots a_{K, j_K} \, T(v_{j_1}, \dots, v_{j_K})$$

$$= \sum_{i_1, \dots, i_K = 1}^{n} T(v_{i_1}, \dots, v_{i_K}) \cdot \varphi_{i_1} \otimes \dots \otimes \varphi_{i_K}(w_1, \dots, w_K)$$

Thus $T = \sum_{i_1, \dots, i_K = 1}^{n} T(v_{i_1}, \dots, v_{i_K}) \, \varphi_{i_1} \otimes \dots \otimes \varphi_{i_K}$

Consequently, $\{\varphi_{i_K} \otimes \dots \otimes \varphi_{i_K}\}$ span $\mathcal{J}^K(V)$!

We still need to establish independence.

⑤

If $\sum_{i_1,\ldots,i_k=1}^{n} a_{i_1,\ldots,i_k} \cdot \varphi_{i_1} \otimes \ldots \otimes \varphi_{i_k} = 0$,

we apply both sides to $(v_{j_1},\ldots, v_{j_k})$,

yielding $\underline{\underline{a_{j_1\ldots j_k} = 0}}$ $\rightsquigarrow$ linear independence

pull-back:

$f: V \longrightarrow W$ linear induces

$f^*: \mathcal{T}^k(W) \longrightarrow \mathcal{T}^k(V)$

$f^*T(v_1,\ldots, v_k) = T(f(v_1),\ldots, f(v_k))$

Exercise: $f^*(S\otimes T) = f^*S \otimes f^*T$

Generalized inner product:

$T \in \mathcal{T}^k(V)$, T symmetric, ie $T(v,w) = T(w,v)$

$v, w \in V$

$\underbrace{T(v,v) > 0 \text{ if } v \neq 0}_{\text{positive-definite}}$

$\langle , \rangle$ = usual inner product, ie

$\langle v, w\rangle = \sum_{i=1}^{n} v^i w^i$.

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Theorem: If T is an inner product on V, there is a basis $v_1, v_2, \ldots, v_n$ for V $\ni$ $T(v_i, v_j) = \delta_{ij}$.
Consequently there is an isomorphism $f : \mathbb{R}^n \to V$ $\ni$ $T(f(x), f(y)) = \langle x, y \rangle$ for $x, y \in \mathbb{R}^n$.
In other words $f^* T = \langle \, , \, \rangle$.

Proof: Let $w_1, \ldots, w_n$ be any basis for V. Define

$$w_1' = w_1,$$

$$w_2' = w_2 - \frac{T(w_1', w_2)}{T(w_1', w_1')} w_1',$$

$$w_3' = w_3 - \frac{T(w_1', w_3)}{T(w_1', w_1')} w_1' - \frac{T(w_2', w_3)}{T(w_2', w_2')} w_2'$$

& so on.

In other words,

$$w_k' = w_k - \sum_{i=1}^{k-1} \frac{T(w_i', w_k)}{T(w_i', w_i')} w_i'$$

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It follows that

$$T(w_\ell', w_m') =$$

$$T\left(w_\ell - \sum_{i=1}^{\ell-1} \frac{T(w_i', w_\ell)}{T(w_i', w_i')} w_i',\ w_m - \sum_{j=1}^{m-1} \frac{T(w_j', w_m)}{T(w_j', w_j')} w_j'\right)$$

$$= T(w_\ell, w_m) + \sum_{i=1}^{\ell-1}\sum_{j=1}^{m-1} \frac{T(w_i', w_\ell)}{T(w_i', w_i')} \frac{T(w_j', w_m)}{T(w_j', w_j')} T(w_i', w_j')$$

$$- \sum_{i=1}^{\ell-1} \frac{T(w_i', w_\ell)}{T(w_i', w_i')} T(w_i', w_m)$$

symmetry

$$- \sum_{j=1}^{m-1} \frac{T(w_j', w_m)}{T(w_j', w_j')} T(w_j', w_\ell)$$

$= 0$ as everything cancels.

Setting $v_i = \dfrac{w_i'}{\sqrt{T(w_i', w_i')}}$.

Set $f : \mathbb{R}^n \longrightarrow V$ $f(e_i) = v_i$ and the final statement is recovered by a direct calculation.

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Alternating k-tensors: $\sim \Lambda^k(V)$

$$\omega(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = - \omega(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \quad \forall v_1, \dots, v_k \in V$$

$$\mathrm{Alt}(T)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) T(v_{\sigma(1)}, \dots, v_{\sigma(k)}),$$

where S_k = permutations on k letters.

Theorem:

1. If $T \in \mathcal{T}^k(V)$, $\mathrm{Alt}(T) \in \Lambda^k(V)$.
2. If $\omega \in \Lambda^k(V)$, then $\mathrm{Alt}(\omega) = \omega$.
3. If $T \in \mathcal{T}^k(V)$, then $\mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T)$

Proof: (i,j) = permutation that interchanges i and j and leaves all other numbers fixed. If $\sigma \in S_k$, let $\sigma' = \sigma \cdot (i,j)$.

Then

$$Alt(T)(v_1, \dots, v_j, \dots, v_i, \dots, v_k) =$$

$$\frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) T(v_{\sigma(1)}, \dots, v_{\sigma(j)}, \dots, v_{\sigma(i)}, \dots, v_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) T(v_{\sigma'(1)}, \dots, v_{\sigma'(i)}, \dots, v_{\sigma'(j)}, \dots, v_{\sigma'(k)})$$

$$= \frac{1}{k!} \sum_{\sigma' \in S_k} -sgn(\sigma') T(v_{\sigma'(1)}, \dots, v_{\sigma'(k)})$$

$$= -Alt(T)(v_1, \dots, v_k)$$

This proves (1). To prove 2), observe that if $\omega \in \Lambda^k(V)$ and $\sigma = (i,j)$, then

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = sgn(\sigma)\, \omega(v_1, \dots, v_k)$$

It follows that

$$Alt(\omega)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma)\, \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) \cdot sgn(\sigma) \cdot \omega(v_1, \dots, v_k)$$

$$= \omega(v_1, \dots, v_k).$$ This proves 2

To prove 3, we simply use 1 and 2.

Dimensionality of $\Lambda^k(V)$ calculation begins w/ observation that $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^\ell(V)$ does not guarantee that $\omega \otimes \eta \in \Lambda^{k+\ell}(V)$.

To remedy this, we introduce the wedge product

$$\omega \wedge \eta = \frac{(k+\ell)!}{k!\,\ell!} Alt(\omega \otimes \eta)$$

does this coefficient look familiar?

Simple observations:

$$(\omega_1+\omega_2)\wedge\eta = \omega_1\wedge\eta + \omega_2\wedge\eta,$$

$$\omega\wedge(\eta_1+\eta_2) = \omega\wedge\eta_1 + \omega\wedge\eta_2,$$

$$\alpha\omega\wedge\eta = \omega\wedge\alpha\eta = \alpha(\omega\wedge\eta)$$

$$\omega\wedge\eta = (-1)^{kl}\,\eta\times\omega$$

$$f^*(\omega\wedge\eta) = f^*(\omega)\wedge f^*(\eta)$$

Harder (to be discussed later):

$$(\omega\wedge\eta)\wedge\theta = \omega\wedge(\eta\wedge\theta)$$

Theorem: If $S\in\mathcal{T}^k(V)$, $T\in\mathcal{T}^l(V)$, and $\mathrm{Alt}(S)=0$, then

(1) $\mathrm{Alt}(S\otimes T) = \mathrm{Alt}(T\otimes S) = 0.$

(2) $\mathrm{Alt}(\mathrm{Alt}(\omega\otimes\eta)\otimes\theta) = \mathrm{Alt}(\omega\otimes\eta\otimes\theta)$

$$= \mathrm{Alt}(\omega\otimes\mathrm{Alt}(\eta\otimes\theta)).$$

(3) If $\omega\in\Lambda^k(V)$, $\eta\in\Lambda^l(V)$, and $\theta\in\Lambda^m(V)$, then $(\omega\wedge\eta)\wedge\theta = \omega\wedge(\eta\wedge\theta) =$

$$\frac{(k+l+m)!}{k!\,l!\,m!}\,\mathrm{Alt}(\omega\otimes\eta\otimes\theta)$$

Proof:

(1) $(k+l)!\,\mathrm{Alt}(S\otimes T)(v_1,\dots,v_{k+l})$

$$= \sum_{\sigma\in S_{k+l}} \operatorname{sgn}\sigma \cdot S(v_{\sigma(1)},\dots,v_{\sigma(k)})\, T(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)})$$

If $G\subset S_{k+l}$ consists of σ that leave $k+1,\dots,k+l$ fixed, then

$$\sum_{\sigma\in G} \operatorname{sgn}\sigma\, S(v_{\sigma(1)},\dots,v_{\sigma(k)})\cdot T(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)})$$

$$= \sum_{\sigma'\in S_k} \operatorname{sgn}\sigma' \cdot S(v_{\sigma'(1)},\dots,v_{\sigma'(k)})\cdot T(v_{k+1},\dots,v_{k+l})$$

$$=0$$

Suppose $\sigma_0\notin G$. Let $G\cdot\sigma_0 = \{\sigma\cdot\sigma_0 : \sigma\in G\}$, and let $v_{\sigma_0(1)},\dots,v_{\sigma_0(k+l)} = w_1,\dots,w_{k+l}$.

Then $\displaystyle\sum_{\sigma\in G\cdot\sigma_0} \operatorname{sgn}\sigma\, S(v_{\sigma(1)},\dots,v_{\sigma(k)})\cdot T(v_{\sigma(k+1)},\dots,\sigma(k+l))$

$$= \operatorname{sgn}\sigma_0 \sum_{\sigma'\in G} \operatorname{sgn}\sigma'\, S(w_{\sigma'(1)},\dots,w_{\sigma'(k)})\, T(w_{k+1},\dots,w_{k+l})$$

$$=0$$

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Why are we done? $\sigma \cap \sigma \cdot \sigma_0 \overset{\text{why?}}{=} \phi$

We just continue in this way, breaking $S_{k+\ell}$ into disjoint subsets. Over each subset the sum is 0.

This completes the proof!

Proof of (2):

We have $\mathrm{Alt}(\mathrm{Alt}(\eta \otimes \theta) - \eta \otimes \theta)$

$= \mathrm{Alt}(\eta \otimes \theta) - \mathrm{Alt}(\eta \otimes \theta) = 0$

By part (1),

$$0 = \mathrm{Alt}(\omega \otimes [\mathrm{Alt}(\eta \otimes \theta) - \eta \otimes \theta])$$

$$= \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta)) - \mathrm{Alt}(\omega \otimes \eta \otimes \theta)$$

Proof of (3)

$$(\omega \wedge \eta) \wedge \theta = \frac{(k+\ell+m)!}{(k+\ell)!\, m!} \mathrm{Alt}((\omega \wedge \eta) \otimes \theta)$$

$$= \frac{(k+\ell+m)!}{(k+\ell)!\, m!} \frac{(k+\ell)!}{k!\, m!} \mathrm{Alt}(\omega \otimes \eta \otimes \theta)$$

Theorem: The set of all $\left\{ \varphi_{i_1} \wedge \varphi_{i_2} \wedge \dots \wedge \varphi_{i_k} \;:\; 1 \le i_1 < i_2 < \dots < i_k \le n \right\}$

is a basis for $\Lambda^k(V)$, so its dimension

is $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

Proof: If $\omega \in \Lambda^k(V) \subset \mathcal{T}^k(V)$, then

$$\omega = \sum_{i_1, \dots, i_k} a_{i_1 \dots i_k} \, \varphi_{i_1} \otimes \dots \otimes \varphi_{i_k}$$

Thus

$$\omega = \mathrm{Alt}(\omega) = \sum_{i_1, \dots, i_k} a_{i_1 \dots i_k} \underbrace{\mathrm{Alt}(\varphi_{i_1} \otimes \dots \otimes \varphi_{i_k})}_{\substack{\equiv \\ \text{constant multiple} \\ \text{of } \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}}}$$

It follows that $\{\varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}\}$ span and independence follows by Theorem 4.1

Theorem: Let $v_1, \ldots, v_n$ be a basis for V, and let $\omega \in \Lambda^n(V)$. If $w_i = \sum_{j=1}^{n} a_{ij} v_j$ are n vectors in V, then

$$\omega(w_1, \ldots, w_n) = \det(a_{ij})\, \omega(v_1, \ldots, v_n)$$

proof: Let $\eta \in \mathcal{J}^n(\mathbb{R}^n)$ be given by

$$\eta((a_{11}, \ldots, a_{1n}), \ldots, (a_{n1}, \ldots, a_{nn})) =$$

$$\omega\left(\sum a_{1j} v_j, \ldots, \sum a_{nj} v_j\right)$$

Since $\eta \in \Lambda^n(\mathbb{R}^n)$ (why?), $\Lambda^n(\mathbb{R}^n)$ is 1-dimensional (previous theorem) and $\det \in \Lambda^n(\mathbb{R}^n)$ (Math 173),

$$\eta = \underbrace{\lambda}_{\text{constant}} \cdot \det$$

and $\lambda = \eta(e_1, \ldots, e_n) = \omega(v_1, \ldots, v_n)$ ✓

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This shows that $\omega \in \Lambda^n(V)$ splits bases of V into two disjoint groups, one where $\omega(v_1, \dots, v_n) > 0$ & one where $\omega(v_1, \dots, v_n) < 0$.

Either group is called orientation, denoted by $[v_1, \dots, v_n]$. The opposite orientation is $-[v_1, \dots, v_n]$

Cross product:

$v_1, \dots, v_{n-1} \in \mathbb{R}^n$

$$\varphi(w) = \det \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ w \end{pmatrix}$$

Since $\varphi \in \Lambda^1(\mathbb{R}^n)$, $\exists!\ z \in \mathbb{R}^n \ni$

$$\langle w, z \rangle = \varphi(w) = \det \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ w \end{pmatrix}$$

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This z is denoted by
$z = v_1 \times v_2 \times \dots \times v_{n-1}$ and is called the cross product.

<u>Fields and forms</u>

$p \in \mathbb{R}^n$ $\qquad \{\underbrace{(p, v)}_{\text{fixed}} : v \in \mathbb{R}^n\}$

$\qquad\qquad\qquad\qquad \text{III}$

tangent space of $\mathbb{R}^n$ at $\underline{p}$

We can make it into a vector space:

$(p,v) + (p,w) = (p, v+w)$

$a \cdot (p,v) = (p, av)$

Inner product of $\mathbb{R}^n_p$:

$\langle v_p, w_p \rangle_p = \underbrace{\langle v, w \rangle}_{\text{usual inner product}}$

$[(e_1)_p, \dots, (e_n)_p] =$ usual orientation

Vector field: selecting a vector at each $\mathbb{R}^n_p$. What does that mean?

F = function $\ni$ $F(p) \in \mathbb{R}^n_p$ for each $p \in \mathbb{R}^n$.

For each p $\exists$ numbers
$F^1(p), \ldots, F^n(p) \ni$

$$F(p) = F^1(p) \cdot (e_1)_p + \ldots + F^n(p) \cdot (e_n)_p$$

Each $F^i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called a component function.

Recall that $(e_i)_p = (p, e_i)$

__Define:__

$$(F+G)(p) = F(p) + G(p)$$

$$\langle F, G \rangle (p) = \langle F(p), G(p) \rangle,$$

$$f \cdot F(p) = f(p)\, F(p)$$

cross product

$$(F_1 \times \ldots \times F_{n-1})(p) = F_1(p) \times \ldots \times F_{n-1}(p)$$

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<u>Divergence:</u> $\operatorname{div} F = \sum_{i=1}^{n} \underbrace{D_i}_{\text{partial derivative}} F^i$

$\underbrace{\nabla}_{\text{formal symbol}} = \sum_{i=1}^{n} D_i \cdot e_i$

In this notation, $\operatorname{div} F = \langle \nabla, F \rangle$

If $n = 3$,

$$\nabla \times F(p) = (D_2 F^3 - D_3 F^2)(e_1)_p + (D_3 F^1 - D_1 F^3)(e_2)_p + (D_1 F^2 - D_2 F^1)(e_3)_p$$

$\nabla \times F$ is called a <u>curl</u>

Differential forms:

Consider a function ω w/
$\omega(p) \in \Lambda^k(\mathbb{R}^n_p)$

Such a function is called a k-form on $\mathbb{R}^n$, or just a differential form.

If $\varphi_1(p), \dots, \varphi_n(p)$ is the dual basis to $(e_1)_p, \dots, (e_n)_p$, then

$$\omega(p) = \sum_{i_1 < \dots < i_k} \underbrace{\omega_{i_1 \dots i_k}(p)}_{\text{functions}} \cdot \left[\varphi_{i_1}(p) \wedge \dots \wedge \varphi_{i_k}(p)\right]$$

A function is considered a 0 form, and $f \cdot \omega$ is also written $f \wedge \omega$.

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If $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ diff, then $Df(p) \in \Lambda^1(\mathbb{R}^n)$

Define a 1-form df by

$$df(p)(v_p) = Df(p)(v)$$

Illustration:

Let π^i denote the projection onto the x^i variable. It is customary to denote π^i by x^i.

Observe that $dx^i(p)(v_p) = d\pi^i(p)(v_p) = D\pi^i(p)(v) = v^i$, it follows that $dx^1(p), \dots, dx^n(p)$ is a dual basis to $(e_1)_p, \dots, (e_n)_p$.

Thus every k-form ω can be written

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

If f is a function, df is of particular interest to us

Theorem: If $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ diff, then

$$df = D_1 f \, dx^1 + \dots + D_n f \, dx^n$$

$$df = \frac{\partial f}{\partial x^1} dx^1 + \dots + \frac{\partial f}{\partial x^n} dx^n$$

Proof:

$$df(p)(v_p) = Df(p)(v) =$$

$$\sum_{i=1}^{n} v^i \cdot D_i f(p) = \sum_{i=1}^{n} dx^i(p)(v_p) \cdot D_i f(p)$$

What about functions $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$?

Then $Df(p): \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Define $\underbrace{f_* : \mathbb{R}^n_p \longrightarrow \mathbb{R}^m_{f(p)}}$

push-forward

$$f_*(v_p) = \left(Df(p))(v)\right)_{f(p)}$$

We now define $\underset{\text{pull-back}}{f^*\omega}$ for a k-form on $\mathbb{R}^m$:

$$(f^*\omega)(p) = f^*(\omega(f(p))), \text{ i.e}$$

$$f^*\omega(p)(v_1, \ldots, v_k) = \omega(f(p))(f_*(v_1), \ldots, f_*(v_k))$$

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Theorem: If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ diff, then

1) $f^*(dx^i) = \sum_{j=1}^{n} D_j f^i\, dx^j = \sum_{j=1}^{n} \frac{\partial f^i}{\partial x^j} dx^j$

$$\left.\begin{array}{l} ② \; f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2) \\ ③ \; f^*(g \cdot \omega) = (g \circ f) \cdot f^*\omega \\ ④ \; f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta \end{array}\right\} \text{much easier}$$

Proof: $f^*(dx^i)(p)(v_p) = dx^i(f(p))(f_* v_p)$

$= dx^i(f(p))\left(\sum_{j=1}^{n} v^j \cdot D_j f^1(p), \dots, \sum_{j=1}^{n} v^j D_j f^m(p)\right)_{f(p)}$

$= \sum_{j=1}^{n} v^j D_j f^i(p) = \sum_{j=1}^{n} D_j f^i(p) \cdot dx^i(p)(v_p)$

Corollary: $f^*(P\, dx^1 \wedge dx^2 + Q\, dx^2 \wedge dx^3)$

$= (P \circ f)\left[f^*(dx^1) \wedge f^*(dx^2)\right]$

$+ Q \circ f\left[f^*(dx^2) \wedge f^*(dx^3)\right]$

Note: $dx^i \wedge dx^i = (-1)\cdot dx^i \wedge dx^i = 0$

Theorem: If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ diff, then

$$f^*(h\,dx^1 \wedge \dots \wedge dx^n) = h\circ f\,(\det f')\,dx^1 \wedge \dots \wedge dx^n$$

Proof:

$$f^*(h\,dx^1 \wedge \dots \wedge dx^n) = (h\circ f)\,f^*(dx^1 \wedge \dots \wedge dx^n)$$

so it is enough to show that

$$f^*(dx^1 \wedge \dots \wedge dx^n) = (\det f')\,dx^1 \wedge \dots \wedge dx^n$$

Let $p \in \mathbb{R}^n$ & $A = \{a_{ij}\}$ be a matrix of $f'(p)$

omit p...

$$f^*(dx^1 \wedge \dots \wedge dx^n)(e_1, \dots, e_n) =$$

$$dx^1 \wedge \dots \wedge dx^n(f_* e_1, \dots, f_* e_n) =$$

$$dx^1 \wedge \dots \wedge dx^n\left(\sum_{i=1}^n a_{i1} e_i, \dots, \sum_{i=1}^n a_{in} e_i\right)$$

$= \det(a_{ij})\,dx^1 \wedge \dots \wedge dx^n(e_1, \dots, e_n)$ by Theorem 4-6.

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Key operation:

If $\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$\sim$ k-form

$$d\omega = \sum_{i_1 < \dots < i_k} d\omega_{i_1 \dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$= \sum_{i_1 < \dots < i_k} \sum_{\alpha=1}^{n} D_\alpha (\omega_{i_1, \dots, i_k}) \cdot dx^\alpha \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

Examples:

$\omega = dx^1 \wedge dx^2 \qquad n = 3 \qquad k = 2$

$$d\omega = \sum_{i_1 < i_2} \sum_{\alpha=1}^{3} D_\alpha (\omega_{i_1, i_2}) \, dx^\alpha \wedge dx^{i_1} \, dx^{i_2}$$

$$\equiv 0$$

$\omega = x^3 dx^1 \wedge dx^2$

$d\omega = dx^3 \wedge dx^1 \wedge dx^2 = dx^1 \wedge dx^2 \wedge dx^3$ ✓

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$\overbrace{P\,dx^1 + Q\,dx^2}^{=\,\omega}$ on $\mathbb{R}^2$ $\quad k=1 \quad n=2$

$$d\omega = \sum_i \sum_{\alpha=1}^{2} D_\alpha(P)\, dx^\alpha \wedge dx^1$$

$$+ \sum_i \sum_{\alpha=1}^{2} D_\alpha(Q)\, dx^\alpha \wedge dx^2$$

$$= D_2 P\, dx^2 \wedge dx^1 + D_1 Q\, dx^1 \wedge dx^2$$

$$= (D_1 Q - D_2 P)\, dx^1 \wedge dx^2$$

Playing ω against df:

$\omega = \sum_{i=1}^{n} \omega_i\, dx^i$ $\quad$ 1-form

Suppose $df = \sum_{i=1}^{n} D_i f \cdot dx^i$ $\qquad f(0)=0$

$$f(x) = \int_0^1 \frac{d}{dt} f(tx)\, dt$$

$$= \int_0^1 \sum_{i=1}^{n} D_i f(tx) \cdot x^i\, dt = \int_0^1 \sum_{i=1}^{n} \omega_i(tx) \cdot x^i\, dt$$

In other words, to find f given ω,

Let $\underline{I}(\omega)(x) = \int_0^1 \sum_{i=1}^{n} \omega_i(tx) \cdot x^i \, dt$

$\int$ makes sense only if ω defined on $A \subseteq \mathbb{R}^n \ni$ whenever $x \in A$, the line segment from 0 to x is contained in A. Such A's are called star-shaped!

Theorem: (Poincare) If $A \subset \mathbb{R}^n$ is an open set, star-shaped w/ respect to 0, then every closed form on A is exact.

Proof:

The strategy is to define $\underline{I}$ from ℓ-forms to $(\ell-1)$-forms $\ni \underline{I}(0) = 0$ and

$\omega = \underline{I}(d\omega) + d(\underline{I}\omega) \quad \forall \omega$

It follows that $\omega = d(\underline{I}\omega)$ if $d\omega = 0$.

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Define

$$\underline{I}\omega(x) = \sum_{i_1 < \dots < i_l} \sum_{\alpha=1}^{l} (-1)^{\alpha-1} \left(\int_0^1 t^{l-1} \omega_{i_1 \dots i_l}(tx)\, dt \right) x^{i_\alpha}$$

$$dx^{i_1} \wedge \dots \wedge \underbrace{\widehat{dx^{i_\alpha}}}_{\text{omitted}} \wedge \dots \wedge dx^{i_l}$$

The proof that $\omega = \underline{I}(d\omega) + d(\underline{I}\omega)$ is a nasty calculation.

$$d(\underline{I}\omega) = l \sum_{i_1 < \dots < i_l} \left(\int_0^1 t^{l-1} \omega_{i_1 \dots i_l}(tx)\, dt \right) dx^{i_1} \wedge \dots \wedge dx^{i_l}$$

$$+ \sum_{i_1 < \dots < i_l} \sum_{\alpha=1}^{l} \sum_{j=1}^{n} (-1)^{\alpha-1} \left(\int_0^1 t^{l} D_j(\omega_{i_1 \dots i_l})(tx)\, dt \right) x^{i_\alpha}$$

$$dx^{j} \wedge dx^{i_1} \wedge \dots \wedge \widehat{dx^{i_\alpha}} \wedge \dots \wedge dx^{i_l}$$

Also, $d\omega = \sum_{i_1 < \dots < i_l} \sum_{j=1}^{n} \left(D_j(\omega_{i_1, \dots, i_l}) \right) dx^{j} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_l}$

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$$\underline{I}(d\omega) = \sum_{i_1 < \dots < i_\ell} \sum_{j=1}^{n} \left(\int_0^1 t^\ell D_j(\omega_{i_1 \dots i_\ell})(tx)\, dt \right) x^j \, dx^{i_1} \wedge \dots \wedge dx^{i_\ell}$$

$$- \sum_{i_1 < \dots < i_\ell} \sum_{j=1}^{n} \sum_{\alpha=1}^{\ell} (-1)^{\alpha-1} \left(\int_0^1 t^\ell D_j(\omega_{i_1 \dots i_\ell})(tx)\, dt \right) x^{i_\alpha} dx^j \wedge dx^{i_1} \wedge \widehat{dx^{i_\alpha}} \wedge dx^{i_\ell}$$

We add and cancel the triple sum, obtaining

$$d(\underline{I}\omega) + \underline{I}(d\omega) =$$

$$\sum_{i_1 < \dots < i_\ell} \ell \cdot \left(\int_0^1 t^{\ell-1} \omega_{i_1 \dots i_\ell}(tx)\, dt \right)$$

$$+ \sum_{i_1 < \dots < i_\ell} \sum_{j=1}^{n} \left(\int_0^1 t^\ell x^j D_j(\omega_{i_1 \dots i_\ell})(tx)\, dt \right)$$

$$= \sum_{i_1 < \dots < i_\ell} \left(\int_0^1 \frac{d}{dt} \left[t^\ell \omega_{i_1 \dots i_\ell}(tx) \right] dt \right)$$

$$dx^{i_1} \wedge \dots \; dx^{i_\ell}$$

$$= \sum_{i_1 < \dots < i_l} \omega_{i_1 \dots i_l} \, dx^{i_1} \wedge \dots \wedge dx^{i_l} = \omega$$

We now backtrack a bit and nail down some properties of the d-operator.

Theorem: i) $d(\omega + \eta) = d\omega + d\eta$

ii) ω k-form, η - l-form, then
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

iii) $d(d\omega) = 0$, ie $d^2 = 0$

iv) If ω is a k-form on $\mathbb{R}^m$ and
$f: \mathbb{R}^n \to \mathbb{R}^m$ diff, then
$$f^*(d\omega) = d(f^* \omega)$$

Proof: i) is immediate

ii) True if $\omega = dx^{i_1} \wedge \dots \wedge dx^{i_k}$
& $\eta = dx^{j_1} \wedge \dots \wedge dx^{j_l}$ since
all terms vanish.

If ω is a 0-form, this is immediate from the definition (check!)

The result follows (why?)

Proof of iii)

$$d\omega = \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^{n} D_\alpha(\omega_{i_1 \ldots i_k})\, dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

so $d(d\omega) =$

$$\sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} D_{\alpha,\beta}(\omega_{i_1 \ldots i_k})\, dx^\beta \wedge dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

Here is the fun part:

the terms $D_{\alpha,\beta}(\omega_{i_1, \ldots, i_k})\, dx^\beta \wedge dx^\alpha \wedge \cdots$

& $D_{\beta,\alpha}(\omega_{i_1 \ldots i_k})\, dx^\alpha \wedge dx^\beta \wedge \cdots$

cancel in pairs!

Proof of iv): Suppose that we have it for 0-forms. Then we can proceed by induction.

$$f^*(d(\omega \wedge dx^i)) = f^*(d\omega \wedge dx^i + (-1)^k \omega \wedge d(dx^i))$$

why is it enough to consider this form?

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$$= f^*(d\omega \wedge dx^i) = f^*(d\omega) \wedge f^*(dx^i)$$

$$= d(f^*\omega \wedge f^*(dx^i)) \quad \text{by ii) and iii)}$$

$$= d(f^*(\omega \wedge dx^i))$$

Singular n-cube

in $A \subseteq \mathbb{R}^n$

$c : [0,1]^n \longrightarrow A$

A singular 0-cube in A is a function $f : \{0\} \longrightarrow A$, i.e a **point** in A.

A singular 1 cube is a map

$c : [0,1] \longrightarrow A$

think of it as a **curve**.

For example, $\underline{I}^n : [0,1]^n \longrightarrow \mathbb{R}^n$

$$\underbrace{\underline{I}^n(x) = x \quad x \in [0,1]^n}_{\text{standard } n\text{-cube}}$$

A weighted sum of singular n-cubes in A is called a chain, e.g. $c_1, c_2, \ldots, c_k$ singular n-cubes, $\alpha_1, \ldots, \alpha_k$ constants, then

chain $= \alpha_1 c_1 + \ldots + \alpha_n c_n$

Boundary of $\partial \underline{I}^n$:

For each i, $1 \le i \le n$, define two singular $(n-1)$-cubes $\underline{I}^n_{(i,0)}$ and $\underline{I}^n_{(i,1)}$ as follows:

If $x \in [0,1]^{n-1}$, then

$$\underline{I}^n_{(i,0)}(x) = \underline{I}^n(x^1, \ldots, x^{i-1}, 0, x^i, \ldots, x^{n-1})$$

$$= (x^1, \ldots, x^{i-1}, 0, x^i, \ldots, x^{n-1})$$

$$I^n_{(i,1)}(x) = I^n(x^1, \dots, x^{i-1}, 1, x^i, \dots, x^{n-1})$$
$$= (x^1, \dots, x^{i-1}, 1, x^i, \dots, x^{n-1})$$

$n = 2 \quad x \in [0,1]$

$$I^2_{(1,0)}(x) = I^2(0, x) = (0, x)$$
$$I^2_{(1,1)}(x) = I^2(1, x) = (1, x)$$
$$I^2_{(2,0)}(x) = I^2(x, 0) \qquad I^2_{(2,1)}(x) = (x, 1) \checkmark$$

Theorem: If c is an n-chain in A, then $\partial(\partial c) = 0$, i.e $\partial^2 = 0$

the proof is a straightforward calculation.

~~Theorem:~~

But what is ∂c?

$I^n_{(i,0)}$ = $(i,0)$-face of I^n

$I^n_{(i,1)}$ = $(i,1)$-face of I^n

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$$\partial \underline{I}^n = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} \underline{I}^n_{(i,\alpha)}$$

$$\partial \underline{I}^2 = \sum_{i=1}^{2} \sum_{\alpha=0,1} (-1)^{i+\alpha} \underline{I}^2_{(i,\alpha)}$$

For a general $c : [0,1]^n \to A$, define
$c_{(i,\alpha)} = c \circ \underline{I}^n_{(i,\alpha)}$ &

$$\partial c = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)},$$

finally,

$$\partial\left(\sum a_i c_i\right) = \sum a_i \partial(c_i)$$

Fundamental Theorem of Calculus

$\omega = k$-form on $[0,1]^k$. Then
$\omega = f\, dx^1 \wedge \dots \wedge dx^k$

Define $\int_{[0,1]^k} \omega = \int_{[0,1]^k} f$

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Equivalently,

$$\int_{[0,1]^K} f\,dx^1\wedge\ldots\wedge dx^K = \int_{[0,1]^K} f(x^1,\ldots,x^K)\,dx^1\ldots dx^K$$

Define $\displaystyle\int_c \omega = \int_{[0,1]^K} c^*\omega$

In particular,

$$\int_{I^K} f\,dx^1\wedge\ldots\wedge dx^K = \int_{[0,1]^K} (I^K)^*(f\,dx^1\wedge\ldots\wedge dx^K)$$

$$= \int_{[0,1]^K} f(x^1,\ldots,x^K)\,dx^1\ldots dx^K$$

0-form definition: ω (0-form) is a function

$$\int_c \omega = \omega(c(0))$$

$$\underbrace{\int_c \omega}_{K\text{-chain}} = \sum a_i \int_{c_i} \omega$$

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Stokes: If ω is a $(k-1)$-form on an open set $A \subseteq \mathbb{R}^n$ and c is a k-chain in A, then

$$\int_c d\omega = \int_{\partial c} \omega.$$

Proof: Suppose that $c = I^k$ and ω is a $(k-1)$-form on $[0,1]^k$. Then ω is a sum of terms of the forms of the type

$$f\, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k,$$

and it suffices to consider those.

$$\int_{[0,1]^{k-1}} I^{k\,*}_{(j,\alpha)} \left(f\, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k \right)$$

$$= \begin{cases} 0 & j \neq i \\ \int_{[0,1]^k} f(x^1, \dots, \alpha, \dots, x^k)\, dx^1 \dots \widehat{dx^i} \dots dx^k & j = i \end{cases}$$

It follows that

$$\int_{\partial I^k} f\, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$$

$$= \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} \int_{[0,1]^{k-1}} I^{k*}_{(j,\alpha)} \left(f\, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \right)$$

$$= (-1)^{i+1} \int_{[0,1]^k} f(x^1, \dots, 1, \dots, x^k)\, dx^1 \dots dx^k$$

$$+ (-1)^i \int_{[0,1]^k} f(x^1, \dots, 0, \dots, x^k)\, dx^1 \dots dx^k$$

On the other hand,

$$\int_{I^k} d\left(f\, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots dx^k\right)$$

$$= \int_{[0,1]^k} D_i f\; dx^i \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$$

$$= (-1)^{i-1} \int_{[0,1]^k} D_i f.$$

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By Fubini and FTC in one variable, we have

$$\int_{I^k} d(f\,dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k)$$

$$= (-1)^{i-1} \int_0^1 \dots \left(\int_0^1 D_i f(x^1, \dots, x^k)\,dx^i \right) dx^1 \dots$$

$$= (-1)^{i-1} \int_0^1 \dots \int_0^1 \left[f(x^1, \dots, 1, \dots, x^k) - f(x^1, \dots, 0, \dots, x^k) \right] dx^1 \dots \widehat{dx^i} \dots dx^k$$

$$= (-1)^{i-1} \int_{[0,1]^k} f(x^1, \dots, 1, \dots, x^k)\,dx^1 \dots \widehat{dx^i} \dots dx^k$$

$$+ (-1)^{i-1} \int_{[0,1]^k} f(x^1, \dots, 0, \dots, x^k)\,dx^1 \dots dx^k$$

Thus $\displaystyle\int_{I^k} d\omega = \int_{\partial I^k} \omega$

Exercise: $\displaystyle\int_{\partial c} \omega = \int_{\partial I^k} c^* \omega$ (easy)

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It follows that since

$$\int_{\partial c} \omega = \int_{\partial I^k} c^* \omega \,, \quad \text{if } c = \text{singular } k\text{-cube}$$

$$\int_{c} d\omega = \int_{I^k} c^*(d\omega) = \int_{I^k} d(c^*\omega)$$

$$= \int_{\partial I^k} c^*\omega = \int_{\partial c} \omega$$

The result for chains follows.