SHARPNESS RESULTS AND KNAPP'S HOMOGENEITY ARGUMENT*

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Introduction

Let S be a smooth compact hypersurface in $\mathbb{R}^n$. Let

$$F_S(\xi) = \int_S e^{i\langle x, \xi \rangle} d\sigma(x) \tag{1}$$

denote the Fourier transform of the surface measure carried by S.

Let $\mathcal{R}f = \hat{f}|_S$, the restriction operator. It is well known (see [T], [G], [S]) that if

$$|F_S(\xi)| \leq C(1+|\xi|)^{-r}, \quad r > 0, \tag{2}$$

then

$$||\mathcal{R}f||_2 \leq C_p ||f||_p, \quad f \in \mathcal{S}(\mathbb{R}^n), \quad \text{for} \quad p \leq p_0 = \frac{2(r+1)}{r+2}, \tag{3}$$

where $\mathcal{S}(\mathbb{R}^n)$ is the standard Schwartz class.

However, it is not in general known whether this result is sharp. More precisely, it is natural to ask the following.

Question A. *Does the estimate* (3) *imply the estimate* (2)*?*

Let

$$Tf(x, x_n) = \int f(x-y, x_n - \Phi(y))\psi(y)dy, \tag{4}$$

where $x, y \in \mathbb{R}^{n-1}$, ψ is a smooth cutoff function, Φ is smooth, $\Phi(0, ..., 0) = 0$, and $\nabla\Phi(0, ..., 0) = (0, ..., 0)$.

It is well known (see [Str]) that if the estimate (2) holds, then

$$||Tf||_{p'} \leq C_p ||f||_p, \quad \text{where} \quad \frac{1}{p} - \frac{1}{2} \leq \frac{r}{2(r+1)}, \tag{5}$$

where p' denotes the conjugate exponent of p.

The key estimate here is

$$||Tf||_{2(r+1)} \leq C||f||_{\frac{2(r+1)}{2r+1}}; \tag{6}$$

the rest follows by interpolation. It is then natural to ask the following.

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Question B. *Does the estimate* (5) *imply the estimate* (2)*?*

The purpose of this paper is to answer questions (A) and B affirmitively in the case of the optimal exponents. We shall employ a multiparameter version of Knapp's homogeneity argument. (See e.g [C] for a similar argument).

More precisely, we will show that if the the estimate (3) holds with $p = \frac{2(n+1)}{n+3}$, then the hypersurface has everywhere non-vanishing Gaussian curvature. Similarly, we will show that if the estimate (5) holds with $p = \frac{n+1}{n}$, then the hypersurface has non-vanishing Gaussian curvature.

We remark here, on the other hand, that non-vanishing Gaussian curvature implies that the estimate (2) holds with $r = \frac{n-1}{2}$ (see e.g. [S]). Thus Question (A) is answered affirmatively in the case $r = \frac{n-1}{2}$. Since the estimate (2) with $r = \frac{n-1}{2}$ implies the estimate (3) with $p = \frac{2(n+1)}{n+3}$, Theorem 2 below shows that the optimal decay of the Fourier transform (i.e. $r = \frac{n-1}{2}$) implies that the hypersurface has non-vanishing Gaussian curvature.

We will also see that if a hypersurface has $\leq k$ non-vanishing principal curvatures at each point, then the exponent p in the estimate (3) can never exceed $\frac{2n+k-2}{6}$. Consequently, the estimate (3) with $p \geq \frac{2n+k-2}{6}$ implies that at least k principal curvatures are non-zero at each point. (See Theorem 3 below). Similarly, we will show that if the estimate (5) holds with $p \geq \frac{2n+k+4}{2n+k+1}$, then at least k principal curvatures are non-zero at each point.

The sharpness of the estimate (3) is known in some cases. For example, if the hypersurface has non-vanishing Gaussian curvature, Knapp's homogeneity argument can be used to show that the exponent $p = \frac{2(n+1)}{n+3}$ is the best possible. Indeed, non-vanishing Gaussian curvature implies that the hypersurface has contact of order two with its tangent plane at every point. Let $f_\delta(x) = g(\delta^{-1}x, \delta^{-2}x_n)$, where $x = (x_1, ..., x_{n-1})$, and g is the characteristic function of the rectangle with sides $(1, ..., 1, C)$, C large, with the long side normal to the hypersurface.

It is not hard to check that $||f_\delta||_p \approx \delta^{(1-\frac{1}{p})(n+1)}$, whereas $||\mathcal{R}f_\delta||_2 \approx \delta^{\frac{n-1}{2}}$. The comparison yields $p \leq \frac{2(n+1)}{n+3}$.

It should be noted that the above example does not verify even a special case of question (A). For example, the above argument does not prove that if the estimate (3) holds with $p = \frac{2(n+1)}{n+3}$, then the estimate (2) holds with $r = \frac{n-1}{2}$. We will show (see Theorem 2 below) that this is indeed the case.

The sharpness of the estimate (5) can also be verified in some cases. By testing T against a characteristic function of a small ball it is not hard to check that if T is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then $(\frac{1}{p}, \frac{1}{q})$ must be contained in the triangle with the endpoints $(0,0)$, $(1,1)$, and $(\frac{n}{n+1}, \frac{1}{n+1})$. However, as before this does not prove that if the estimate (5) holds with $p = \frac{n+1}{n}$, then the estimate (2) holds with $r = \frac{n-1}{2}$. We will show (see Theorem 5 below) that this is indeed the case.

Statement of results

Theorem 1. *Let* $S = \{(x, x_n) \in \mathbb{R}^n : x_n = \Phi(x)\}$*, where* $x = (x_1, ..., x_{n-1})$*,* Φ *is a* *smooth function which does not vanish on a set of positive measure,* $\Phi(0,...,0) = 0$, *and* $\bigtriangledown\Phi(0,...,0) = (0,...,0)$. *Suppose that the estimate* (3) *holds. Let* G *be any continuous function which does not vanish on a set of positive measure satisfying* $G(0,...,0) = 0$. *Then*

$$(|G(\delta)|)^r \geq CR(\delta)^{r+1}|\delta_1\delta_2...\delta_{n-1}|, \tag{7}$$

where $R(\delta) = |\{x \in [-1,1]^{n-1} : |\Phi(\delta_1 x_1, ..., \delta_{n-1}x_{n-1})| \leq C|G(\delta)|\}|$.

Remark. If $G(\delta)$ is chosen to be $\Phi(\delta)$, and Φ is increasing in each variable separately, Theorem 1 says that the estimate (3) implies that $(|\Phi(\delta)|)^r \geq C\delta_1\delta_2...\delta_{n-1}$. The same estimate would be true, of course, if we just assume that $R(\delta)$ is bounded below, which is a much weaker assumption. To prove Theorem 2, Theorem 3, Theorem 5, and Theorem 6 below we shall use Theorem 1 with

$$G(\delta) = \sup_{\{x\in[-1,1]^{n-1}\}} |\Phi(x_1\delta_1, ..., x_{n-1}\delta_{n-1})|. \tag{8}$$

Theorem 2. *Suppose that the estimate* (3) *holds with* $p = \frac{2(n+1)}{n+3}$. *Then the hypersurface* S *has everywhere non-vanishing Gaussian curvature.*

Theorem 3. *Suppose that the estimate* (3) *holds with* $p \geq \frac{2n+k-2}{6}$. *Then the hypersurface* S *has at least* k *non-vanishing principal curvatures at each point.*

Remark. The conclusion of Theorem 3 can be motivated as follows. If the hypersurface has exactly k non-vanishing principal curvatures at a point, then after perhaps applying a rotation we can write it as a graph of the function $x_1^2 + ... + x_k^2 + A(x)$, where A is a higher order remainder. It is not hard to believe that the best possible estimate (2) is obtained if $A(x) = |x''|^3$, where $x'' = (x_{k+1}, ..., x_{n-1})$. This gives us the estimate (2) with $r = \frac{k}{2} + \frac{n-1-k}{3}$. The conclusion of Theorem 3 is the consequence of the fact that $\frac{2(r+1)}{r+2} = \frac{2n+k-2}{6}$.

Theorem 4. *Let* $\delta y = (\delta_1 y_1, ..., \delta_{n-1}y_{n-1})$ *and* $g_\delta(s) = |\{y \in supp(\psi) : |s - |\Phi(\delta y)/\Phi(\delta)|| \leq C\}|$. *Suppose that the estimate* (5) *holds. Then for* $|\delta|$ *sufficiently small,*

$$(|\Phi(\delta)|)^r \geq CP_\delta||g_\delta||_{L^{p'}(ds)}. \tag{9}$$

Theorem 5. *Suppose that the estimate* (5) *holds with* $r = \frac{n-1}{2}$. *Let* $S = \{(x, x_n) : x \in supp(\psi), x_n = \Phi(x)\}$. *Then* S *has everywhere non-vanishing Gaussian curvature.*

Theorem 6. *Suppose that the estimate* (5) *holds with* $p \geq \frac{2n+k+4}{2n+k+1}$. *Then the hypersurface has at least* k *non-vanishing principal curvatures at each point.*

(See the remark after Theorem 3 for the motivation of the conclusion of Theorem 6).

Proof of Theorem 1

Let $\delta x = (\delta_1 x_1, ..., \delta_{n-1}x_{n-1})$, and $\delta^{-1}x = (\delta_1^{-1}x_1, ..., \delta_{n-1}^{-1}x_{n-1})$. Let $\hat{f}_\delta(x, x_n) = g(\delta^{-1}x, \frac{x_n}{|G(\delta)|})$, where g is the characteristic function of a rectangle with sides of length $(1, 1, ..., 1, C)$. Let $P_\delta = |\delta_1\delta_2...\delta_{n-1}|$. It is not hard to see that

$$||f_\delta||_p \approx (P_\delta|G(\delta)|)^{(1-1/p)}. \tag{10}$$

On the other hand,

$$||\mathcal{R}f_\delta||_2^2 = \int \left| g\left(\delta^{-1}x, \frac{\Phi(x)}{|G(\delta)|}\right)\right|^2 dx =$$

$$P_\delta \int \left| g\left(x, \frac{\Phi(\delta x)}{|G(\delta)|}\right)\right|^2 dx \approx CP_\delta R(\delta), \tag{11}$$

where $R(\delta)$ is defined in the statement of the theorem.

Comparing the estimates (10) and (11) we see that (3) can hold only if

$$(|G(\delta)|)^r \geq CP_\delta R^{r+1}(\delta), \tag{12}$$

for $|\delta|$ sufficiently small. This completes the proof of Theorem 1.

Proof of Theorem 2

Let $G(\delta) = \sup_{\{x\in[-1,1]^{n-1}\}} |\Phi(\delta_1 x_1, ..., \delta_{n-1}x_{n-1})|$. It follows that $R(\delta) \equiv 1$, and so

$$(G(\delta))^r \geq CP_\delta, \tag{13}$$

where $r = \frac{n-1}{2}$ by assumption.

After perhaps applying a rotation, we can use Taylor's theorem to write

$$\Phi(x) = a_1x_1^2 + a_2x_2^2 + ... + a_kx_k^2 + A(x), \tag{14}$$

where $A(x)$ is a higher order remainder term, and $k \leq n-1$. If $k = n-1$, then in a sufficiently small neighborhood of the origin the determinant of the Hessian matrix of Φ never vanishes, which would verify the claim of Theorem 2. We shall henceforth assume that $k < n-1$.

It is not hard to check that

$$(G(\delta))^{\frac{n-1}{2}} \leq (a_1\delta_1^2 + ... + a_k\delta_k^2 + C|\delta|^3)^{\frac{n-1}{2}}, \tag{15}$$

$|\delta|$ small.

We must show that the estimate (13) cannot hold if $k < n-1$. It suffices to show that the right hand side of (15) is not bounded below by CP_δ. We may assume that $A(x)$ is not identically 0, and that $A(x)$ depends on x_{n-1}, for otherwise the contradiction is immediate. Let $\delta_j = \delta_{n-1}^{\frac{3}{2}}$. If the right hand side were bounded below by CP_δ, we could use the fact that $A(x)$ is a higher order remainder term to force an inequality

$$|\delta_{n-1}|^{\frac{3(n-1)}{2}} \geq C|\delta_{n-1}|^{\frac{3n-4}{2}}, \tag{16}$$

δ_{n-1} small, which is not true. This shows that the estimate (8) cannot hold unless $k = n-1$. This implies that there exists a small neighborhood of the origin where S has non-vanishing Gaussian curvature. This completes the proof.

Proof of Theorem 3

We must show that if Φ is as in the estimate (14) above, with k denoting the number of non-vanishing principal curvatures, then the estimate

$$(G(\delta))^r \geq CP_\delta \tag{17}$$

can only hold if $r \leq \frac{k}{2} + \frac{n-1-k}{3} = \frac{2n+k-2}{6}$.

Let $\delta = (\delta', \delta'')$, where $\delta' = (\delta_1, ..., \delta_k)$, and $\delta'' = (\delta_{k+1}, ..., \delta_{n-1})$.

Let $\delta_j = |\delta''|^{\frac{3}{2}}$. The estimate (16) cannot hold if the inequality

$$|\delta''|^{3r} \geq C|\delta''|^{(\frac{3k}{2}+(n-1-k))} \tag{18}$$

is not satisfied. However, the estimate (17) can only hold if $r \leq \frac{k}{2} + \frac{n-1-k}{3} = \frac{2n+k-2}{6}$. This completes the proof.

Proof of Theorem 4

Let $\delta^{-1}y = (\delta_1^{-1}y_1, ..., \delta_{n-1}^{-1}y_{n-1})$. Let f denote the characteristic function of the rectangle with sides of length $(1, 1, ..., 1, C)$, C large. Let $\tau_\delta f(x, x_n) = f(\delta x, |\Phi(\delta)|x_n)$, and $\tau_\delta^{-1} f(x, x_n) = f(\delta^{-1}x, |\Phi(\delta)|^{-1}x_n)$. Let $f_\delta(x, x_n) = \tau_\delta^{-1} f(x, x_n)$. Let

$$T_\delta f(x, x_n) = \int f(x - y, x_n - \Phi(y))\psi(\delta^{-1}y)dy. \tag{19}$$

After making a change of variables we see that

$$T_\delta f_\delta(x, x_n) = P_\delta \tau_\delta^{-1} T_\delta^* f(x, x_n), \tag{20}$$

where

$$T_\delta^* f(x, x_n) = \int f(x-y, x_n - \Phi(\delta y)/|\Phi(\delta)|)\psi(y)dy. \tag{21}$$

It is not hard to see that

$$||f_\delta||_p \approx P_\delta^{\frac{1}{p}} (|\Phi(\delta)|)^{\frac{1}{p}}. \tag{22}$$

Also,

$$||T_\delta f_\delta||_{p'} = ||P_\delta \tau_\delta^{-1} T_\delta^* f||_{p'} = P_\delta P_\delta^{\frac{1}{p'}} (|\Phi(\delta)|)^{\frac{1}{p'}} ||T_\delta^* f||_{p'} \approx$$

$$P_\delta P_\delta^{\frac{1}{p'}} (|\Phi(\delta)|)^{\frac{1}{p'}} ||g_\delta||_{L^{p'}(ds)}, \tag{23}$$

where g_δ is defined above.

Comparing the estimates (22) and (23) yields the assertion of the theorem.

Proof of Theorem 5 and Theorem 6

Let $G(\delta) = \sup_{x\in[-1,1]^{n-1}} |\Phi(\delta x)|$. The proof of Theorem 4 shows that if the estimate (5) holds then

$$(G(\delta))^r \geq CP_\delta. \tag{24}$$

The proof of Theorem 5 and Theorem 6 now follows in the same way as the proofs of Theorem 2 and Theorem 3.

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