Proofs from the Book: Infinity of primes

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- A prime factorization of the integer $n > 1$ is the expression

$$n = p_1^{a_1} \cdot p_2^{a_2} \dots p_k^{a_k},$$

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where p_k's are distinct primes.

- Moreover, the prime factorization is unique!

Uniqueness of prime factorization

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Lemma

(Bezout's identity) Let a, b be integers with the greatest common divisor d. Then there exist integers x, y such that

$$ax + by = d.$$

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Euclid's lemma

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- Finally, we shall use Euclid's lemma to establish the uniqueness of the prime number factorization.

Bezout and Euclid

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- In particular, $S_{a,b}$ is not empty.

- By the well-ordering principle, $S_{a,b}$ has the least element
$$d = as + bt.$$

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- Observe that
$$r = a - dq = a - q(as + bt) = a(1 - qs) - bqt,$$
which implies that $r \in S_{a,b} \cup \{0\}$.

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- But $r < d$ and d is the least element in $S_{a,b}$, so $r = 0$ and hence d is a divisor of a. In the same way, d is a divisor of b.

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- This completes the proof of Bezout's identity.

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- Suppose that p does not divide a. Then by Bezout's identity, there exist x, y such that
$$px + ay = 1.$$
- Multiplying both sides by b yields
$$bpx + bay = b.$$
- Observe that $\boxed{bpx}$ is divisible by p because p is present and $\boxed{bay}$ is divisible by p because p divides ab by assumption. This implies that p divides b, and Euclid's lemma is proved.

Existence and uniqueness of prime number factorization

Theorem

Every positive integer n can be written in the form

$$p_1^{a_1} p_2^{a_2} \dots p_k^{a_k},$$

where each p_j is prime, $a_j \geq 1$, and

$$p_1 < p_2 < \cdots < p_k.$$

Moreover, this representation of n is unique.

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- if n is prime, there is nothing to prove.
- If n is not prime, $n = ab$, where $a < n, b < n$.
- By the induction hypothesis, a is a product of primes and so is b, so $n = ab$ is also a product of primes.

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- We see p_1 divides $q_1 q_2 \dots q_k$, so p_1 divides some q_i by Euclid's lemma.
- Without loss of generality, p_1 divides q_1, which implies that $p_1 = q_1$ since they are both prime.

Proof of uniqueness (concluded)

- Going back to factorization of n, we may cancel p_1 and q_1, which yields

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- This completes the proof of uniqueness of the prime number factorization.

Euclid's proof of the infinity of primes

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- We conclude that m must be a prime number, which is a contradiction since we assumed that

$$p_1, \ldots, p_n$$

is a complete list of primes.

Sam Northshield's proof of the infinity of primes

- Suppose that the set of primes $\mathbb{P}$ is finite. Then

$$0 < \prod_{p \in \mathbb{P}} \sin\left(\frac{\pi}{p}\right)$$

since all the angles $\frac{\pi}{p}$ are in the first quadrant.

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- On the other hand,

$$\prod_{p \in \mathbb{P}} \sin\left(\frac{\pi}{p}\right) = \prod_{p \in \mathbb{P}} \sin\left(\frac{\pi}{p} + \frac{2\pi \prod_{p' \in \mathbb{P}} p'}{p}\right)$$

Sam Northshield's proof (concluded)

- $$= \prod_{p \in \mathbb{P}} \sin\left(\frac{\pi \left(1 + 2 \prod_{p' \in \mathbb{P}} p'\right)}{p} \right) = 0.$$

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- Why is it 0?

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- Why is it 0?

- Because
$$1 + 2 \prod_{p' \in \mathbb{P}} p'$$
must be divisible by some $p \in \mathbb{P}$ by the virtue of the fact that every number is a product of primes.

Fermat numbers proof

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- If we can show that all the Fermat numbers are relatively prime (no divisors in common), then there must be infinitely many primes.

- To this end, we are going to prove that

$$\prod_{k=0}^{n-1} F_k = F_n - 2.$$

Fermat and friends

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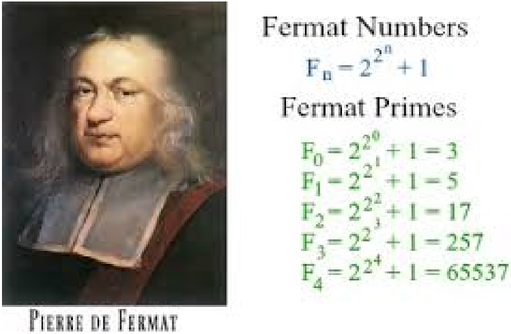

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- This proves that F_n's are relatively prime provided that the recurrence above holds.
- We now turn our attention to the proof of the recurrence.

Proof of the Fermat recurrence

- We proceed by induction. If $n = 1$, we have

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- $$= (2^{2^n} - 1)(2^{2^n} + 1) = 2^{2^{n+1}} - 1 = F_{n+1} - 2.$$

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- This is a two-way infinite arithmetic progression in $\mathbb{Z}$.

- Define a subset O of $\mathbb{Z}$ to be **open** if either O is empty, or for every $a \in O$, there exists $b > 0$ such that

$$N_{a,b} \subset O.$$

Properties of open and closed sets

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$$N_{a,b} = N_{a+kb,b}.$$

Properties of open and closed sets

- We say that $O \subset \mathbb{Z}$ is **closed** if $\mathbb{Z} \backslash O$ is open.

- Every set $N_{a,b}$ is **open** since given any $a' \in N_{a,b}$, i.e $a' = a + kb$ for some k,

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- By the same argument, the union of any number (finite or infinite) of $N_{a,b}$'s is **open**.

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- Suppose that O_1 and O_2 are both **open** and consider $a \in O_1 \cap O_2$.
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- Then $N_{a,b_1} \subset O_1$ and $N_{a,b_2} \subset O_2$ for some $b_1, b_2 > 0$.
- But then

$$N_{a,b_1 b_2} \subset O_1 \cap O_2,$$

so $O_1 \cap O_2$ is **open**.

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- hence $N_{a,b}$ is a complement of an **open** set, so it is **closed**!

Primes enter the picture

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- Suppose that the set of primes $\mathbb{P}$ is finite. Then the right hand side is a union of finitely many **closed sets**.
- If $\bigcup_{p \in \mathbb{P}} N_{0,p}$ is **closed**, we are done because then $\{-1, 1\}$ is **open**, which is impossible since by definition, **open** sets contain an infinite two-sided arithmetic progression.

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- Since the intersection of finitely many **open** sets is open, as we showed above, we conclude that

$$\bigcup_{p\in\mathbb{P}} N_{0,p} \text{ is } \textbf{closed} \text{ and we are done!}$$

DeMorgan Laws

- We shall state these for subsets of the integers, but these laws are really universal. Let $A_1, A_2, \dots, A_n \subset \mathbb{Z}$. Then

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- It follows that $m \in \mathbb{Z} \backslash A_i$ for some i, which means that

$$m \in \bigcup_{i=1}^{n} \mathbb{Z} \backslash A_i.$$

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- Now suppose that $m \in \bigcup_{i=1}^{n} \mathbb{Z} \setminus A_i$. Then $m \in \mathbb{Z} \setminus A_i$ for some i.
- This implies that $m \notin \bigcap_{i=1}^{n} A_i$, so we conclude that

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- We have shown that the left hand side is a subset of the right hand side, and vice-versa, so the proof is complete.

DeMorgan Laws in pictures

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