MATH 238, FALL 2023, HOMEWORK #1

ALEX IOSEVICH

Problem #1 We proved in class that if A is an n by n matrix with 1 and 0 entries, such that if $j \neq j'$, then

$$a_{ij}a_{ij'} = 1 \text{ for at most one value of } i.$$

Then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \leq \sqrt{2} n^{\frac{3}{2}}.$$

Prove that under these assumptions, it is impossible to prove an estimate of the form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \leq C n^{a}$$

with $a < \frac{3}{2}$ and C independent of n.

Problem #2: State and prove an analogous result for n by m matrices, i.e if A is an n by n matrix with 1 and 0 entries, such that if $j \neq j'$, then

$$a_{ij}a_{ij'} = 1 \text{ for at most one value of } i,$$

then a suitable bound on $\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}$ holds. Find such a bound and prove it.

Problem #3: Let P be a collection of n points in $\mathbb{R}^3$. Let L denote a collection of m two-dimensional affine subspaces of $\mathbb{R}^3$ satisfying the hypothesis that if H, H', H'' are planes in L, then $H \cap H' \cap H''$ contains at most one point from P. Let $I(P, L)$ denote the total number of incidences between P and L. State and prove the best bound you can prove for $I(P, L)$.

Problem #4: i) Fill in all the details for the proof that n points in $\mathbb{R}^d$, $d \geq 2$, determine at least $C_d n^{\frac{1}{d}}$ distances. In the course of writing your proof, please estimate C_d.

ii) How does the problem change if we take P to be a finite set with n elements, Q to be a finite set with m elements, and define

$$\Delta(P, Q) = \{|x - y| : x \in P, y \in Q\}?$$

State and prove a variant of i) in this setting.

Problem #5: We say that a bounded subset K of $\mathbb{R}^d$, $d \geq 2$, is convex if given $x, y \in K$, $(1-t)x + ty \in K$ for every $t \in [0,1]$. Suppose that K is also symmetric with respect to the origin, in the sense that $-x \in K$ whenever $x \in K$. Define $||x||_K$ to be the largest positive number t such that $tx \in K$. Given a finite set $P \subset \mathbb{R}^d$, define

$$\Delta_K(P) = \{||x-y||_K : x, y \in P\}.$$

i) Prove that if $d = 2$ and K is a symmetric (with respect to the origin) convex polygon with finitely many sides, then if P is a finite set with n elements, then

$$|\Delta_K(P)| \geq cn^{\frac{1}{2}}.$$

ii) (Harder) Prove that if $d \geq 2$ and K is a symmetric (with respect to the origin) symmetric polyhedron finitely many sides, then if P is a finite set with n elements, then

$$|\Delta_K(P)| \geq c_d n^{\frac{1}{d}}.$$

iii) Prove that if K is a symmetric (with respect to the origin) convex polyhedron in $\mathbb{R}^d$, $d \geq 2$, the estimate in part ii) is best possible, i.e there is no estimate of the form

$$|\Delta_K(P)| \geq c_d n^a$$

is true if $a > \frac{1}{d}$ and c_d is independent of n.

Problem #6: Let P be a finite subset of $\mathbb{R}^2$ with n elements. Define

$$\prod(P) = \{x \cdot y : x, y \in P\},$$

where

$$x \cdot y = x_1 y_1 + x_2 y_2.$$

i) Is it true that

$$|\prod(P)| \geq cn^{\frac{1}{2}}$$

with c independent of n? Prove or disprove.

ii) State and prove a higher dimensional version of i).

Problem #7: (Exploration) Prove that

$$|\{a^2 + b^2 : 0 \leq a, b \leq \sqrt{n}\}| \leq C \frac{n}{\sqrt{log(n)}},$$

where C is independent of n. More precise estimate are available, but this will do for now.

Please feel free to look this up. What is important is that you learn to search for information and figure it out on the fly.

Problem #8: ii) Write out all the details of Moser's proof of the fact that if P is a finite points set with n elements in $\mathbb{R}^2$, then

$$|\Delta(P)| \geq cn^{\frac{2}{3}}.$$

ii) Use Moser's argument to obtain an improvement over the $|\Delta(P)| \geq c_d n^{\frac{1}{d}}$ bound in higher dimensions.