Math 201

Final Exam ANSWERS

December 13, 2015

1. (1 points)

(a) $\binom{15}{4}$.

(b) The number of subsets where none of $1, 2, 3, 4, 5$ appear is $\binom{10}{4}$, hence the number of subsets where at least one of $1, 2, 3, 4, 5$ appears is $\binom{15}{4} - \binom{10}{4}$.

2. (1 points)

(a) 4/52.

(b) The number of all possible orderings is 52!. The number of orderings where the first ace is on the 8th position can be counted as follows: there cannot be any aces in the first 7 slots, so they have to come from the remaining 48 cards. There are $48 \cdot 47 \cdot \cdots \cdot 42 = \frac{48!}{41!}$ ways to choose the first 7 cards. The 8th card must be an ace, and there are 4 choices. They remaining 44 cards can come in any order, so there are 44! ways to choose the rest. Thus, there are $\frac{48!}{41!} 4 \cdot 44!$, which implies the probability of the first ace being in the 8th position is

$$\frac{48! \cdot 4 \cdot 44!}{41! \cdot 52!}.$$

3. (1 points)

$P(A \cup B) = P(A) + P(B) - P(AB) = P(A) + P(BA^c) = P(A) + P(B|A^c)P(A^c) = P(A) + P(B|A^c)(1 - P(A)) = 0.7 + 0.2(1 - 0.7) = 0.76$.

4. (1 points)

(a) Let X be the number of defects. X is $\sim Ber(100, 0.05)$, so the expected number is $100 * 0.05 = 5$.

(b) $P(X \leq 1) = P(X = 0) + P(X = 1) = 0.95^{100} + \binom{100}{1} 0.95^{99} 0.05^1$.

(c) X can be approximated by a $Poisson(100 \cdot 0.05) = Poisson(5)$ random variable. Thus $P(X \leq 4) = \sum_{i=0}^{4} P(X = i) \approx \sum_{i=0}^{4} e^{-5} \frac{5^i}{i!}$.

5. (1 points)

(a) $1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 cx^n dx = cx^{n+1}/(n+1)|_0^1 = c/(n+1)$, so $c = (n+1)$.

(b) $P(X > a) = \int_a^1 (n+1)x^n dx = x^{n+1}|_a^1 = 1 - a^{n+1}$.

(c) $\lim_{n\to\infty} P(X > a) = \lim_{n\to\infty} 1 - a^{n+1} = 1$ since $0 < a < 1$.

6. (1 points)

$P(X > 0) = P(\frac{X-\mu}{\sigma} > \frac{0-\mu}{\sigma}) = P(\frac{X-\mu}{\sigma} > -1)$. Since $\frac{X-\mu}{\sigma}$ is a standard normal, we get $P(X > 0) = \Phi(-1)$.

7. (1 points)

(a) It is easy to see that $f(x,y) \geq 0$. It remains to show $\iint f(x,y)dxdy = 1$. We have $\iint f(x,y)dxdy = \int_0^1 \int_0^{1-y} 24xydxdy = \int_0^1 12(1-y)^2 ydy = \int_0^1 12y - 24y^2 + 12y^3 dy = (6y^2 - 8y^3 + 3y^4)|_0^1 = 6 - 8 + 3 = 1$.

(b) We have $P(Y > 1/2) = \int_{1/2}^1 \int_0^{1-y} 24xydxdy = (6y^2 - 8y^3 + 3y^4)|_{1/2}^1 \neq 0$. Similarly $P(X > 1/2) \neq 0$, however $P(X > 1/2, Y > 1/2) = 0$, so X and Y are not independent.

8. (1 points)

(a) $M_{X+Y}(t) = E[e^{(X+Y)t}] = E[e^{Xt}e^{Yt}]$. If X and Y are independent, then so are e^{Xt} and e^{Yt}. Thus $E[e^{Xt}e^{Yt}] = E[e^{Xt}]E[e^{Yt}] = M_X(t)M_Y(t)$.

(b) The moment generating function of X is $M_X(t) = E[e^{Xt}] = \sum_{k=0}^{\infty} e^{kt}e^{-\lambda_1}\frac{\lambda_1^k}{k!} = e^{-\lambda_1}\sum_{k=0}^{\infty}\frac{(\lambda_1 e^t)^k}{k!} = e^{-\lambda_1}e^{\lambda_1 e^t} = e^{\lambda_1(e^t-1)}$. Similarly $M_Y(t) = e^{\lambda_2(e^t-1)}$. Using part (a) we get $M_{X+Y}(t) = e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)}$.

(c) Since $e^{(\lambda_1+\lambda_2)(e^t-1)}$ is the moment generating function of a $Poisson(\lambda_1 + \lambda_2)$ random variable, that must be the distribution of $X + Y$.

9. (1 points)

(a) $Cov(X_i - \bar{X}, \bar{X}) = Cov(X_i, \bar{X}) - Cov(\bar{X}, \bar{X}) = \sum_{j=1}^n Cov(X_i, X_j/n) + Var(\bar{X})$. In the remaining sum all the terms are zero except the one where $j = i$ since the X_i's are independent. Thus, the sum is $Cov(X_i, X_i/n) = Var(X_i)/n$. For the same reason, $Var(\bar{X}) = \sum_{j=1}^n Var(X_j/n) = nVar(X_i/n) = Var(X_i)/n$. Hence $Cov(X_i - \bar{X}, \bar{X}) = 0$.

(b) No. For example take $n = 2$ and X_1, X_2 i.i.d. $Ber(1/2)$. Then $P(X_1 - \bar{X} = 0, \bar{X} = 0) =$

$P(\frac{X_1-X_2}{2} = 0, \frac{X_1+X_2}{=} 0) = P(X_1 = 0, X_2 = 0) = 1/4$, $P(\bar{X} = 0) = \P(\frac{X_1+X_2}{2} = 0) = P(X_1 = 0, X_2 = 0) = 1/4$ and $P(X_1 - \bar{X} = 0) = P(X_1 = X_2) = 1/2$, so $P(X_1 - \bar{X} = 0, \bar{X} = 0) \neq P(X_1 - \bar{X} = 0)P(\bar{X} = 0)$, which implies $X_1 - \bar{X}$ and $\bar{X}$ are not independent. However, they are uncorrelated, as part (a) shows.

10. (1 points)

(a) Let S_n be the sum of the 10 dice. Let X_1 the value of the first die roll. We have $E[S_n] = 10E[X_1] = 10 \cdot 3.5 = 35$ and $Var(S_n) = 10Var[X_1] = 10(E[X_1^2] - E[X_1]^2) = 10(\sum_{i=1}^{6} i^2/6 - (7/2)^2) = 175/6$. Thus $P(30 < S_n < 40) = P(-\frac{5}{\sqrt{175/6}} < \frac{S_n - E[S_n]}{\sqrt{Var[S_n]}} < \frac{5}{\sqrt{175/6}})$, which, by the central limit theorem can be approximated by $\Phi(\frac{5}{\sqrt{175/6}}) - \Phi(-\frac{5}{\sqrt{175/6}}) = \Phi(\sqrt{6/7}) - \Phi(-\sqrt{6/7})$.

(b) $P(X_1 \dots X_{100} \leq a^{100}) = P(\log_a(X_1 \dots X_{100}) \leq \log_a a^{100}) = P(\log_a X_1 + \dots + \log_a X_{100} \leq 100)$. The random variables $Y_i = \log_a X_i$ are iid, so the central limit theorem apples to them. Let $\mu = E[Y_1] = \sum_{i=1}^{6} \frac{\log_a i}{6} = \frac{\log_a 21}{6}$ and $\sigma^2 = Var[Y_1] = \sum_{i=1} \frac{(\log_a i)^2}{6}$. Then $P(X_1 \dots X_{100} \leq a^{100}) = P(\frac{Y_1 + \dots + Y_{100} - 100\mu}{\sigma} \leq \frac{100 - 100\mu}{\sigma}) \approx \Phi(\frac{100 - 100\mu}{\sigma})$.