# Math 201: Introduction to Probability

Midterm 1

February 28, 2019

**NAME (please print legibly):** ______________________________
**Your University ID Number:** ______________________________

**Instructions:**

1. Indicate your instructor with a check in the appropriate box:

| | | |
|---|---|---|
| Krishnan | MW 14:00 | |
| Tucker | MW 10:25 | |

2. Read the notes below:

   - **The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.**
   - **Notes of any kind are strictly forbidden.**
   - **Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
   - **You do not need to simplify complicated numerical expressions such as $\binom{100}{30}$ and $50!$ to a number.**
   - **You are responsible for checking that this exam has all 12 pages.**

3. Read the following Academic Honesty Statement and sign:

   I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

   Signature:______________________________

| QUESTION | VALUE | SCORE |
|---|---|---|
| 1 | 15 | |
| 2 | 15 | |
| 3 | 15 | |
| 4 | 15 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 20 | |
| TOTAL | 100 | |

**1. (15 points)** Three balls are chosen without replacement from an urn that contains 2 red, 2 green, and 2 yellow balls (for a total of six balls).

(a) What is the chance that of the three balls chosen, exactly two are green?

**Solution.** The sample space is all sets of 3 balls out of the 6, which has size $\binom{6}{3}$. The number of ways of choosing three balls out of which exactly 2 is green is $\binom{2}{2} \cdot \binom{4}{1}$, so we obtain a chance of

$$\frac{\binom{2}{2} \cdot \binom{4}{1}}{\binom{6}{3}} = \frac{4}{20} = \frac{1}{5}$$

(b) What is the chance that all three balls are the same color?

**Solution.** Since there are only three balls of each color, there is no way this can happen so 0.

(c) What is the chance that all three balls are different colors?

**Solution.** There are two ways of choosing each of the three balls (since there are two balls fo each color), so we obtain

$$\frac{2^3}{\binom{6}{3}} = \frac{8}{20} = \frac{2}{5}.$$

**2. (15 points)** Suppose we have subsets $A$, $B$, and $C$ of a sample space such that $P(A) = .5$, $P(B) = .2$, $P(C) = .7$, $P(A \cup B) = .6$ , and $P(A \cup C) = .7$.

(a) Calculate $P(A \cap B)$

**Solution.** We use inclusion exclusion to caculate. We have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

which gives $.6 = .5 + .2 = P(A \cap B)$ so $P(A \cap B) = .1$.

(b) Are $A$ and $B$ independent?

**Solution.** Yes, because $P(A)P(B) = P(A \cap B)$ since $.5 \cdot .2 = .1$.

(c) Are $A$ and $C$ independent? (Recall the data from the previous page: $P(A) = .5$, $P(B) = .2$, $P(C) = .7$, $P(A \cup B) = .6$ , and $P(A \cup C) = .7$.)

**Solution.** No, since $P(A \cup C) = P(C)$, we must have that $A$ is contained in $C$, so $A \cap C = A$, so

$$P(A \cap C) \neq P(A)P(C).$$

**3. (15 points)** Suppose Ram and Laxman play a game: they take turns shooting arrows at a bullseye. Ram goes first, and if he misses, Laxman goes next. If Laxman misses as well, the round ends, and the next round begins where Ram again goes first. Ram hits the bulls eye with probability 1/3, and Laxman hits with probability 1/2. The game ends when one of them hits the bullseye. Let $X$ be the total number of rounds played.

(a) What kind of random variable is $X$? Name its distribution and give its parameter values. Also write down the pmf (probability mass function) of $X$.

**Solution.** In any given round, there is a $(1-1/3)(1-1/2) = 1/3$ chance that neither player wins, and thus a 2/3 chance that one of the players wins. Thus, the chance that one of the players wins in the $k$-th round is $(1/3)^{k-1} \cdot (2/3)$ ($k-1$ failures followed by one success). So

$$P(X = k) = (1/3)^{k-1}(2/3)$$

This is the geometric distribution $Geom(2/3)$.

Note that the solution above is when one thinks of a "round" as both players taking a turn. If you think of a "round" as a single player's turn, you get a different answer that depends on whether $k$ is even or odd. You get $P(X = k) = (1/3)^{(k+1)/2}$ for odd $k$ and $P(X = k) = (1/3)^{k/2}$ for $k$ even (note that this is not technically a gemoetric distribution.)

(b) Find the probability that Ram wins in the end.

**Solution.** The chance that Ram wins in the $k$-th round is $(1/3)^{k-1} \cdot (1/3)$ (as above, $k-1$ rounds where no one wins followed by Ram winning). So we get

$$\sum_{k=1}^{\infty}(1/3)^{k-1}(1/3) = \frac{1}{3}\sum_{n=0}^{\infty}(1/3)^n = \frac{1/3}{1-(1/3)} = \frac{1}{2}.$$

On this problem, you needed to evaluate $\sum_{n=0}^{\infty}(1/3)^n$ to get full credit (you lost one point if you did not evaluate it).

**4. (15 points)** There are 30,000 offshore oil rigs in North America, and about 50 rig inspectors in the Minerals Management Service, a government oversight agency. Being severely understaffed, the agency can only survey a randomly selected group of 1,000 rigs annually. Given that a rig is inspected, there is a 1 percent (0.01 probability) that it will have a catastrophic failure the following year. If it's not inspected, there is a 5 percent chance that it will fail. Label the rigs with numbers $\{1, 2, \ldots, 30,000\}$.

(a) **5 points** Let $I_1$ be the event the rig number 1 is inspected. What is $P(I_1)$? **Solution:** Different rigs are equally likely to be inspected. There are 1000 out of 30000 inspected so $P(I_1) = \frac{1000}{30000} = \frac{1}{30}$. The more cumbersome way is the following: there are $\binom{30000}{1000}$ ways of choosing a sample of 1000 rigs. The number of ways of choosing a sample that contains rig 1 is $\binom{29999}{999}$. Therefore the probability of capturing rig 1 in the sample is

$$\binom{29999}{999} \Big/ \binom{30000}{1000} = \frac{29999!\,999!}{(29999-999)!} \frac{1000!(30000-1000)!}{30000!} = \frac{1000}{30000}$$

(b) **5 points** Let $E_1$ be the event that rig number 1 will fail. What is $P(E_1)$?

$$P(E_1) = P(E_1|I_1)P(I_1) + P(E_1|I_1^c)P(I_1^c) = \frac{1}{100}\frac{1}{30} + \frac{5}{100}\frac{29}{30} = \frac{146}{3000}$$

(c) **5 points** What is the probability that rig number 1 was inspected given that it failed the following year? *Hint: express the probability first in terms of* $E_1$ *and* $I_1$. **Solution:**

$$P(I_1|E_1) = \frac{P(E_1|I_1)P(I_1)}{P(E_1)} = \frac{\frac{1}{100}\frac{1}{30}}{146/3000} = \frac{1}{146}$$

**5. (10 points)** Consider the square in the plane consisting of all $(x, y)$ such that $-1 \leq x, y \leq 1$, and let $Q$ be a point chosen uniformly at random inside the square. Let $X$ be the distance from $Q$ to $(0, 0)$. Calculate $P(X > 1)$.

**Solution.** This square is $2 \times 2$, so its area is 4. The set of points within it that are at a distance of 1 or less from the origin is simply the circle of radius 1. Thus the area of points $Q$ where that are a distance of more than 1 is $4 - \pi$. This means that

$$P(X > 1) = \frac{4 - \pi}{4}.$$

**6. (10 points)** A random sample (without replacement) of 1000 Americans are polled about their voting preferences. They vote either Democrat or Republican. Democrats make up 49% of the total number of voters. The total number of voters is 150 million. Give an expression for the probability that exactly 550 people in the sample of 1000 people are Democrats.

**Solutions:** This is a hypergeometric problem, since it is sampling without replacement. However, you do not really need to know anything about the hypergeometric distribution. Let $N = 150 \times 10^6$ be the total number of people. Simply find the ways of selecting 550 Democrats, 450 Republicans from $0.49N$ Democrats and $0.51N$ Republicans respectively.

$$\frac{\binom{0.49N}{550}\binom{0.51N}{450}}{\binom{N}{1000}}$$

The binomial coefficients in the numerator were worth 4 points each. The denominator was worth 2 points.

A lot of you did this using sampling with replacement. In this case, the answer is

$$\binom{1000}{550} 0.49^{550} 0.51^{450}$$

You got six points for this, since numerically, this is very very close to the correct answer. In fact, later in the course, we will see how the hypergeometric distribution is approximated by the binomial in a certain parameter regime.

You got only 2 points if you wrote the pmf of the binomial incorrectly and dropped the binomial coefficient. If the binomial coefficient is present and you got it wrong, then you got 3 points total on this problem.

**7. (20 points)** Jane must get at least three of the four problems on the exam correct to get an A. She has been able to do 80% of the problems on old exams, so she assumes that the probability she gets any problem correct is 0.8. She also assumes that the results on different problems are independent. Let $X$ describe the number of problems she gets correct on the exam.

(a) **6 points** Write down the pmf (probability mass function) for $X$.
**Solution:** This is $X \sim \text{Binomial}\, 4, 0.8$ and so

$$P(X = k) = \binom{4}{k} 0.8^k 0.2^{4-k}$$

You got six points if you got the pmf correct. If you missed the binomial coefficient, you got only 3 points.

(b) **7 points** What is the probability she gets an A?
**Solution:** She could get 3 problems or 4 problems right to get an $A$. So

$$\begin{aligned} P(A) &= P(X = 3) + P(X = 4) \\ &= \binom{4}{3} 0.8^3 0.2^{4-3} + \binom{4}{4} 0.8^4 0.2^{4-4} \end{aligned}$$

If you missed the $P(X = 4)$ then you only got 3 points. If you used the incorrect pmf from part $a$, you were still given 7 points as long as you got the idea right.

(c) **7 points** If she gets the first problem correct, what is the probability she gets an A?

**Solution:** Since the last three problems are independent of the first problem, $X' \sim \text{Binomial}(3, 0.8)$ is the random variable describing the number of problems she gets right on the last three problems. Then

$$\begin{aligned} P(A|X=1) &= P(X' \geq 2) \\ &= \binom{3}{2} 0.8^2 0.2 + 0.8^3 \end{aligned}$$

You got 2 or 3 points if you just wrote the form of the conditional probability, depending on how clear your presentation was.