THE ZEROS OF HURWITZ'S ZETA-FUNCTION ON $\sigma = 1/2$

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Dedicated to Professor Emil Grosswald

1. Introduction.

Let $s = \sigma + it$ be a complex variable. For a fixed α, $0 < \alpha \leq 1$, Hurwitz's zeta-function is defined in the half-plane $\sigma > 1$ by

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} (n+\alpha)^{-s},$$

and except for a simple pole at $s = 1$, may be analytically continued throughout the complex plane. The resemblance of $\zeta(s,\alpha)$ to Riemann's zeta-function, $\zeta(s)$, is in certain ways superficial. For besides the two cases $\zeta(s,1/2) = (2^s-1)\zeta(s)$ and $\zeta(s,1) = \zeta(s)$, $\zeta(s,\alpha)$ possesses neither a functional equation nor an Euler product. It is therefore not surprising that the zeros of these functions are distributed differently. For instance, we note the following:

1. While $\zeta(s)$ has no zeros in $\sigma > 1$, $\zeta(s,\alpha)$ has infinitely many (provided $\alpha \neq 1/2$ or 1). In particular the analogue of the Riemann hypothesis for $\zeta(s,\alpha)$ is false. This was proved by Davenport and Heilbronn [3] when α is rational ($\neq 1/2$ or 1) or transcendental, and by Cassels [1] when α is an algebraic irrational. One may also prove a quantitative version of this result [2; p. 1780]. Namely, for any $\delta > 0$, the number of zeros of $\zeta(s,\alpha)$ ($\alpha \neq 1/2$ or 1) in the rectangle $1 < \sigma < 1+\delta$, $0 < t < T$ is $\asymp T$ for sufficiently large T.

2. Let σ_1,σ_2 be fixed with $1/2 < \sigma_1 < \sigma_2 < 1$. Then $\zeta(s,\alpha)$ has infinitely many zeros in the strip $\sigma_1 < \sigma < \sigma_2$ when α is rational ($\neq \frac{1}{2}$ or 1) or transcendental. The rational case is due to S.M. Voronin [8] (see also S.M. Gonek [5]), the transcendental case to S.M. Gonek [5]. Here too one can show that the number of zeros up to height T is $\asymp T$ for all large T. On the other hand, well-known zero-density estimates imply that $\zeta(s)$ has at most $o(T)$ zeros in such a rectangle.

Pursuing these contrasts further, one might naturally ask whether the line $\sigma = 1/2$ is special to $\zeta(s,\alpha)$ as it is to $\zeta(s)$. We know that as T tends to infinity, the number of zeros of either function in the strip $0 < t < T$ is $\sim \frac{T}{2\pi} \log T$. For $\zeta(s)$, N. Levinson [7] showed that more than 1/3 of these zeros lie on $\sigma = 1/2$; it is widely held that the correct proportion is 1. In this paper, our purpose is to show that for certain values of α the proportion of zeros of $\zeta(s,\alpha)$ on $\sigma = 1/2$ is definitely less than 1. Specifically, we shall prove the following result.

THEOREM. Let $\alpha = \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}$ or $\frac{5}{6}$. There is a positive constant $c < 1$ such that the number of zeros of $\zeta(s,\alpha)$ (counted according to their multiplicities) on the segment $[1/2, 1/2 + iT]$ is $\le (c+o(1)) \frac{T}{2\pi} \log T$ as T tends to infinity.

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2. An Auxiliary Lemma.

To prove our theorem we require information about the number of zeros common to two L-functions. This is provided by the lemma below which is essentially due to A. Fujii [4; Theorem 1].

Recall that two Dirichlet characters not induced by the same primitive character are called *inequivalent*. We denote by $L(s,\chi)$ the Dirichlet L-function with character χ.

LEMMA. Suppose χ_1 and χ_2 are inequivalent characters. Let $\rho_1 = \beta_1 + i\gamma_1$ denote a zero of $L(s,\chi_1)$ with $0 < \beta_1 < 1$, and write $m_i(\rho_1)$ for the multiplicity of ρ_1 as a zero of $L(s,\chi_i)$ $(i = 1,2)$. Then there exists a positive constant $c < 1$ such that

$$\sum_{0 \le \gamma_1 \le T}' \min_{i=1,2} m_i(\rho_1) \le (c+o(1)) \frac{T}{2\pi} \log T \tag{1}$$

as T tends to infinity, where $\sum'$ means the sum is over distinct zeros ρ_1.

PROOF. We see from the proof of Theorem 1 in Fujii [4; §3,2] that for distinct primitive characters χ_1, χ_2 there exists a positive constant $c_1 < 1$ such that as T tends to infinity

$$\sum_{\substack{0 \le \gamma_1 \le T \\ m_1(\rho_1) > m_2(\rho_1)}}' 1 \ge (c_1 + o(1)) \frac{T}{2\pi} \log T. \tag{2}$$

Indeed, (2) holds even when χ_1, χ_2, or both χ_1 and χ_2 are imprimitive as long as they are inequivalent. To see this, note that if χ_i^* induces χ_i $(i = 1,2)$ and χ_1, χ_2 are inequivalent, then χ_1^*, χ_2^* are distinct primitive characters. (Of course if χ_i is primitive $\chi_i = \chi_i^*$.) Therefore (2) is true for the pair $L(s,\chi_1^*)$, $L(s,\chi_2^*)$. But $L(s,\chi_i)$ and $L(s,\chi_i^*)$ have the same zeros in $0 < \sigma < 1$. Hence (2) is valid for the pair $L(s,\chi_1)$, $L(s,\chi_2)$ as well. (In the statement of his theorem, Fujii assumes χ_1 and χ_2 have the same modulus. However, he later points out (in §4) that this assumption is unnecessary.) Now

$$\sum_{0 \le \gamma_1 \le T}' \min_{i=1,2} m_i(\rho_1) = \sum_{\substack{0 \le \gamma_1 \le T \\ m_1(\rho_1) \le m_2(\rho_1)}}' m_1(\rho_1) + \sum_{\substack{0 \le \gamma_1 \le T \\ m_1(\rho_1) > m_2(\rho_1)}}' m_2(\rho_1)$$

$$\le \sum_{\substack{0 \le \gamma_1 \le T \\ m_1(\rho_1) \le m_2(\rho_1)}}' m_1(\rho_1) + \sum_{\substack{0 \le \gamma_1 \le T \\ m_1(\rho_1) > m_2(\rho_1)}}' (m_1(\rho_1) - 1)$$

$$= \sum_{0 \le \gamma_1 \le T}' m_1(\rho_1) - \sum_{\substack{0 \le \gamma_1 \le T \\ m_1(\rho_1) > m_2(\rho_1)}}' 1.$$

The first sum on the last line is the total number of zeros of $L(s,\chi_1)$ in $0 < \sigma < 1$, $0 < t < T$, and is therefore equal to $(1+o(1)) \frac{T}{2\pi} \log T$ as T tends to infinity. Using this and (2) we conclude that

$$\sum_{0 \le \gamma_1 \le T}' \min_{i=1,2} m_i(\rho_1) \le (1-c_1+o(1)) \frac{T}{2\pi} \log T.$$

This establishes (1) with $c = 1-c_1$.

3. Proof of the Theorem.

For the sake of convenience, we carry out the proof of the Theorem only for $\alpha = 1/3$ and $2/3$. The modifications required to prove the other cases are minor and will be discussed at the end of this section. Throughout we write $e(x)$ for $e^{2\pi i x}$.

We begin with the identity (see Davenport and Heilbronn [3; p. 181])

$$\zeta(s, \frac{a}{q}) = \frac{q^s}{\phi(q)} \sum_{\chi} \bar{\chi}(a)L(s,\chi), \tag{3}$$

where $1 \le a < q$, $(a,q) = 1$, and the sum is over all $\phi(q)$ characters mod q. Take $q = 3$ and assume that a is either 1 or 2. We are then summing over $\phi(3) = 2$ characters in (3), both of which are real. Thus

$$\frac{2}{3^s} \zeta(s, \frac{a}{3}) = L(s,\chi_0) + \chi(a)L(s,\chi),$$

where χ_0 and χ are the principal and nonprincipal characters, respectively, mod 3. Since $L(s,\chi_0) = (1-3^{-s})\zeta(s)$, the last equation becomes

$$\frac{2}{3^s} \zeta(s, \frac{a}{3}) = (1-3^{-s})\zeta(s) + \chi(a)L(s,\chi). \tag{4}$$

REMARK. As will become apparent, it is essential to our proof that the sum in (3) reduce to two terms. This is why the reduced fraction α in the Theorem must have denominator 3,4 or 6.

Now write

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) \tag{5}$$

and

$$\xi(s,\chi) = (\frac{\pi}{3})^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2})L(s,\chi). \tag{6}$$

Using (5) and (6) to replace $\zeta(s)$ and $L(s,\chi)$ in (4) by $\xi(s)$ and $\xi(s,\chi)$, and then multiplying both sides of (4) by $(\frac{\pi}{3})^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2})$, we find (after simplifying) that

(7) $$\sqrt{\frac{12}{\pi}}\,(3\pi)^{-s/2}\Gamma(\tfrac{s+1}{2})\zeta(s,\tfrac{a}{3}) = \sqrt{\frac{12}{\pi}}\;\frac{(3^{s/2}-3^{-s/2})}{s(s-1)}\;\frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s}{2})}\,\xi(s)+\chi(a)\xi(s,\chi).$$

We write this more briefly as

(8) $$A(s)\zeta(s,\tfrac{a}{3}) = B(s)\xi(s) + \chi(a)\xi(s,\chi),$$

where

(9) $$A(s) = \sqrt{\frac{12}{\pi}}\;(3\pi)^{-s/2}\Gamma(\tfrac{s+1}{2})$$

and

(10) $$B(s) = \sqrt{\frac{12}{\pi}}\;\frac{(3^{s/2}-3^{-s/2})}{s(s-1)}\;\frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s}{2})}\,.$$

Since $A(s)$ never vanishes, the zeros of the right-hand side of (8) are precisely those of $\zeta(s,\frac{a}{3})$. Thus, $\zeta(1/2 + it_0, \frac{a}{3}) = 0$ if and only if the terms on the right-hand side of (8) cancel or vanish for $s = 1/2 + it_0$. Since $B(s) \neq 0$ on $\sigma = 1/2$ we see that $1/2 + it_0$ is a zero of $\zeta(s,\frac{a}{3})$ if and only if:

I. $\xi(1/2 + it_0) \neq 0$, $\xi(1/2 + it_0,\chi) \neq 0$, *and* $B(1/2 + it_0) = -\chi(a)\,\dfrac{\xi(1/2 + it_0,\chi)}{\xi(1/2 + it_0)}$,

or

II. $\xi(1/2 + it_0) = \xi(1/2 + it_0,\chi) = 0$.

Writing $N(T)$ for the number of zeros (counting multiplicities) of $\zeta(s,\frac{a}{3})$ on $[1/2, 1/2 + iT]$ $(T > 0)$, $N_I(T)$ for the number of these zeros arising from condition I, and $N_{II}(T)$ for the number arising from II, we see that

(11) $$N(T) = N_I(T) + N_{II}(T).$$

We estimate $N(T)$ by combining estimates for $N_I(T)$ and $N_{II}(T)$.

First consider $N_I(T)$. From the relation $\overline{\xi(s,\chi)} = \xi(\bar{s},\chi)$ (χ is real) and the functional equation

$$\xi(1-s,\chi) = \frac{i\sqrt{3}}{\tau(\chi)}\ \xi(s,\chi),$$

where $\tau(\chi) = \sum_{n=1}^{3} \chi(n)e(\frac{n}{3})$, one easily finds that $\xi(1/2 + it,\chi)$ is real. Similarly $\xi(1/2 + it)$ is real. Thus if t_0 satisfies I, $B(1/2 + it_0)$ is real. If $T \geq T_0 > 0$ and if $N_I'(T_0,T)$ denotes the number of solutions of

$$\arg B(1/2 + it) \equiv 0 \pmod{\pi}$$

with $t \in [T_0,T]$, it follows that $N_I'(T_0,T)$ is an upper bound for the number of distinct $t_0 \in [T_0,T]$ that satisfy I. We now prove that there exists a T_0 such that $N_I'(T_0,T) \ll T$ for all $T \geq T_0$, and that $1/2 + it_0$ is a simple zero of $\zeta(s, \frac{a}{3})$ if t_0 satisfies I and $t_0 \geq T_0$. These two assertions and the fact that $\zeta(s, \frac{a}{3})$ has only finitely many zeros on $[1/2, 1/2 + iT_0]$ clearly imply that

(12) $$N_I(T) \ll T \qquad (T \geq T_0).$$

To estimate $N_I'(T_0,T)$ we examine $\frac{d}{dt} \arg B(1/2 + it)$. (The derivative exists for all t since $B(s)$ is analytic and nonzero in $0 < \sigma < 1$.) By (10)

$$\arg B(1/2 + it) = \arg\left(\frac{-1}{t^2 + 1/4} \right) + \arg e\left(\frac{t \log 3}{4\pi}\right)$$

$$+ \arg\left(1- \frac{1}{\sqrt{3}}\, e\left(\frac{-t \log 3}{2\pi}\right)\right)$$

$$+ \arg\left(\Gamma\left(\frac{3}{4} + i\ \frac{t}{2}\right)/\Gamma\left(1/4 + i\ \frac{t}{2}\right)\right)$$

or

(13) $$\arg B(1/2 + it) = \pi + \frac{t \log 3}{2} + \arctan\left(\frac{\sin(t \log 3)}{\sqrt{3} - \cos(t \log 3)}\right)$$

$$+ \arg\left(\Gamma\left(\frac{3}{4} + i\ \frac{t}{2}\right)/\Gamma\left(1/4 + i\ \frac{t}{2}\right)\right),$$

where the choice of arguments is immaterial. The sum of the derivatives of the first three terms on the right-hand side of (13) is equal to

$$\frac{\log 3}{4-2\sqrt{3}\cos(t\log 3)}.$$

Observing that

$$\frac{d}{dt}\arg\Gamma(\sigma+it) = \mathrm{Re}\,\frac{\Gamma'}{\Gamma}(\sigma+it)$$

and using the formula (see Ingham [6; p. 57])

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(\frac{1}{|s|}\right)$$

which is valid in $|\arg s| < \pi-\delta$ for any $\delta > 0$, we find that

$$\frac{d}{dt}\arg\left(\Gamma\left(\frac{3}{4}+i\,\frac{t}{2}\right)/\Gamma\left(1/4+i\,\frac{t}{2}\right)\right) \ll \frac{1}{t+1}$$

for $t \geq 0$. Thus

$$\frac{d}{dt}\arg B(1/2+it) = \frac{\log 3}{4-2\sqrt{3}\cos(t\log 3)} + O\left(\frac{1}{t+1}\right) \quad (t \geq 0).$$

From this we see that there exists a $T_0 > 0$ such that $\frac{d}{dt}\arg B(1/2+it)$ is bounded and greater than zero for $t \geq T_0$. That is, $\arg B(1/2+it)$ is an increasing function with bounded derivative on $[T_0,\infty)$. Clearly this implies that

$$N_I'(T_0,T) \ll T \qquad (T \geq T_0).$$

Now suppose that $1/2 + it_0$ is a zero of $\zeta(s, \frac{a}{3})$ arising from condition I and that $t_0 \geq T_0$ (T_0 as above). Differentiating the right-hand side of (8) with respect to t and evaluating at $s = 1/2 + it_0$, we obtain

$$(14)\quad \xi(1/2+it_0)\left(\frac{d}{dt}\right)_{t_0}B(1/2+it)$$

$$+ B(1/2+it_0)\left(\frac{d}{dt}\right)_{t_0}\xi(1/2+it)+\chi(a)\left(\frac{d}{dt}\right)_{t_0}\xi(1/2+it,\chi).$$

The second and third terms are real since $\chi(a)$, $\frac{d}{dt}\xi(1/2 + it)$, $\frac{d}{dt}\xi(1/2 + it,\chi)$ and $B(1/2 + it_0)$ are. (Recall that $B(1/2 + it_0)$ is real whenever t_0 satisfies I.)

If we write $B(1/2 + it) = |B(1/2 + it)|e(\frac{\arg B(1/2 + it)}{2\pi})$, the first term in (14) becomes

$$(15)\quad \xi(1/2 + it_0)e(\frac{\arg B(1/2 + it_0)}{2\pi})\{(\frac{d}{dt})_{t_0}|B(1/2 + it)| + i(\frac{d}{dt})_{t_0} \arg B(1/2 + it)\}.$$

Since t_0 satisfies I, $e(\frac{\arg B(1/2 + it_0)}{2\pi}) = \pm 1$ and $\xi(1/2 + it_0)$, which is real, does not equal zero. Also $(\frac{d}{dt})_{t_0} \arg B(1/2 + it) > 0$ for $t_0 \geq T_0$ (this is how T_0 was chosen), and $\frac{d}{dt}|B(1/2 + it)|$ is real for all t. It follows that (15) and therefore (14) have nonvanishing imaginary parts. Thus $1/2 + it_0$ is a simple zero of the right-hand side of (8) or, what is the same thing, of $\zeta(s, \frac{a}{3})$. This finally establishes (12).

We now turn to $N_{II}(T)$. Let $m(z)$, $m_1(z)$, and $m_2(z)$ be the multiplicities of the point z as a zero of $\zeta(s, \frac{a}{3})$, $\zeta(s)$, and $L(s,\chi)$ repsectively. By (5), $\zeta(s)$ and $\xi(s)$ have the same zeros in $0 < \sigma < 1$; the same is true for $L(s,\chi)$ and $\xi(s,\chi)$ in light of (6). Thus t_0 satisfies II if and only if $1/2 + it_0$ is a common zero of $\zeta(s)$ and $L(s,\chi)$. In particular, $\frac{1}{2} + it_0$ is a zero of $\zeta(s)$ on $\sigma = 1/2$. Letting $\rho = \beta + i\gamma$ denote a typical zero of $\zeta(s)$, we then have

$$(16)\qquad N_{II}(T) = \sideset{}{'}\sum_{\substack{0 \leq \gamma \leq T \\ \beta = 1/2}} m(\rho),$$

where as usual $\sum'$ means the sum is over distinct zeros ρ. In order to estimate this we need to consider the numbers $m(\rho)$. From (8) and the fact that $B(s) \neq 0$ on $\sigma = 1/2$, it immediately follows that

$$m(1/2 + i\gamma)\begin{cases} = \min\limits_{i=1,2} m_i(1/2 + i\gamma) & \text{if } m_1(1/2 + i\gamma) \neq m_2(1/2 + i\gamma) \\ \geq m_1(1/2 + i\gamma) & \text{if } m_1(1/2 + i\gamma) = m_2(1/2 + i\gamma). \end{cases}$$

However, the lower bound this provides for $m(1/2 + i\gamma)$ in the case $m_1(1/2 + i\gamma) = m_2(1/2 + i\gamma)$ is of no use to us since we seek an upper bound for $N_{II}(T)$. We remedy this by proving that, except for finitely many γ, if $m_1(1/2 + i\gamma) = m_2(1/2 + i\gamma)$ then $m(1/2 + i\gamma) = m_1(1/2 + i\gamma)$ or $m_1(1/2 + i\gamma) + 1$, with the latter holding at most $O(T)$ times for $\gamma \in [0,T]$.

To show this set $m_1(1/2 + i\gamma) = m_2(1/2 + i\gamma) = k \geq 1$. Then the k^{th} derivative of the right-hand side of (8) with respect to t evaluated at $s = 1/2 + i\gamma$ is

$$(17) \qquad B(1/2 + i\gamma)(\tfrac{d}{dt})^k_\gamma \xi(1/2 + it) + \chi(a)(\tfrac{d}{dt})^k_\gamma \xi(1/2 + it,\chi).$$

Since the zeros of $B(s)\xi(s) + \chi(a)\xi(s,\chi)$ are those of $\zeta(s, \frac{a}{3})$, we see that $m(1/2 + i\gamma) > k$ if and only if (17) vanishes. By the definition of k, the k^{th} derivatives of the two ξ-functions are nonzero at $1/2 + i\gamma$. Hence (17) vanishes only if its terms cancel. Since $\chi(a)$, $(\frac{d}{dt})^k\ \xi(1/2 + it)$, and $(\frac{d}{dt})^k \xi(1/2 + it,\chi)$ are real, this occurs only if $B(1/2 + i\gamma)$ is real. But we have already seen that $B(1/2 + it)$ is real at most $O(T)$ times on $[0,T]$. Thus $m_1(1/2 + i\gamma) = m_2(1/2 + i\gamma)$ implies that $m(1/2 + i\gamma) = m_1(1/2 + i\gamma)$ ($= k$) except for possibly $O(T)$ values of $\gamma \in [0,T]$. Suppose now that (17) does vanish at $1/2 + i\gamma$ (so that $B(1/2 + i\gamma)$ is real). Taking the $k+1^{st}$ derivative of the right-hand side of (8) with respect to t and evaluating at $s = 1/2 + i\gamma$, we obtain

$$(18) \quad (k+1)[(\tfrac{d}{dt})^k_\gamma \xi(1/2 + it)][(\tfrac{d}{dt})_\gamma B(1/2 + it)] + B(1/2 + i\gamma)(\tfrac{d}{dt})^{k+1}_\gamma \xi(1/2 + it) + \chi(a)(\tfrac{d}{dt})^{k+1}_\gamma \xi(1/2 + it,\chi).$$

As in our analysis of (14), we find that the second and third terms are real and that the first has nonvanishing imaginary part when γ is large. Thus (18) is nonzero and $m(1/2 + i\gamma) = k+1 = m_1(1/2 + i\gamma) + 1$ (for large γ).

To summarize: there exists a $T_0 > 0$ such that if $1/2 + i\gamma$ is a zero of $\zeta(s)$ with $\gamma \geq T_0$, then

$$m(1/2 + i\gamma) = \min_{i=1,2} m_i(1/2 + i\gamma) \text{ or } \min_{i=1,2} m_i(1/2 + i\gamma) + 1;$$

the second case occurs at most $O(T)$ times on $[T_0,T]$.

We can now bound $N_{II}(T)$. Writing (16) as

$$N_{II}(T) = \sum_{\substack{T_0 \le \gamma \le T \\ \beta = 1/2}}' m(\rho) + O(1)$$

and using the previous result, we have

$$N_{II}(T) = \sum_{\substack{T_0 \le \gamma \le T \\ \beta = 1/2}}' \min_{i = 1,2} m_i(\rho) + O(T)$$

$$= \sum_{\substack{0 \le \gamma \le T \\ \beta = 1/2}}' \min_{i = 1,2} m_i(\rho) + O(T)$$

$$\le \sum_{0 \le \gamma \le T}' \min_{i = 1,2} m_i(\rho) + O(T),$$

where the final sum is over the distinct zeros ρ of $\zeta(s)$ with $0 < \beta < 1$, $0 \le \gamma \le T$. Applying the Lemma to the last sum (note that $\zeta(s)$ is an L-function) we see that as T tends to infinity

(19) $$N_{II}(T) \le (c+o(1)) \frac{T}{2\pi} \log T,$$

where c is a positive constant < 1.

The proof of the Theorem for $\alpha = 1/3$ and $2/3$ now follows from (11), (12), and (19).

Our proof carries over to the cases $\alpha = 1/4$, $3/4$. $1/6$, and $5/6$ with only slight changes in the formulae. For instance, if $\alpha = a/4$, $a = 1$ or 3, then corresponding to (8), (9), and (10) we have

$$A(s)\zeta(s, \tfrac{a}{4}) = B(s)\xi(s) + \chi(a)\xi(s,\chi),$$

$$A(s) = \frac{4}{\sqrt{\pi}} (4\pi)^{-s/2}\Gamma(\tfrac{s+1}{2}),$$

and

$$B(s) = \frac{4}{\sqrt{\pi}} \quad \frac{(2^s-1)}{s(s-1)} \quad \frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s}{2})} ,$$

where χ is the nonprincipal character mod 4.

When $\alpha = \frac{a}{6}$, a = 1 or 5, the situation is only slightly more complicated. The nonprincipal character χ mod 6 is induced by the primitive character χ^* mod 3. Also, for the principal character χ_0 mod 6 we have $L(s,\chi_0) = (1-2^{-s})(1-3^{-s})\zeta(s)$. Thus, in place of (4) we obtain

$$\frac{2}{6^s} \zeta(s, \frac{a}{6}) = (1-2^{-s})(1-3^{-s})\zeta(s)+\chi^*(a)(1+2^{-s})L(s,\chi^*),$$

and instead of (8), (9), (10) we have

$$A(s)\zeta(s, \frac{a}{6}) = B(s)\xi(s) + \chi^*(a)\xi(s,\chi^*),$$

$$A(s) = \sqrt{\frac{12}{\pi}} \quad \frac{(12\pi)^{-s/2}}{(1+2^{-s})} \quad \Gamma(\frac{s+1}{2}),$$

and

$$B(s) = \sqrt{\frac{12}{\pi}} \quad \frac{(3^{s/2}-3^{-s/2})(1-2^{-s})}{s(s-1)(1+2^{-s})} \quad \frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s}{2})} .$$

In either case $A(s) \neq 0$ for $0 < \sigma < 1$ and $\frac{d}{dt} \arg B(\frac{1}{2} + it)$ is bounded and > 0 for all large t.

4. A Conjecture.

We expect the Lemma, and therefore the Theorem, to be far from best possible. Indeed, it is generally held that no two L-functions with inequivalent characters have common zeros in $0 < \sigma < 1$. On this assumption we would have $N_{II}(T) \ll T$ instead of (19) and this along with (11) and (12) implies that $N(T) \ll T$. It is plausible to suppose that these bounds are valid for other rational values of α so we make the following

CONJECTURE. If α is rational, $0 < \alpha < 1$, and $\alpha \neq 1/2$, then $\zeta(s,\alpha)$ has $\ll T$ zeros on $[1/2, 1/2 + iT]$.

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