This document contains

- Abstracts of papers in the order in the list of publications
- Abstracts of thesis of students

The numerals refer to the list of publications

Basic notation:
K: a function field of one variable with field of constants $\mathbf{F}_q$.
∞: a place of K.
A: the ring of elements of K integral outside ∞.
Think of K, ∞, A as analogues of $\mathbf{Q}$, ∞ and $\mathbf{Z}$.

(1) This paper describes some of the results of the thesis (2). Highlights of part of the thesis containing first transcendence results on the Carlitz zeta function have not been published anywhere else. (Since they were soon improved by (5)).

(2) The thesis consists of three parts: (i) Gamma functions: $\Gamma : \mathbf{Z}_p \to K_\infty^*$ and interpolations $\Gamma_v : \mathbf{Z}_p \to A_v^*$ at finite places v of K were defined. Functional equations, such as the reflection and multiplication formulae, were proved, by reducing them to manipulations of base q digits of p-adic numbers. The special values of Γ were related to the periods $\tilde{\pi}$ (analogue of $2\pi i$) of Drinfeld modules, e.g. $\Gamma(1/2) = \sqrt{\tilde{\pi}}$. The special values of Γ_v were related to Gauss sums defined in (ii) below, giving elementary proof of an analogue, for $A = \mathbf{F}_q[T]$, of Gross-Koblitz formula, for which only known proofs involve p-adic cohomology.

(ii) Gauss sums: Mixing the classical cyclotomic theory and Drinfeld's cyclotomic theory, an analogue of the Gauss sum, taking values in function fields, was defined. Analogues of classical theorems such as Hasse-Davenport theorem, Stickelberger theorem, Gross-Koblitz theorem, Weil's theorem on Jacobi sums as Hecke characters etc. were proved, with different kind of proofs.

(iii) Zeta functions: Using approximation methods, irrationality and transcendence results for $\zeta(k)$ and $\zeta(k)/\tilde{\pi}^k$, for some special odd k's were proved for the first time; where for $k \in \mathbf{N}$,

$$\zeta(k) = \sum_{n \text{ monic} \in \mathbf{F}_q[T]} \frac{1}{n^k} \in \mathbf{F}_q((1/T))$$

(3) Contains essentially the 'Gauss sum' part of the thesis, but with many proofs different than in (2).

(4) This is an extract of the different kind of proof for analogue of Gross-Koblitz formula in the thesis (2), together with some related background.

(5) It is shown that $\zeta(k)$ is essentially logarithm of explicit algebraic point on the k-th tensor power (defined by Anderson) of Carlitz module (analogue of motive $\mathbf{Z}(k)$). In particular, using analogues of classical Hermite-Lindemann, Gelfond-Schneider transcendence results on logarithms, which were then proved by Jing Yu, it follows that

Theorem : $\zeta(k)$ is transcendental for all k and $\zeta(k)/\tilde{\pi}^k$ is transcendental for k odd.

There is also a v-adic version for this. Note that for k even, Carlitz had already proved analogue of Euler's theorem that $\zeta(k)/\tilde{\pi}^k$ is rational. Also note that $\zeta(1)$ is analogue of Euler's gamma constant.

(6) In these lectures for students and teachers, we have tried to motivate and explain various analogies, in spite of major differences, between arithmetic of numbers and polynomials.

(7) This is a brief account (at the occasion of Fields medal award) of Drinfeld's work on quantum groups, Drinfeld modules and Langlands conjectures and other topics.

(8) Goss has studied zeta values at negative integers, defined as $\zeta(-k) := \sum_{i=0}^{\infty}(\sum n^k) \in \mathbf{F}_q[T]$, where n runs through monic polynomials of degree i. In this paper, the divided power series corresponding to the measure μ on A_v such that $\int_{A_v} x^k d\mu = \zeta(-k)$, was calculated. The answer turns out to be simple, but quite different than its classical counterpart, suggesting some hidden phenomenon.

(9) Two gamma functions are considered, one defined in the thesis with domain in characteristic zero and the other with domain in characteristic p. These are interpolated, their functional equations, algebraicity and transcendence questions for special values are studied. Their arithmetic is related to the classical cyclotomic theory and Drinfeld's cyclotomic theory resp. Some general conjectures such as Chowla-Selberg phenomenon are given with partial evidence.

(10) In the thesis, an analogue of the Gauss sums was defined for general A. In (2-3), it was shown that the Gauss sums for $A = \mathbf{F}_q[T]$ satisfied analogues of many classical theorems. Here it is shown that even though the cyclotomic theory for general A is quite similar, the behavior of Gauss sums is wildly different. For example, classically and in the case $A = \mathbf{F}_q[T]$, the norm of the Gauss sum made up from P-torsion is essentially P. But

e. g. for $A = \mathbf{F}_2[x,y]/y^2 + y = x^5 + x^3 + 1$ it is $P(P^\sigma)^2 P^{\sigma^2}$, where σ is automorphism of A given by $\sigma(x) = x + 1$, $\sigma(y) = y + x^2$. Similar behavior holds for all class number one cases.

(11) Carlitz defined a good analogue $e(z)$ of the exponential function e^z (later greatly generalized by Drinfeld). It is shown here that $e(z)$ and $e = e(1)$ have very interesting continued fractions, though quite different from the classical pattern. Classically, we have Euler's formula

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, \cdots]$$

Here e.g. for $q = 2$ and with $[n] = T^{2^n} - T$, we have 'doubling pattern':

$$e = [1, [1], [2], [1], [3], [1], [2], [1], [4], \cdots, [5], \cdots]$$

Interesting continued fractions are also proved for analogues of π, Euler's gamma constant etc.

(12) Arithmetically interesting quantities such as factorial and gamma functions, binomial coefficients, zeta functions, in the context of function fields, are related to quantities connected with Drinfeld modules. This is applied to obtain algebraicity of Drinfeld exponentials of some special zeta values. The transcendence properties of exponential, then imply transcendence of such values and their ratios with periods $\tilde{\pi}$. This generalizes (5) in some instances of higher genus. The connection between zeta values and multilogarithms in (2, 5) and here is quite different in spirit from the relations between the classical relative zeta functions and multilogarithms recently investigated by Zagier.

Exa. $A = \mathbf{F}_4[x,y]/y^2 + y = x^3 + \zeta_3$. Then $exp(\zeta(1)) = x^8 + x^4 + x^2 + x$. Note $\zeta(1)$ is analogue of Euler's gamma constant.

(13) This is an expository article on the work on gamma functions in function field. It also contains announcement of results of (14) and also explains another gamma function.

(14) Jacobi sums, in the context of function fields, introduced and studied by combining two types of cyclotomic theories in (I), (II) are shown to be related to Shtukas studied by Drinfeld and theta divisors. This is applied to obtain some results on the factorizations of the analogues of Gauss sums and prove an analogue of Gross-Koblitz formula in general case, generalizing $A = \mathbf{F}_q[T]$ case of (II). For this purpose, a different generalization of the gamma function in $\mathbf{F}_q[T]$ case is introduced and interpolated, at finite as well as infinite places.

(15) The valuations of the function field Gauss sums at the infinite places are shown to be related to Weierstrass gaps. This generalizes the result of (2-3) for $\mathbf{F}_q[T]$, where the valuations are all $-1/(q-1)$, in direct analogy with the well-known classical result that all the absolute values of Gauss sums are $q^{1/2}$. The additional twist in the general case is existence of finitely many exceptional primes. The sign of quadratic and higher order Gauss sums is determined, giving results in the direction of Gauss' sign theorem and the work of Cassels and Matthews.

(16) We have described (sketches of proofs) and put in perspective of Drinfeld's theory, some theorems and conjectures relating class numbers and zeta values at positive and negative integers (there are two distinct theories in contrast to the classical case), analogues of results and conjectures of Kummer and Vandiver, class group vs. module questions, growth rates of class numbers, zeta measures and other aspects of Iwasawa theory.

(17) We introduce and study analogues of hypergeometric functions in the settings of function fields over finite fields. We show analogues of differential equations, integral representations, transformation formulas, continued fractions and show that analogues of various special functions and orthogonal polynomials occur as specializations. There are two analogues: one with characteristic zero domain and one with characteristic p domain.

(18) Orders of vanishing and leading terms of the special zeta values are known to contain deep arithmetic information. This is true also for the characteristic p zeta functions introduced by Carlitz and Goss. Goss showed that these vanish to at least to orders expected by analogies and raised a question whether they are exact. We show that there can be extra vanishing. The pattern involving q-digits that control these extra vanishings gives us a challenge to put these zeros in a general framework. Also, the results have strong implications for the Drinfeld's cyclotomic theory.

(19) Developing ideas of (11) further, we show that analogues of Hurwitz numbers $(ae^{2/n}+b)/(ce^{2/n}+d)$, with a, b, c, d, n integers have continued fractions with very interesting patterns involving block reversing and block repeating moves, in contrast with the classical arithmetic progressions patterns. Hence even though the patterns and the proofs are completely different, for some reason analogous numbers have interesting patterns. For $q=2$, situation is highly interesting and analogue of 2 in $2/n$ seems to be t, $t+1$ or t^2+t then, for the reasons of rationality of torsion points of carlitz module. We settle the $q>2$ case and give suggesting examples for the case $q=2$.

(20) Voloch gave a nice proof of the transcendence of the period of the

Tate elliptic curve in finite characteristic, by analyzing the Galois action in Igusa tower. This is analogous to Schneider's classical result or rather to Mahler-Manin conjectures. We give another proof of this result based on the transcendence criterion of Christol involving notions of recognizable sequences and automata.

(21) Our earlier work connecting function field gamma values to periods has corollaries about transcendence of gamma values at fractions, parallel to Chudnovsky's results about gamma values at fractions with denominators 2, 3, 4, 6. Using automata criterion of Christol, which is a completely different method, we show transcendence of many gamma values and monomials in them, with no restrictions on denominators, but some restrictions on the numerators. There is no parallel theorem in the classical case.

(22) We settle (most of) the remaining (and most interesting) case of $q = 2$ of (19), by proving an universal inductive scheme of block repetition and reversal patterns. The patterns depend subtly on prime factorization of one of the parameters of the Hurwitz family.

(23) We describe the automata method, with examples and short sketches of proofs, and explain the two applications to number theory that we have made, namely (20) and (21). Finally, we prove a new result on the transcendence of some values of interpolations of function field gamma function at finite primes.

(24) We propose a computational classification of finite characteristic numbers (Laurent series with coefficients in a finite field) and prove that some classes have good algebraic properties. This provides tools from the theories of computation, formal languages and formal logic for finer study of transcendence and algebraic independence questions. Using them, we place some well-known transcendental numbers occurring in number theory in the computational hierarchy.

(25) For each rational diophantine approximation exponent in a certain range, we provide an explicit family of continued fractions of algebraic power series in finite characteristic (together with the algebraic equations they satisfy) which have that exponent. We can take exponent arbitrarily near 2. We also provide some non-quadratic examples with bounded sequence of partial quotients.

(26) Building on Krichever-Drinfeld dictionary, Anderson has introduced and studied an analogue of solitons for rational function field, with applications to the values of gamma and zeta functions and Hecke characters. We give a simple approach which gives algebraic equations for solitons, some concrete examples and information on its divisor, Galois action etc. We dis-

cuss how the generalization of our approach to general function field differs from the generalization to Anderson's approach and arithmetic applications of both cases.

(27) Using the techniques of Automata theory, we give another proof of the function field analogue of Mahler-Manin conjecture and prove transcendence results for the power series associated to higher divisor functions $\sigma_k(n) = \sum d|nd^k$ and discuss algebraic dependence relations between such series.

(28) In this paper, we develop the theory of hypergeometric functions introduced in (17) further by explaining the noncommutative aspects and various analogies satisfied by it. In particular, we give analogues of Kummer solutions at ∞, by defining a suitable analogue of $(a)_{-n}$. The analogies do not always work as expected naively.

(29) These articles deal with many issues of basics of algebraic number theory and cyclotomic theory in particular, and give many examples, proofs and comments. Possible Fermat's false first proof of his last theorem is reproduced from Euler.

(30) See table of contents on my home page.

(31) It is well-known that while the analogue of Liouville's theorem on diophantine approximation holds in finite characteristic, the analogue of Roth's theorem fails quite badly. We associate certain curves over function fields to given algebraic power series and show that bounds on the rank of Kodaira-Spencer map of this curves imply bounds on the exponents of the power series, with more 'generic' curves (in the deformation sense) giving lower exponents. If we transport Vojta's conjecture on height inequality to finite characteristic by modifying it by adding suitable deformation theoretic condition, then we see that the numbers giving rise to general curves approach Roth's bound. We also prove a hierarchy of exponent bounds for approximation by algebraic quantities of bounded degree.

(32) We describe the emerging theory of L-functions and modular forms in the setting of function fields over finite fields. We give a quick introduction to Drinfeld modules and higher dimensional motives, to arithmetic and analytic properties of finite characteristic zeta functions, modular forms etc.

(33) We give an overview of applications of the concepts and techniques of the theory of integrable systems to the number theory in finite characteristic. The applications include explicit class field theory and Langlands conjectures, how geometry of theta divisor controls factorization of analogues of Gauss sums, special values of Gamma, Zeta and L-functions, analogues of Weil's and Stickelberger's theorems and control of intersection of the Jaco-

bian torsion with the theta divisor. The techniques are Krichever-Drinfeld dictionary and theory of solitons, Akhiezer-Baker and tau functions developed in this context by Greg Anderson.

(34) We describe the theory of elliptic curves and Drinfeld modules in the function field setting. Both these objects share some of the properties of the elliptic curves familiar in the number fields setting. We develop the basics, describe analogies, give examples, survey and compare the main results and some open questions.

(35) In contrast to Roth's theorem that all algebraic irrational real numbers have approximation exponent two, the distribution of the exponents for the function field counterparts is not even conjecturally understood. We describe some recent progress made on this issue. An explicit continued fraction is not known even for a single non-quadratic algebraic real number. We provide many families of explicit continued fractions, equations and exponents for non-quadratic algebraic laurent series in finite characteristic, including non-Riccati examples with both bounded or unbounded sequences of partial quotients.

(36) See the table of contents on my web page. The new material not published elsewhere includes sections on multizeta, log-algebraicity, continued fraction -approximation connection etc.

(37) Expository survey talk focusing on recent developments, especially relations and algebraic independence results on gamma and zeta values.

(38) For diophantine approximation of algebraic numbers, straight analog of Roth's theorem in the number field case fails in function fields and the situation is not even conjecturally understood. We describe the partial progress on this and related issues of continued fractions for algebraic quantities. We provide some new examples of families and some speculation. We also describe progress on transcendence and algebraic independence issues, especially for gamma, zeta and multizeta values, using higher dimensional generalizations of Drinfeld modules.

(39) See the table of contents on my web page.

(40) For diophantine approximation of algebraic real numbers by rationals, Roth's celebrated theorem settles the issue of approximation exponent completely in the number field case. But in spite of strong analogies between number fields and function fields, its naive analog fails in function fields of finite characteristic, and the situation is not even conjecturally understood. We describe the recent progress on this and related issues of explicit continued fractions for algebraic quantities.

(41) This is announcement of (49) with skecth of proofs.

(42) We study the sum of integral powers of monic polynomials of given degree over a finite field. The combinatorics of cancellations are quite complicated. We prove several results on the degrees of these sums giving direct or recursive formulas, congruence conditions and degree bounds for them. We point out a 'duality' between values for positive and negative powers. We show that despite the combinatorial complexity of the actual values, there is an interesting kind of a recursive formula (at least when the finite field is the prime field) which leads to many interesting structural facts, such as Riemann hypothesis for Carlitz-Goss zeta function, monotonicity in degree, non-vanishing and special identity classification for function field multizeta, as easy consequences.

(43) We provide a period interpretation for the multizeta values (in the function field context) in terms of explicit iterated extensions of tensor powers of Carlitz motives (mixed Carlitz-Tate motives). We give examples of combinatorially involved relations that these multizeta values satisfy.

(44) Despite the failure of naive analogs of the sum shuffle or integral shuffle relations, and despite the lack of understanding of analogs of many classical structures that exist in the corresponding theory in the number field case, the multizeta defined by the author are proved (and conjectured) to satisfy many interesting and combinatorially involved identities. The connections of this multizetas with iterated extensions of Carlitz-Tate motives, analogs of Ihara power series and Deligne-Soule cocycles etc. make it an interesting challenge to understand all the identities and discover the other relevant underlying structures.

(45) We give a quick survey of some important recent results and open problems in the area of function field arithmetic, which studies geometric analogs of arithmetic questions. We sketch related developments and try to trace the multiple influences of works of John Tate in this context. We mainly, but not fully, limit ourselves to topics where these are clearly visible. Also, we focus mainly on results simple to state, and leave variants or generalizations to the references.

(46) We consider the values at proper fractions of the Arithmetic Gamma function and the values at positive integers of the Zeta function for $F_q[\theta]$ and provide complete algebraic independence results for them.

(47) Despite the failure of naive analogs of the sum shuffle or the integral shuffle relations, we prove the existence of 'shuffle' relations for the multizeta values (for a general A, with a rational prime at infinity) introduced by the author in the function field context. This makes the F_p-span of the multizeta values into an algebra. We effectively determine and prove all the

F_p-coeffcients identities (but not the $F_p(t)$-coefficients identities).

(48) We construct many families of non-quadratic algebraic Laurent series with continued fractions having bounded partial quotient sequence, (the diophantine approximation exponent for approximation by rationals is thus 2 agreeing with Roth values), and with the diophantine approximation exponent for approximation by quadratics being arbitrarily large. In contrast, the Schmidt's value (analog of Roth value for approximations by quadratics, in the number field case) is 3. We calculate diophantine approximation expoenents for approximations by rationals for function field analogs of π, e and Hurwitz numbers (which are transcendental), and also give interesting lower bouns (which may be the actual value) for the exponent for approximation by quadratics for the latter two. We do this by exploting the situation when 'folding' or 'negative reversal' patterns of the relevant continued fractions become 'repeating' or 'half-repeating' in even or odd characteristic respectively.

(49) We prove a simple transcendence criterion suitable for function field arithmetic. We apply it to show the transcendence of special values at nonzero rational arguments (or more generally, at algebraic arguments which generate extension of the rational function field with less than q places at infinity) of the entire hypergeometric functions in the function field (over F_q) context, and to obtain a new proof of the transcendence of special values at non-natural p-adic integers of the Carlitz-Goss gamma function. We also characterize when hypergeometric functions are algebraic in the balanced case, giving an analog of the result of F. R. Villegas, based on Beukers-Heckman results in the classical hypergeometric case.

(50) Chowla (1930) gave infinitely many counter-examples to Ramanujan's (1911) conjecture/claim on numerators of Bernoulli numbers and later (1986) made another conjecture. We point out that in some sense infinitely many counter-examples to Ramanujan's as well as Chowla's conjectures are already 'known', but somehow missed by them, editors and commentators of their collected works published later, referees etc. We also discuss results for two analogs in the function field case and compare the situtation.

(51) We prove the existence of infinitely many composite numbers simultaneously passing all elliptic curve primality tests assuming a weak form of a standard conjecture on the bound on the least prime in (special) arithmetic progressions. Our results are somewhat more general than both the 1999 dissertation of the first author (written under the direction of the third author) and a 2010 paper on Carmichael numbers in a residue class written by Banks and the second author.

(52) We present several elementary theorems, observations and questions related to the theme of congruences satisfied by binomial coefficients and factorials modulo primes (or prime powers) in the setting of polynomial ring over a finite field. When we look at the factorial of n or the binomial coefficient 'n choose m' in this setting, though the values are in a function field, n and m can be usual integers, polynomials or mixed! Thus, there are several interesting analogs of the well-known theorems of Lucas, Wilson etc., with quite different proofs and new phenomena.

(53) In this expository article, we survey progress, and mention open questions, speculations and conjectures regarding the study of various degrees of irrationality, algebraicity, and transcendence from various angles, for general as well as special quantities in function field arithmetic. Though we often mention the number field case and characteristic zero function field case, we mostly concentrate on function fields in finite characteristic, and often specialize to those over finite fields.

(54) We consider analog, when $\mathbf{Z}$ is replaced by $\mathbf{F}_q[t]$, of Wilson primes, namely the primes satisfying Wilson's congruence $(p-1)! \equiv -1$ to modulus p^2 rather than the usual prime modulus p. In contrast to the only three known such primes in $\mathbf{Z}$ case, we prove that there are infinitely many of these, for example, $\wp = t^{3*13^n} - t^{13^n} - 1$ are Wilson primes for $\mathbf{F}_3[t]$.

(55) We study the valuation at an irreducible polynomial v of the v-adic power sum, for exponent k (or $-k$), of polynomials of a given degree d in $F_q[t]$, as a sequence in d (or k). Understanding these sequences have immediate consequences, via standard Newton polygon calculations, for the zero distribution of correspondinf v-adic Goss zeta functions. We concentrate on v of degree one and two and give several results and conjectures describing these sequences. As an application, we show, for example, that the naive Riemann hypothesis statement which works in several cases, needs modification, even for a prime of degree two. In the appendix, we give an elementary proof of (and generalize) a product formula of Richard Pink for the leading term of the Goss zeta function.

(56) We construct families of non-quadratic algebraic laurent series (over finite fields of any characteristic) which have only bad rational approximations so that their exponent is as near to 2 as we wish and at the same time have very good quadratic approximations so that their quadratic exponent is close to the Liouville bound and thus can be arbitrarily large. In contrast, in the number field case, the Schmidt's exponent (analog of Roth exponent of 2 for rational approximation) for approximation by quadratics is 3. We do this by exploiting the symmetries if the relevant continued fractions. We then

generalize some of the aspects from the degree $2(=p^0+1)$-approximations to degree p^n+1-approximation. We also calculate the rational approximation exponent of an analog of π.

(57) We consider analog, when $\mathbf{Z}$ is replaced by $\mathbf{F}_q[t]$, of Wilson primes, namely the primes satisfying Wilson's congruence $(p-1)! \equiv -1$ to modulus p^2 rather than the usual prime modulus p. We fully characterize these primes by connecting these or higher power congruences to other fundamental quantities such as higher derivatives, higher difference quotients as well as higher Fermat quotients. For example, in characteristic $p>2$, we show that a prime $\wp$ of $\mathbf{F}_q[t]$ is a Wilson prime, if and only if, its second derivative with respect to t is zero, and in this case, further, that the congruence holds automatically modulo $\wp^{p-1}$. For $p=2$, the power $p-1$ is replaced by $4-1=3$. For every q, we show that there are infinitely many such primes.

(58) We prove and conjecture several realtions between multizeta values for $F_q[t]$, focusing on zeta-like values, namely those whose ratio with the zeta value of the same weight is rational (or equivalently algebraic). In particular, we describe them conjecturally fully for $q=2$ or more generally for any q for 'even' weight ('eulerian' tuples). We provide some data in support of the guesses.

(59) These are lecture notes of 12 hour lectures in advanced courses in mathematics series at CRM, Barcelona 2010. They are almost self-contained and also cover many latest developments.

(60) We consider analogs of fundamental congruences of Fermat and Wilson in the setting of Carlitz-Drinfeld modules and show that they are linked together when we look at them modulo higher powers of primes. They are also linked with the divisibilities of zeta values. The link with the zeta value carries over to the number field case, with the zeta value at 1 being replaced by Euler's constant.

(61) In this expository survey, for the special volume 'Second Decade of Finite Fields and their applications', we explain some results and conjectures on various aspects of the study of the sums of integral powers of monic polynomials of a given degree over a finite field. The aspects include non-vanishing criteria, formulas and bounds for degree and valuations at finite primes, explicit formulas of various kind for the sums themselves, factorizations of such sums, generating functions for them, relations between them, special types of interpolations of the sums by algebraic functions, and the resulting connections between the motives constructed from them and the zeta and multisite special values. We mention several applications to the function field arithmetic.

(62) We prove some interesting multiplicative relations which hold between the coefficients of the logarithmic derivatives obtained in a few simple ways from $\mathbf{F}_q$-linear formal power series. Since the logarithmic derivatives connect power sums to elementary symmetric functions via the Newton identities, we establish, as applications, new identities between important quantities of function field arithmetic, such as the Bernoulli-Carlitz fractions and power sums as well as their multi-variable generalizations. Resulting understanding of their factorizations has arithmetic significance, as well as applications to function field zeta and multizeta values evaluations and relations between them. Using specialization/generalization arguments, we provide much more general identities on linear forms providing a switch between power sums for positive and negative powers.

(63) In this expository note, we describe some interesting encounters with Ramanujan's mathematics. They are related to higher congruences between modular forms and factorizations of Bernoulli numbers.

(64) In contrast to the 'universal' multizeta shuffle relations, when the chosen infinite place of the function field over $\mathbf{F}_q$ is rational, we show that in the non-rational case, only certain interesting shuffle relations survive, and the $\mathbf{F}_q$-linear span of the multizeta values does not form an algebra. This is due to the subtle interactions between the larger finite field $\mathbf{F}_\infty$, the residue field of the completion at infinity where the signs live and $\mathbf{F}_q$, the field of constants where the coefficients live. We study the classification of these special relations which survive.

(65) We give a survey of the recent developments in the understanding of the multizeta values for function fields. These include motivic interpretations, study of algebraic relations and non-relations, conjectures on dimensions and basis of various spaces of multizeta values and their variants.

(66) We study variations in slope zero multiplicities of components of the Diedonné module (or equivalently the p-divisible group) of the Jacobians of $\wp$-th Carlitz cyclotomic covers of $F_q(t)$, as we vary the primes $\wp$ of $F_q[t]$ of characteristic p. We also give some applications to the question of ordinariness and of p-ranks of the factors of these Jacobians. We do this by proving and guessing some interesting structural patterns in prime factorizations of power sums representing the leading terms of the Goss zeta function at negative integers.

(67) We generalize results of D. Kaprekar, H. Hasse and G. D. Prichett from 3 and 4 digits to 5 and 7 digit numbers (in any base) characterizing all Kaprekar constants in this case. For a fixed base, and any number of digits, we reduce the infinite computation needed apriori to a finite effective

computation. We also give several other results and conjectures for general bases and digits, and mention some interesting open questions.

(68) The purpose of this expository article is to explain diverse new tools that automata theory provides to tackle transcendence problems in function field arithmetic. We collect and explain various useful results scattered in computer science, formal languages, logic literature and explain how they can be fruitfully used in number theory, dealing with transcendence, refined transcendence and classification problems.

(69) This is a brief report on recent work of the author (some joint with Greg Anderson) and his student on multizeta values for function fields. This includes definitions, proofs and conjectures on the relations, period interpretations in terms of mixed Carlitz-Tate t-motives and related motivic aspects. We also verify Taelman's recent conjectures in special cases.

(70???) Analog of Ihara power series, Deligne-Soule cocycles are constructed using Anderson's solitons, t-motives and providing connections between Gauss-Jacobi sums, Zeta values, Cyclotomic module, corresponding algebraic points and motivic extensions, Gamma values, absolute Galois group, analog of Fermat Jacobians etc.

(71??) The primes or prime polynomials (over finite fields) are supposed to be distributed 'irregularly', despite nice asymptotic or average behavior. We provide some conjectures/guesses/hypotheses with 'evidence' of surprising symmetries in prime distribution. At least when the characteristic is 2, we provide conjectural rationality and characterization of vanishing for families of interesting infinite sums over irreducible polynomials over finite fields. The cancellations responsible do not happen degree by degree or even for degree bounds for primes or prime powers, so rather than finite fields being responsible, interaction between all finite field extensions seems to be playing a role and thus suggests some interesting symmetries in the distribution of prime polynomials. Primes are subtle, so whether there is actual vanishing of these sums indicating surprising symmetry (as guessed optimistically), or whether these sums just have surprisingly large valuations indicating only some small degree cancellation phenomena of low level approximation symmetry (as feared sometimes pessimistically!), remains to be seen. In either case, the phenomena begs for an explanation. **Updates at the end**

(72???) We prove some (and conjecture more) relations between the multizeta values for positive genus function fields of class number one, focusing on the zeta-like values, namely those whose ratio with the zeta value of the same weight is rational (or conjecturally equivalently algebraic). These are the first known relations between multizetas, which are not with prime field coefficients. We seem to have one universal family. We also find that, interestingly, the mechanism with which the relations work is quite different from the rational function field case, raising interesting questions about the expected motivic interpretation in higher genus.

Abstracts of PhD thesis of students

For links 'available at' type the filename after my homepage url at Rochester

(1) Javier Diaz-Vargas (1996): On zeros of characteristic p zeta functions

We present a simpler proof of Wan's theorem that 'non-trivial' zeros of characteristic p zeta function of Goss interpolating Carlitz zeta function for $F_p[t]$ satisfies Goss' analog of the Riemann hypothesis that they all lie on a 'real line' in 'complex plane' and give partial results for $F_q[t]$, for q not prime. For trivial zeros for Goss zeta functions for general function fields, we give many examples of extra vanishing than that suggested by naive analogies. We give applications to non-vanishing of certain class group components for cyclotomic function fields. We give examples of function fields, where all primes of degree more than two are 'irregular' in the sense of Drinfeld-Hayes cyclotomic theory.

Available at VargasThesis.pdf (or dvi)

(2) Aaron Ekstrom (1999): On the infinitude of elliptic carmichael numbers

Under the assumption (much weaker than the current conjectures) that the smallest prime congruent to $-1 \bmod q$ is at most $q \exp((\log q)^{1-\epsilon})$, for large enough q, we show that there are infinitely many elliptic Carmichael numbers (i.e. composite numbers passing all primality tests for all CM eeliptic curves and points on them). We give many examples, show they are square-free, give bounds on their number and show that there are no strong elliptic Carmichael numbers.

Available at EkstromThesis.pdf (or dvi)

(3)Justin Miller (2007): On p-adic continued fractions and quadratic irrationals

Provides new results, computational evidence about known and new (introduced in the thesis) p-adic continued fraction algorithms and a function field analog with regard to questions about convergence, finiteness for rationals, periodicity for quadratic irrationals, symmetry patterns etc. In particular, a new quadratic algorithm introduced seems (numerical evidence, proved only for $p = 2, 3$) to provide periodicity for $p < 37$, improving results of Browkin and a new symmetry different from the real case or other p-adic cases is observed.

Available at MillerThesis.pdf

(4) Huei Jeng Chen (2011): Distribution of Diophantine approximation exponents for algebraic quantities in finite characteristic:

In contrast to Roth's (Uchiyama's respectively) theorem that algebraic real numbers (algebraic power series in characteristic zero respectively) have Diophantine approximation exponents qual to 2, Mahler had shown that the Liouville bound is best possible in finite characteristic. Schmidt and Thakur proved that given any rational number μ between 2 and $q+1$, where q is a power of a prime p, there exists (explicitly given) algebraic Laurent series α in characteristic p, with Diophantine approximation exponent μ and with degree at most $q+1$. We first refine this result by showing that degree of α can be prescribed to be $q+1$. Next, we describe how the exponents of α's are asymptotically distributed with their heights in the case of algebraic elements of class IA for function fields over finite fields. A result of Thakur says that most elements α have exponents near 2. We refine this result and give more precise descriptions of description of approximation exponents of such elements α of class IA. In the last chapter, we compute the continued fractions and approximation exponents of certain families of elements related to Carlitz torsion.

We note correction here that in Thm 1.4 on pa. 6 and Thm. 2.4 (same theorem of Lasjaunias, de Mathan quoted) finite characteristic field assumption should be replaced by finite field. (Thanks to Alain Lasjaunias for pointing this out).

Final version of her thesis is at ChenThesis.pdf

(5) Aleksandar Petrov (2012): On A-expansions of Drinfeld modular forms

In this dissertation, we introduce the notion of Drinfeld modular forms with A-expansions, where instead of the usual Fourier expansion in t^n (t being the uniformizer at infinity), parametrized by $n \in N$, we look at expansions in t_a, parametrized by $a \in A = \mathbf{F}_q[T]$. We construct an infinite family of such eigenforms. Drinfeld modular forms with A-expansions have many desirable properties that allow us to explicitly compute the Hecke action. The applications of our results include: (i) various congruences between Drinfeld eigenforms; (ii) interesting relations between the usual Fourier expansions and A-expansions, and resulting recursive relations for special families of forms with A-expansions; (iii) the computation of the eigensystems of Drinfeld modular forms with A-expansions; (iv) many examples of failure of multiplicity one result, as well as a restrictive multiplicity one result for Drinfeld modular forms with A-expansions; (v) the proof of diagonalizability of the Hecke action in 'non-trivial' cases; (vi) examples of eigenforms that can be represented as 'non-trivial' products of eigenforms; (vii) an extension of a result of Böckle and Pink concerning the Hecke properties of the space of cuspidal modulo double cuspidal forms for $\Gamma_1(T)$ to the groups $GL_2(\mathbf{F}_q[T])$ and $\Gamma_0(T)$.

Final version of his thesis is available at PetrovThesis.pdf

(6) José Alejandro Lara Rodriguez (Jan 2013): Relations between multizeta values, and Bernoulli-Carlitz numbers

In this thesis, Lara gives explicit formula for product of two zetas as sum of multizetas, and deduces several conjectures made by me and him about the function field multizeta values. He also investigates which relations survive in higher genus curves when infinity is not a rational place. He also completes the determination of the denominators of analog of B_n/n.

Full abstract: D. Thakur introduces and studies the multizeta values for function fields with constant field F_q (for a general A, with a rational place at infinity) and proves the existence of relations between them ([Tha04, Sec. 5.10], [Tha09b], [Tha10]). In particular, he proves that the product $\imath\zeta(a)\zeta\imath(b)$ is a linear combination of multizeta values. For $q = 2$, a full conjectural description of how the product of two zeta values can be described as the sum of multizeta was given by Thakur. Here, we shall prove part of the latter conjecture and also we shall prove a generalization of this conjecture made by the author of this thesis. Moreover, for general q, we shall prove closed formulas as well as a recursive recipe to express $\imath\zeta(a)\imath\zeta(b)$ as a sum of multizeta values. Several of the conjectures formulated in [Lar10, Tha09b] for small values or for special families of a about how to write $\imath\zeta(a)\imath\zeta(b)$ as an F_p-linear combination of multizeta values, will be proved. Also, we shall prove the parity conjecture formulated in [Tha09b]. When the place at infinity is not rational, we generalize the multizeta value definition and provide evidence that, in contrast with the rational case, the product of zeta values is not always a sum of multizeta. Also, we shall prove that for special families of a and b, $\imath\zeta(a)\imath\zeta(b)$ can be expressed as a sum of multizeta values. On the other hand, in 1935, L. Carlitz introduced analogues of Bernoulli numbers for $F_q[t]$ [Car35, Car37]. These are now called Bernoulli-Carlitz numbers Bm. He proved a von Staudt-Clausen type theorem, with a much more subtle statement than the classical one, describing their denominators completely. As an analog of the important relative B_m/m of the usual Bernoulli number B_m, Thakur considered an analog $B_m(m-1)!_C/m!_C$, where $m!_C$ is the Carlitz factorial. He described their denominator fully, except when $q = 2$ and m has a particular form. In this work we shall completely describe the latter. Also, we shall see that a group of symmetries recently discovered by D. Goss [Gos11] may be realized as symmetries of our results.

Final version of his thesis is at LaraThesis.pdf

(7) Somjit Datta (2013): On Jacobian theta functions, Rogers series and identities of Rogers-Ramanujan type.

Final version of his thesis is available at DattaThesis.pdf

(8) George Todd (2015): Linear relations between multizeta values.

In this dissertation, we discuss Fq(t)-linear relations beteen the multizeta values in function fields defined by Thakur. He proved that the product of multizeta values is a Fp-linear combination of multizeta values of the same weight analagous to the classical stuffle product for classical multizeta values. However, there is no known analog of the shuffle product for Thakur multizeta values from which to derive Fq(t)- linear relations. In this work, we introduce several families of maps between the space of relations of the power sums from which the multizeta values are defined. We describe the Fq(t)-linear relations currently in the literature in terms of these maps and provide many new relations. The main results of the dissertation are a conjectural characterization of all Fq(t)-linear relations between Thakur multizeta values as well as the dimension of the Fq(t)-span of multizeta values of a fixed weight, in addition to proving several cases under which the two are equivalent. These two conjectures provide the function field analog of the conjectures provided by Zagier and others dealing with similar issues for the classical multizeta values.

Final version of his thesis is available at ToddThesis.pdf

(9) Shuhui Shi (2017): Multiple Zeta Values for $\mathbf{F}_q[t]$.

In this thesis, we study structural relations between multiple zeta values over Fq[t] and their variants. In Chapter 2, we show that these values at negative integers of depth at least 2 only vanish at "trivial" zeros. In Chapter 3, we define a product on the space underlying the space of multiple zeta values, reflecting the shuffle relations. We also give a coproduct and conjecture that this gives a Hopf algebra structure on this space. We give a partial proof and a computational evidence for the associativity and coassociativity. In Chapter 4, we introduce several versions of truncated multiple zeta values depending on the choice of a prime and prove some "universal" relations among them which work for almost all primes. We also give a conjecture on the dimensions of the Fq(t)-spaces spanned by these adelic truncated values of given weight.

Final version of her thesis is available at ShiThesis.pdf

(10) Qibin Shen (2019): v-adic multiple zeta values over function fields.

In this thesis, we study structural relations between interpolated v-adic multiple zeta values over function fields. In chapter 2, we show that when K = Fq(t), the interpolated t-adic MZVs only vanish at 'trivial zeros" and based on numerical data we produced, we conjecture that this result can be generalized to any interpolated v-adic MZVs. In chapter 3, we proved a universal family of linear relations of interpolated v-adic MZVs, which is conjectured to generate all linear relations over Fp which work for all v. In chapter 4, we conjectured and proved some results on the relations between interpolated and Chang-Mishiba's motivic v-adic MZVs. In chapter 5, based on computations with our application of LLL method, we give a conjecture on the dimensions of the Fq(t)-span by these v-adic MZVs of the given weight.

Final version of his thesis is available at ShenThesis.pdf

Two papers based on (1) are published as JNT 59(1996) 313-318 and JNT 117 (2006) 241-262. (Available at javierRHjnt.pdf and javierzerosjnt.pdf)

Paper improving (2) is a joint publication (51). (available at elliptic-sub2.pdf: UPdate)

Paper of Jose Alejandro Lara-Rodrigues based on his Masters thesis done at U Arizona (2009) has appeared as JNT 130(2010) 1013-1023 (available at larajnt2010.pdf), and papers based on (6) (published versions of these are also included in his thesis) are JNT 131 (2011) 2081-2099 (available at larajnt2011.pdf), JNT 132 (2012) 495-501 and J. Ramanujan Soc. 27 No. 3 (2012) 275-293 (available at BCjntdraft.pdf and JRMSspecialdraft.pdf).

Paper of Somjit Dutt in JNT is available at JNTsomjit.pdf.