$RO(D_{2p})$-graded Slice Spectral Sequence of $H\underline{\mathbf{Z}}$

by

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Biographical Sketch

The author was born and raised in Hohhot, Inner Mongolia in 1987. He attended Peking University, where he received three degrees: Bachelor of Science in Mathematics and Bachelor of Arts in Economics in 2010 and Master of Science in Mathematics in 2013. He attended the University of Rochester in Fall 2013, and has been working with Professor Douglas Ravenel since then. He received Master of Arts Degree in 2015 in University of Rochester.

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Abstract

The slice spectral sequence was used by Hill, Hopkins and Ravenel to solve the Kervaire invariant one problem. The regular slice spectral sequence is a variant of the original version of slice spectral sequence developed by John Ullman[Ull13]. In this thesis, we compute the regular slice tower of suspensions by virtual representation spheres of $H\underline{\mathbf{Z}}$. The computation is based on the Mackey functor homotopy of $H\underline{\mathbf{Z}}$.

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Chapter 1

Introduction

Let $G = D_{2p} = \{\gamma, \tau | \gamma^p = e = \tau^2\}$ be the dihedral group of order $2p$. This article gives a computational approach to the D_{2p}-spectrum $H\underline{\mathbf{Z}}$.

In equivariant homotopy theory, Mackey functors are the natural generalization of abelian groups in the role of coefficients. In Chapter 2, we give a brief introduction to Mackey functors, and a symmetric monoidal category structure called box product.

In Chapter 3, we compute $\underline{\pi}_\star H\underline{\mathbf{Z}}$, the $RO(D_{2p})$-graded homotopy groups of $H\underline{\mathbf{Z}}$, in terms of Mackey functors. We also describe the ring structure and divisibility of elements in $\underline{\pi}_\star H\underline{\mathbf{Z}}$.

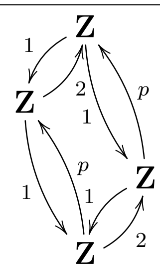
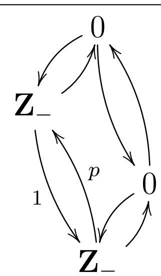
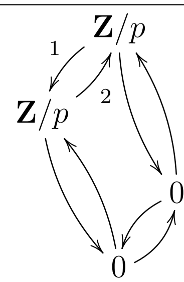
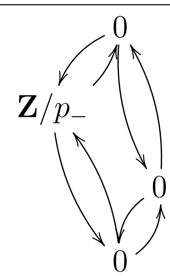
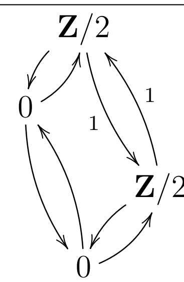
In Chapter 4, we use the slice spectral sequence, which was developed in [HHR16] and modified by Ullman in his thesis [Ull13], to compute the slice spectral sequence for $\Sigma^V \wedge H\underline{\mathbf{Z}}$. Computing the general slice spectral sequence is usually hard. However in specific cases, the structure of cohomological Mackey functor will shed light on it. The main theorem we prove in this chapter is:

Theorem 1.1. *Let $G = D_{2p}$ and let $V - W \in RO(G)$, where both V and W are representations of G. Then there exists $e(V-W) \in ROG)$, such that $S^{V-W} \wedge H\underline{\mathbf{Z}}$*

has a spherical $(|V|-|W|)$*-slice* $S^{e(V-W)} \wedge H\underline{\mathbf{Z}}$*. The other slices are suspension of* $H\underline{B}$*,* $H\underline{B}_-$ *or* $H\underline{D}$*.*

Lewis diagrams of the Mackey functors used can be found in Table 1.1.

Table 1.1: Some G-Mackey functors

Symbol	$\underline{\mathbf{Z}}$	$\underline{\mathbf{Z}}_-$	$\underline{B}$	$\underline{B}_-$	$\underline{D}$
Lewis diagram					

Chapter 2

Mackey Functors

In this chapter, we introduce basic definitions associated with Mackey functors.

2.1 Mackey Functors

Let $G\mathscr{S}$ be the category of G-sets with G-equivariant morphisms and let $\mathscr{A}b$ be the category of finitely generated abelian groups.

We use Dress's definition of Mackey functors [Dre73].

Definition 2.1.1. *A G-**Mackey functor** $\underline{\boldsymbol{M}}$ consists of a covariant functor M_*, and a contravariant functor M^**

$$G\mathscr{S} \to \mathscr{A}b$$

such that:

(i) for disjoint G-sets S and T, M_* and M^* convert disjoint unions of finite G-sets to direct sums of abelian groups.

(ii) for each pull-back diagram of finite G-sets

$$\begin{tikzcd} S_1 \arrow[r, "F"] \arrow[d, "f_1"] & S_2 \arrow[d, "f_2"] \\ T_1 \arrow[r, "H"] & T_2 \end{tikzcd}$$

we have $M_*(F)M^*(f_1) = M^*(f_2)M_*(H)$.

The morphisms between Mackey functors are natural transformations.

The category of all G-Mackey functors and morphisms as mentioned is naturally an abelian category, which we denote by $\mathfrak{M}_G$.

Since every G-set can be decomposed into direct sums of transitive G-sets, i.e. the left G-sets of the form G/H where H is a subgroup of G, a Mackey functor can therefore be determined by its value on all the transitive G-sets together with the restrictions and transfers between them.

For subgroups $K \subseteq H \subseteq G$, there exists a unique G-map $G/K \to G/H$:

- its image under M_* is called the *transfer* Tr_K^H.
- its image under M^* is called the *restriction* Res_K^H.

Definition 2.1.2. *A Mackey functor* $\underline{M}$ *is called a* ***fixed point Mackey functor*** *if* $\underline{M}(G/H) = \underline{M}(G/e)^H$ *for all* $H \subseteq G$. *In a fixed point Mackey functor ,each restriction is injective.*

Furthermore, a fixed point Mackey functor $\underline{T}$ *is called a permutation Mackey functor if* $\underline{T}(G/e)$ *is a free abelian group on a G-set* T.

An example of Mackey functor is the Burnside Mackey functor:

Table 2.1: Some C_2-Mackey functors

Symbol	$\square$	$\overline{\square}$	$\bullet$	◩	$\dot{\square}$	$\widehat{\square}$
Lewis diagram	$\begin{array}{c}\mathbf{Z}\\ {\scriptstyle 1}\downarrow\ \uparrow{\scriptstyle 2}\\ \mathbf{Z}\end{array}$	$\begin{array}{c}0\\ \downarrow\ \uparrow\\ \mathbf{Z}_-\end{array}$	$\begin{array}{c}\mathbf{Z}/2\\ \downarrow\ \uparrow\\ 0\end{array}$	$\begin{array}{c}\mathbf{Z}\\ {\scriptstyle 2}\downarrow\ \uparrow{\scriptstyle 1}\\ \mathbf{Z}\end{array}$	$\begin{array}{c}\mathbf{Z}/2\\ {\scriptstyle 0}\downarrow\ \uparrow{\scriptstyle 1}\\ \mathbf{Z}_-\end{array}$	$\begin{array}{c}\mathbf{Z}\\ {\scriptstyle \Delta}\downarrow\ \uparrow{\scriptstyle \nabla}\\ \mathbf{Z}[C_2]\end{array}$
Lewis symbol	R	R_-	$\langle \mathbf{Z}/2 \rangle$	L	L_-	$R(\mathbf{Z}^2)$

Example 2.1.3. *[HHRa] For a finite group G, the Burnside ring $A(G)$ of G is the Grothendieck completion of the abelian monoid (under disjoint unions) of isomorphism classes of finite G-sets, with multiplication induced by Cartesian product.*

The Burnside Mackey functor is constructed from the Burnside ring as follows: Let $A(S)$ denote the Grothendieck completion of the abelian monoid (under disjoint unions) of isomorphism classes of finite G-sets over S, with multiplication induced by Cartesian product. A finite G-set R over S is a G-set R with a G-set map to S. A map $\alpha : S \to T$ of G-sets induces a map $\alpha_* : \underline{A}(S) \to \underline{A}(T)$ by composition, and $\alpha^* : \underline{A}(T) \to \underline{A}(S)$ by pullback.

2.2 Lewis Diagrams and Some Mackey Functors

To visualize Mackey functors, we use the Lewis diagrams which were first introduced in [Lew88]. The following table extracted from [HHRc, Section 5] shows some Lewis diagrams of C_2-Mackey functors .

The Lewis diagram of the D_{2p}-Mackey functor $\underline{\mathrm{M}}$ where p is an odd prime is

$$\begin{tikzcd}
 & \underline{M}(G/G) \arrow[dl, bend right] \arrow[dr, bend left] & \\
\underline{M}(G/C_p) \arrow[ur, bend right] \arrow[dr, bend right] & & \underline{M}(G/C_2) \arrow[ul, bend left] \arrow[dl, bend left] \\
 & \underline{M}(G/e) \arrow[ul, bend right] \arrow[ur, bend left] &
\end{tikzcd} \tag{2.2.1}$$

Figure 2.1 is a table of some D_{2p}-Mackey functors.

2.3 Box Products, Green Functors, and Cohomological Mackey Functors

The box product of Mackey functors was first described in [Lew80] using Day convolution. A constructive definition was given in [Li15, Definition 3.1]. Li proved in Section 4 of [Li15] that the two definition are equivalent for any finite group.

Definition 2.3.1. *Given two Mackey functors $\underline{M}$ and $\underline{N}$, the **box product** $\underline{M}\Box\underline{N}$ is the Mackey functor defined inductively:*

$$\underline{M}\Box\underline{N}(G/e) = \underline{M}(G/e) \otimes \underline{N}(G/e)$$

$$\underline{M}\Box\underline{N}(G/H) = (\underline{M}(G/H) \otimes \underline{N}(G/H) \bigoplus_{K<H} (\underline{M}\Box\underline{N}(G/K)/W_H K)/FR)$$

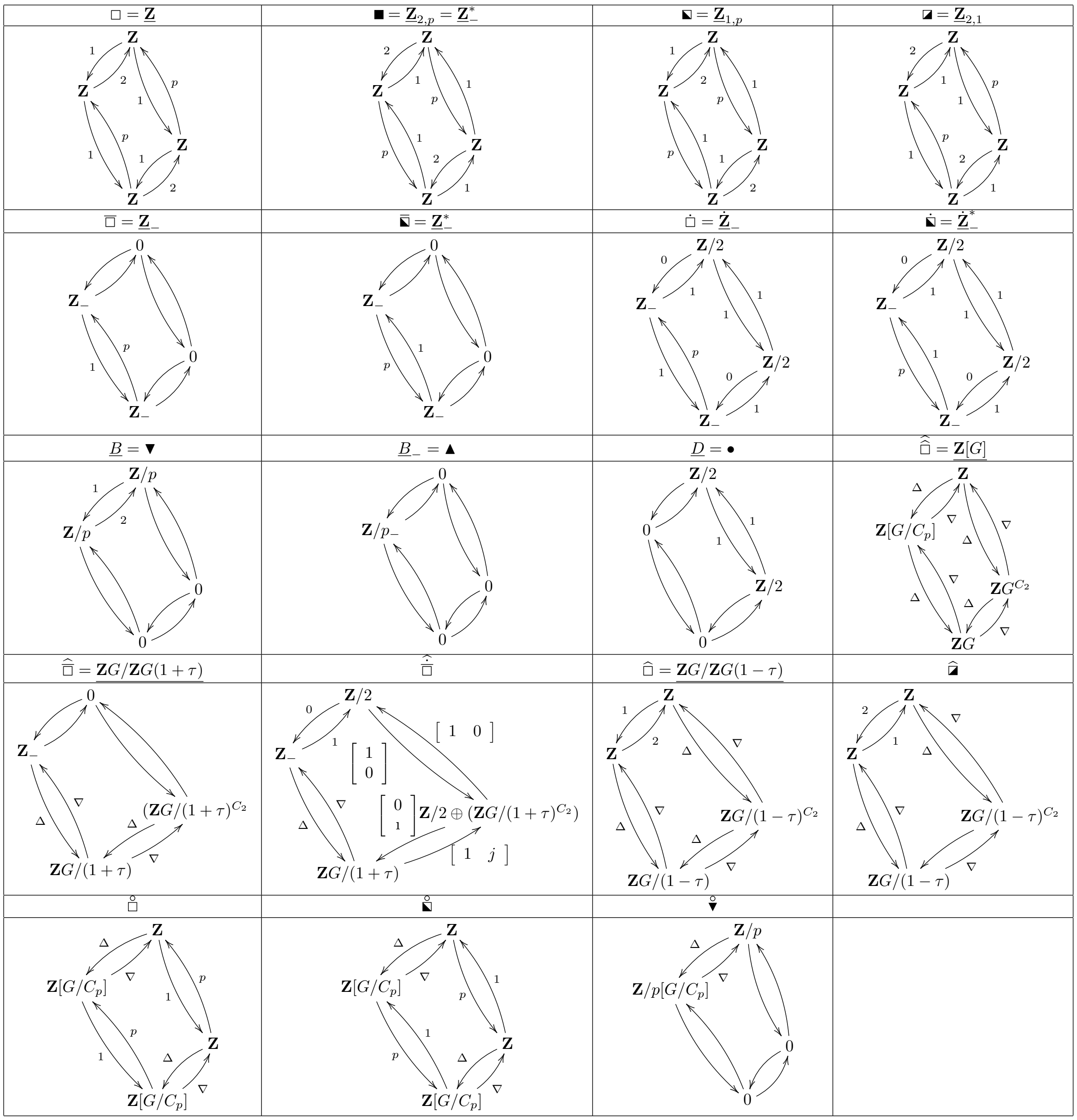


Figure 2.1: Figures for D_{2p} Mackey Functors

where $W_H K$ is the Weyl group $N_H K/K$ and FR is the Frobenius reciprocity submodule generated by elements of the form

$$x \otimes Tr_K^H(y) - Tr_K^H(Res_K^H(x) \otimes y)$$

and

$$Tr_K^H(y) \otimes x - Tr_K^H(y \otimes Res_K^H(x))$$

for all $K < H$, $x \in \underline{M}(G/H)$ and $y \in \underline{N}(G/K)$.

For more concrete examples of computing the box products, one may refer to [Li15].

The box product defines a closed symmetric monoidal category structure on $\mathfrak{M}_G$, where the unit is the Burnside Mackey functor $\underline{A}(G)$. The reader may refer to [HHRa] for more details. We don't discuss general Mackey functors in this article. Instead, we focus on cohomological Mackey functors.

A Mackey functor which behaves like a ring in the closed symmetric monoidal category $(\mathfrak{M}_G, \Box, \underline{A})$ is called a Green functor. The definition of Green functors was first given in [Lew80]. Kristen Mazur claimed in [Maz11, Definition 1.3.2] that there is an equivalent definition. We will use the definition given by Mazur.

Definition 2.3.2. *A G-Mackey functor $\underline{R}$ is called a **Green functor**, if*

(i) $\underline{R}(G/H)$ is a ring for all G/H,

(ii) for $K \leq H \leq G$, the restriction maps $Res_K^H : \underline{R}(G/H) \to \underline{R}(G/K)$ are unit preserving ring homomorphisms,

(iii) $\underline{R}$ satisfies Frobenius reciprocity: for $K \leq H \leq G$,

$$Tr_K^H(x) \cdot y = Tr_K^H(xRes_K^H(y).)$$

A Green functor is commutative, if all the rings $\underline{R}(G/H)$ are commutative.

Remark 2.3.3. *The reference Mazur mentioned is not correct, though the definitions are still equivalent. The reader can refer to [Lew80].*

A special type of Mackey functoris the cohomological Mackey functor.

Definition 2.3.4. *[TW95, 16] A Mackey functor is a* ***cohomological Mackey functor*** *if for any $K \leq H \leq G$, $Tr_K^H Res_K^H = |H : K|$.*

We denote the category of cohomological Mackey functors and natural transformations as $\mathfrak{CM}_G$. It is a full subcategory of $\mathfrak{M}_G$ as an abelian category. However, the Burnside Mackey functor is not a cohomological Mackey functor. So $\mathfrak{CM}_G$ does not inherit the closed symmetric monoidal category structure of $(\mathfrak{M}_G, \Box, \underline{A}(G))$. But luckily, it is still a closed symmetric monoidal category with box product as the binary operation, but the unit is the fixed pointed Mackey functor $\underline{\mathbf{Z}}$ determined by $\underline{\mathbf{Z}}(G/e) = \mathbf{Z}$.

Proposition 2.3.5. *$\underline{\mathbf{Z}}$ is the unit in $\mathfrak{CM}_G$, that is, if $\underline{M}$ is a cohomological Mackey functor, then $\underline{M} \Box \underline{\mathbf{Z}} = \underline{M}$.*

Proof. We use induction for the proof. Clearly this is true for $G/\{e\}$. Let H be a subgroup of G. Assume that the result is true for each G/K where $|K| < |H|$, we will prove it for G/H.

From the assumption we have

$$\begin{aligned}\underline{M}\Box\underline{\mathbf{Z}}(G/H) &= (\underline{M}(G/H)\otimes\underline{\mathbf{Z}}(G/H)\bigoplus_{K<H}(\underline{M}\Box\underline{\mathbf{Z}}(G/K)/W_HK)/FR)\\ &\cong(\underline{M}(G/H)\bigoplus_{K<H}\underline{M}(G/K)/FR)\\ &\cong\underline{M}(G/H)\end{aligned}$$

The third equality comes from the Frobenius relations. $\square$

Therefore, $(\mathfrak{CM}_G,\Box,\mathbf{Z})$ is a closed symmetric monoidal category.

In this article, we work with commutative cohomological Green functors, i.e. Mackey functors that are both commutative cohomological Mackey functors and Green functors.

When X is a ring spectrum, we have the *fixed point Frobenius relation*,

$$\mathrm{Tr}_K^H(\mathrm{Res}_K^H(a)b)=a(\mathrm{Tr}_K^H(b))\qquad\text{for } a\in\underline{\pi}_\star X(G/H)\text{ and } b\in\underline{\pi}_\star X(G/K). \tag{2.3.6}$$

which results from the box product of Green functors. In particular

$$a(\mathrm{Tr}_K^H(b))=0\qquad\text{when } \mathrm{Res}_K^H(a)=0. \tag{2.3.7}$$

When the G-spectrum X is a commutative ring spectrum, for example $H\underline{\mathbf{Z}}$, $\underline{\pi}_\star X$ is a commutative $RO(G)$-graded Green functor. The multiplication of ring spectra induces box product of Green functors.

2.3.A Lemmas for Cohomological Mackey Functors

There are some lemmas for cohomological Mackey functors, which we will use later:

Lemma 2.3.8. *If $\underline{M}$ is a $\mathbf{Z}$-valued cohomological Mackey functor, i.e., $\underline{M}(G/H) = \mathbf{Z}$ for all subgroups of G, then we have the following properties*

(i) $Res^G_{C_2} : \underline{M}(G/G) = \mathbf{Z} \to \underline{M}(G/C_2) = \mathbf{Z}$ *and* $Res^{C_p}_e : \underline{M}(G/C_p) = \mathbf{Z} \to \underline{M}(G/e) = \mathbf{Z}$ *coincide as maps between abelian groups as maps between abelian groups.*

(ii) $Res^G_{C_p} : \underline{M}(G/G) = \mathbf{Z} \to \underline{M}(G/C_p) = \mathbf{Z}$ *and* $Res^{C_2}_e : \underline{M}(G/C_2) = \mathbf{Z} \to \underline{M}(G/e) = \mathbf{Z}$ *coincide as maps between abelian groups.*

(iii) $Tr^G_{C_2} : \underline{M}(G/C_2) = \mathbf{Z} \to \underline{M}(G/G) = \mathbf{Z}$ *and* $Tr^{C_p}_e : \underline{M}(G/e) = \mathbf{Z} \to \underline{M}(G/C_p) = \mathbf{Z}$ *coincide as maps between abelian groups.*

(iv) $Tr^G_{C_p} : \underline{M}(G/C_p) = \mathbf{Z} \to \underline{M}(G/G) = \mathbf{Z}$ *and* $Tr^{C_2}_e : \underline{M}(G/e) = \mathbf{Z} \to \underline{M}(G/C_2) = \mathbf{Z}$ *coincide as maps between abelian groups.*

Proof. We only work on the first and the others can be done in the same way. The composition is $Tr^G_{C_2} \circ Res^G_{C_2} = [G : C_2] = p$. So there are only two possibilities: (a)$Tr^G_{C_2} = p$ and $Res^G_{C_2} = 1$ or (b)$Tr^G_{C_2} = 1$ and $Res^G_{C_2} = p$. For the same reason, $Res^G_{C_p}$ must be either 1 or 2. However, two compositions of restriction maps $Res^{C_2}_e \circ Res^G_{C_2}$ and $Res^{C_p}_e \circ Res^G_{C_p}$ have to coincide. But 1,2 and p are all coprime, which forces $Res^G_{C_2} = Res^{C_p}_e$. □

We will give names to some Mackey functors which will be used:

Example 2.3.9 (Some D_{2p} cohomological Mackey functors)**.** *We define the following four types of* $\mathbf{Z}$*-valued cohomological Mackey functors:*

(i) *We let* $\underline{\mathbf{Z}} = \underline{\mathbf{Z}}_{1,1}$ *denote the* $Mackey functor$ $\square$ *in* (2.2.1)*, which means the restrictions* $Res^{C_2}_{\{e\}}$ *and* $Res^{C_p}_{\{e\}}$ *are both* 1*,*

(ii) *We let* $\underline{\mathbf{Z}} = \underline{\mathbf{Z}}_{2,p}$ *denote the* $Mackey functor$ $\blacksquare$ *in* (2.2.1)*,which means the restriction* $Res^{C_2}_{\{e\}} = 2$*, and* $Res^{C_p}_{\{e\}} = p$*,*

(iii) *We let* $\underline{\mathbf{Z}} = \underline{\mathbf{Z}}_{1,p}$ *denote the* $Mackey functor$ ⬕ *in* (2.2.1)*, which means the restriction* $Res^{C_2}_{\{e\}} = 1$*, and* $Res^{C_p}_{\{e\}} = p$*,*

(iv) *We let* $\underline{\mathbf{Z}} = \underline{\mathbf{Z}}_{2,1}$ *denote the* $Mackey functor$ ◪ *in* (2.2.1)*, which means the restriction* $Res^{C_2}_{\{e\}} = 2$*, and* $Res^{C_p}_{\{e\}} = 1$*.*

Some cohomological Mackey functors which have the (G/e)*value* $\underline{\mathbf{Z}}_-$ *are:*

(i) *if* $\underline{M}(G/G) = 0$ *and* $Res_e^{C_p} = 1$*, then the cohomological Mackey functor is the fixed point Mackey functor* $\underline{\mathbf{Z}}_- = \overline{\square}$*.*

(ii) *if* $\underline{M}(G/G) = 0$ *and* $Res_e^{C_p} = p$*, then the cohomological Mackey functor is denoted by* $\underline{\mathbf{Z}}^*_- = \overline{\text{⬕}}$*.*

(iii) *if* $\underline{M}(G/G) = \mathbf{Z}/2$ *and* $Res_e^{C_p} = 1$*, then the cohomological Mackey functor is denoted by* $\dot{\underline{\mathbf{Z}}}_- = \dot{\square}$*.*

(iv) *if* $\underline{M}(G/G) = \mathbf{Z}/2$ *and* $Res_e^{C_p} = p$*, then the cohomological Mackey functor is denoted by* $\dot{\underline{\mathbf{Z}}}^*_- = \dot{\text{⬕}}$*.*

Some other Mackey functors we are going to use in the next chapters are:

(i) $\underline{B} = \blacktriangledown$ *is defined to be the cokernel of the morphism* $\underline{\mathbf{Z}}_{1,1} \xrightarrow{1} \underline{\mathbf{Z}}_{1,p}$.

(ii) $\underline{B}_{-} = \blacktriangle$ *is defined to be the cokernel of the morphism* $\underline{\mathbf{Z}}_{1,1} \xrightarrow{1} \underline{\mathbf{Z}}_{1,p}$.

(iii) $\underline{D} = \bullet$ *is defined to be the cokernel of the morphism* $\underline{\mathbf{Z}}_{1,1} \xrightarrow{1} \underline{\mathbf{Z}}_{2,1}$.

The morphisms mentioned above are actually the morphisms between $\underline{M}(G/e)$*-values. Since all the Mackey functors listed are fixed point Mackey functors, the morphisms between Mackey functors are completely determined.*

Chapter 3

RO(G)-graded Homotopy of $H\underline{\mathbf{Z}}$

Notation: From now, unless otherwise stated, G denotes D_{2p}, G' denotes the subgroup C_p of G. In this paper there are two different types of homotopy: $\underline{\pi}_* X$ denotes the integer graded homotopy, while $\underline{\pi}_\star X$ denotes the $RO(G)$ graded homotopy of X. i_H^G denotes the forgetful functor.

In this chapter, we describe the $RO(G)$-graded Green functor $\pi_\star H\underline{\mathbf{Z}}$, where $H\underline{\mathbf{Z}}$ is the Eilenberg-Mac Lane spectrum associated with the Mackey functor $\underline{\mathbf{Z}}$. Let $\lambda(k)$ denote the composite of the inclusion of the $2p$-th roots of unity with the degree k-map on S^1. If q is a prime not equal to 2 or p, then $S^{\lambda(q)} \simeq S^{\lambda(1)}$. Therefore $S^{\lambda(q)}$ and $S^{\lambda(1)}$ have the same G-cell structure. This is an equivalence relation called JO-equivalence on Page 4, [HHRb].

3.1 Mackey Functor Homotopy

In this section, we will illustrate how Mackey functors are engaged in equivariant stable homotopy theory.

Definition 3.1.1. *Let X be a G-spectrum. Let V be an orthogonal representation of G, whose 1-point compactification is denoted by S^V. Furthermore we let T_+ denote the suspension spectrum of the union of the G-set T with a disjoint base point. We define*

$$\underline{\pi}_V X(T) = [S^V \wedge T_+, X]^G$$

to be the set of G-equivariant maps between the spectra. And we define the Mackey functor $\underline{\pi}_V X$ to be the ***V-th Mackey functor homotopy of X*** *In particular, when V is an n-dimensional trivial representation, we will just write $\underline{\pi}_n X$.*

To determine the Mackey functor structure maps in $\underline{\pi}_V X$, we need to specify $Res_H^K : \underline{\pi}_V X(G/H) \to \underline{\pi}_V X(G/K)$ and $Tr_H^K : \underline{\pi}_V X(G/K) \to \underline{\pi}_V X(G/H)$. Such maps are induced by the natural basepoint preserving G-map $f : G/K_+ \to G/H_+$ such that $f(xK) = xH$ for $x \in G$.

We can define the Eilenberg-Mac Lane spectrum for a Mackey functor $\underline{M}$ in the same way as the Eilenberg-Mac Lane spectrum for an abelian group in nonequivariant homotopy theory.

Definition 3.1.2. *The* ***Eilenberg-Mac Lane spectrum $H\underline{M}$*** *for the Mackey functor $\underline{M}$ is the G-spectrum such that*

$$\pi_0 H\underline{M} = \underline{M}$$

$$\pi_V H\underline{M} = \emptyset \quad if\ V \neq 0.$$

For each Mackey functor M, $H\underline{M}$ exists([GM95, Section 5]). A specific construction of Mackey functor Eilenberg-Mac Lane spectrum is given in [BO15].

Definition 3.1.3. *A G-spectrum X is called a* ***cohomological G-spectrum****, if*

$\underline{\pi}_n(X)$ *is a cohomological Mackey functor for any integer* n.

Example 3.1.4. *We list a few examples of cohomological* G*-spectra without giving any proof at this moment:*

(i) $H\underline{M}$ *where* $\underline{M}$ *is a cohomological Mackey functor.*

(ii) $S^V \wedge H\underline{\mathbf{Z}}$. *It is because* $\underline{\pi}_* S^V \wedge H\underline{\mathbf{Z}} \cong \underline{H}_*(S^V;\underline{\mathbf{Z}})$. $\underline{H}_*(S^V;\underline{\mathbf{Z}})$ *can be computed via chain complexes of fixed point Mackey functor of* $\underline{\mathbf{Z}}[G]$*-modules. The fixed point Mackey functors of* $\underline{\mathbf{Z}}[G]$*-modules are all cohomological Mackey functors. So are the kernels and cokernels of them.*

3.2 Identifying RO(G)-graded homotopy of $H\underline{\mathbf{Z}}$

In this section, we compute the $RO(G)$-graded Mackey functor homotopy $\underline{\pi}_\star H\underline{\mathbf{Z}}$. Since $H\underline{\mathbf{Z}}$ is a ring spectrum, we need to determine its ring structure.

Computing $\underline{\pi}_V H\underline{\mathbf{Z}}$ where V is a virtual representation of D_{2p} can be done through the following procedure: Let X be a finite G-CW spectrum, then

$$\underline{H}_* X(G/H) = \underline{\pi}_* X \wedge H\underline{\mathbf{Z}}(G/H) \cong \underline{\pi}_*(X \wedge H\underline{\mathbf{Z}})^H \tag{3.2.1}$$

Therefore $\underline{\pi}_V H\underline{\mathbf{Z}} \cong \underline{H}_0 S^{-V}$. In general $\underline{H}_* X(G/H)$ is different from $H_*(X^H)$, because fixed points do not commute with smash products.

Recall that $H\underline{\mathbf{Z}}$ is a commutative ring spectrum, so $\underline{\pi}_\star H\underline{\mathbf{Z}}$ is a commutative $RO(G)$-graded Green functor. The multiplication of ring spectra induces box product of Green functors. Since the Mackey functor homology $\underline{H}_* X$ can be computed via chain complexes of fixed point Mackey functors, $\underline{H}_\star X \cong \underline{\pi}_*(X \wedge H\underline{\mathbf{Z}})$

are actually cohomolocial Green functors. So we can leverage both the properties of cohomological Mackey functors and of Green functors to do our computations.

In particular, when X is a virtual sphere S^{V-W} where V and W are both representations of G, the $RO(G)$-graded homology of S^{V-W} can be computed with chain complexes of $\underline{\mathbf{Z}}[G]$-modules. The group $G = D_{2p}$ has three equivalence classes of JO-equivalent irreducible real representations: the trivial representation 1, the sign representation σ, and the two dimensional representations λ. Though there are actually other 2-dimensional irreducible representations of $G = D_{2p}$, the representation spheres obtained are of the same homotopy type if an integer other than 2 or p is inverted. Such an equivalence is called JO-equivalence in [HHRb]. In this article we always consider JO-equivalent representation spheres.

Applying the forgetful functor i_H^G to a G-spectrum will give an H-spectrum. In particular, there is an isomorphism

$$i_H^G(\underline{\pi}_* X) \cong \underline{\pi}_*(i_H^G X). \tag{3.2.2}$$

where $H \leq G$. We need to point out that i_H^G on the left hand side of the equation is the forgetful functor from $\mathfrak{M}_G$ to $\mathfrak{M}_H$, while on the right hand side it is the forgetful functor from the category of G-spectra to the category of H-spectra. The reader should be aware of the abuse of notations.

Definition 3.2.3. *A representation sphere S^V is* ***orientable****, if τ, the element of order 2 in G, acts on $\underline{H}_{|V|}S^V(G/e)$ trivially. S^V is not orientable if τ acts on $\underline{H}_{|V|}S^V(G/e)$ by multiplying by -1.*

We should point out that $i_{C_2}^G \lambda = S^{1+\sigma}$ as a C_2 representation, therefore S^λ is not orientable. S^V is orientable if and only if $i_{C_2}^G S^V$ is orientable as a C_2-

representation sphere. There are more examples of orientable and nonorientable G-spheres:

Example 3.2.4. *(i) The trivial sphere S^n is always orientable.*

(ii) $S^{n\sigma}$ is orientable if and only if n is even.

(iii) $S^{m\lambda}$ is orientable if and only if m is even.

(iv) S^{V+W} is orientable if S^V and S^W are both orientable or both nonorientable.

Let V be an actual representation of the form $a+b\sigma+c\lambda$ and let W be $l+m\sigma+n\lambda$. The orientability of S^V only depends on the parity of $b+c$. Therefore we can generalize the definition of orientability to virtual representable spheres: S^{V-W} is orientable if and only if $i^G_{C_2}S^{V-W}$ is orientable as a C_2-virtual representation sphere, or equivalently, $b+c+m+n$ is even.

For the reader's convenience, we recall Table 2.1 here. In addition we made a corresponding version of C_p-Mackey functors.

Table 3.1: Some C_2-Mackey functors

Symbol	$\square$	$\overline{\square}$	$\bullet$	◩	$\dot{\square}$	$\widehat{\square}$
Lewis diagram	$\begin{array}{c}\mathbf{Z}\\ {}_1\downarrow\ \uparrow{}_2\\ \mathbf{Z}\end{array}$	$\begin{array}{c}0\\ \downarrow\ \uparrow\\ \mathbf{Z}_-\end{array}$	$\begin{array}{c}\mathbf{Z}/2\\ \downarrow\ \uparrow\\ 0\end{array}$	$\begin{array}{c}\mathbf{Z}\\ {}_2\downarrow\ \uparrow{}_1\\ \mathbf{Z}\end{array}$	$\begin{array}{c}\mathbf{Z}/2\\ {}_0\downarrow\ \uparrow{}_1\\ \mathbf{Z}_-\end{array}$	$\begin{array}{c}\mathbf{Z}\\ {}_\Delta\downarrow\ \uparrow{}_\nabla\\ \mathbf{Z}[C_2]\end{array}$

First let's compute $\underline{H}_*(S^V;\underline{\mathbf{Z}})$ when the group is C_2 or C_p.

The following proposition is taken from ([HHRc, §5]).

Proposition 3.2.5. *By abuse of notation, we use σ to denote the sign representation for both G and C_2. Therefore $i^G_{C_2}(S^\lambda) = S^{1+\sigma}$. Also, we have $i^G_{C_p}(S^\sigma) = S^1$.*

When the subgroup H is C_2 we have the following example:

Let τ denote the generator of C_2. For positive n, the chain complex of $S^{n\sigma}$ has the form

$$
\begin{tikzcd}
0 & 1 & 2 & 3 & & n \\
\mathbf{Z} \arrow[d, bend right, "1"'] & \mathbf{Z} \arrow[l, "2"'] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[l, "0"'] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[l, "2"'] \arrow[d, bend right, "\Delta"'] & \cdots \arrow[l] & \mathbf{Z} \arrow[l, "\epsilon_n"'] \arrow[d, bend right, "\Delta"'] \\
\mathbf{Z} \arrow[u, bend right, "2"'] & \mathbf{Z}[C_2] \arrow[l, "\nabla"'] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[C_2] \arrow[l, "\overline{\tau}_2"'] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[C_2] \arrow[l, "\overline{\tau}_3"'] \arrow[u, bend right, "\nabla"'] & \cdots \arrow[l] & \mathbf{Z}[C_2] \arrow[l, "\overline{\tau}_n"'] \arrow[u, bend right, "\nabla"']
\end{tikzcd}
\tag{3.2.6}
$$

where

$$\overline{\tau}_i = 1 - (-1)^i\overline{\tau} \quad \textit{and} \quad \epsilon_i = 1 - (-1)^i. \tag{3.2.7}$$

The homology Mackey functors are

$$
\begin{tikzcd}
0 & 1 & 2 & 3 & & n \\
\bullet & 0 & \bullet & 0 & & \underline{H}_n \\
\mathbf{Z}/2 \arrow[d, bend right] & 0 \arrow[d, bend right] & \mathbf{Z}/2 \arrow[d, bend right] & 0 \arrow[d, bend right] & \cdots & \underline{H}_n(G/G) \arrow[d, bend right, "\Delta"'] \\
0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & \cdots & \mathbf{Z}[G]/(\overline{\tau}_{n+1}) \arrow[u, bend right, "\nabla"']
\end{tikzcd}
\tag{3.2.8}
$$

where

$$\underline{H}_n(G/G) = \begin{cases} \mathbf{Z} & \textit{for } n \textit{ even} \\ 0 & \textit{for } n \textit{ odd} \end{cases} \qquad \textit{and} \qquad \underline{H}_n = \begin{cases} \square & \textit{for } n \textit{ even} \\ \overline{\square} & \textit{for } n \textit{ odd} \end{cases}$$

For negative n, the chain complex of $S^{n\sigma}$ has the form

$$
\begin{tikzcd}
0 & -1 & -2 & -3 & & -n \\
\mathbf{Z} \arrow[r, "1"] \arrow[d, bend right, "1"'] & \mathbf{Z} \arrow[r, "0"] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[r, "2"] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[r, "0"] \arrow[d, bend right, "\Delta"'] & \cdots \arrow[r, "\epsilon_n"] & \mathbf{Z} \arrow[d, bend right, "\Delta"'] \\
\mathbf{Z} \arrow[r, "\nabla"] \arrow[u, bend right, "2"'] & \mathbf{Z}[C_2] \arrow[r, "\overline{\tau}_{-1}"] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[C_2] \arrow[r, "\overline{\tau}_{-2}"] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[C_2] \arrow[r, "\overline{\tau}_{-3}"] \arrow[u, bend right, "\nabla"'] & \cdots \arrow[r, "\overline{\tau}_{-n+1}"] & \mathbf{Z}[C_2] \arrow[u, bend right, "\nabla"']
\end{tikzcd}
\tag{3.2.9}
$$

Passing to homology we get

$$
\begin{tikzcd}
0 & -1 & -2 & -3 & & -n \\
0 & 0 & 0 & \bullet & & \underline{H}_{-n} \\
0 \arrow[d, bend right] & 0 \arrow[d, bend right] & 0 \arrow[d, bend right] & \mathbf{Z}/2 \arrow[d, bend right] & \cdots & \underline{H}_{-n}(G/G) \arrow[d, bend right, "\Delta"'] \\
0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & \cdots & \mathbf{Z}[C_2]/(\overline{\tau}_{-n+1}) \arrow[u, bend right, "\nabla"']
\end{tikzcd}
\tag{3.2.10}
$$

where

$$
\underline{H}_{-n}(G/G) = \begin{cases} \mathbf{Z} & \text{for } n \text{ even} \\ 0 & \text{for } n = 1 \\ \mathbf{Z}/2 & \text{for } n \geqslant 1 \text{ and } n \text{ odd} \end{cases}
$$

and

$$
\underline{H}_{-n} = \begin{cases} ◪ & \text{for } n \text{ even} \\ \overline{\square} & \text{for } n = 1 \\ \dot{\square} & \text{for } n \geqslant 1 \text{ and odd} \end{cases}
$$

When $H = C_p$, the homology are quite different.

Proposition 3.2.11. *We use $\overline{\lambda}$ to denote the 2-dimensional irreducible represen-*

Table 3.2: Some C_p-Mackey functors

Symbol	$\square$	$\overline{\square}$	$\blacktriangledown$	◩	$\widehat{\square}$
Lewis diagram	$\begin{array}{c}\mathbf{Z}\\ {\scriptstyle 1}\downarrow\uparrow{\scriptstyle p}\\ \mathbf{Z}\end{array}$	$\begin{array}{c}0\\ \downarrow\uparrow\\ \mathbf{Z}_-\end{array}$	$\begin{array}{c}\mathbf{Z}/p\\ \downarrow\uparrow\\ 0\end{array}$	$\begin{array}{c}\mathbf{Z}\\ {\scriptstyle p}\downarrow\uparrow{\scriptstyle 1}\\ \mathbf{Z}\end{array}$	$\begin{array}{c}\mathbf{Z}\\ {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla}\\ \mathbf{Z}[C_2]\end{array}$

tation of C_p. *Therefore* $i^G_{C_p}(S^\lambda) = S^{\overline{\lambda}}$

Let γ *be the generator of* C_p. *For positive* n, *the chain complex of* $S^{n\overline{\lambda}}$ *has the form*

$$\begin{tikzcd}
0 & 1 & 2 & 3 & & 2n \\
\mathbf{Z} \arrow[d, bend right, "1"'] & \mathbf{Z} \arrow[l, "p"'] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[l, "0"'] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[l, "p"'] \arrow[d, bend right, "\Delta"'] & \cdots \arrow[l] & \mathbf{Z} \arrow[l, "0"'] \arrow[d, bend right, "\Delta"'] \\
\mathbf{Z} \arrow[u, bend right, "p"'] & \mathbf{Z}[C_p] \arrow[l, "\nabla"'] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[C_p] \arrow[l, "\overline{\gamma}_2"'] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[C_p] \arrow[l, "\overline{\gamma}_3"'] \arrow[u, bend right, "\nabla"'] & \cdots \arrow[l] & \mathbf{Z}[C_p] \arrow[l, "\overline{\gamma}_{2n}"'] \arrow[u, bend right, "\nabla"']
\end{tikzcd} \tag{3.2.12}$$

where

$$\overline{\gamma}_{2i} = 1 - \gamma \quad \textit{and} \quad \overline{\gamma}_{2i+1} = \sum_{k=0}^{p-1} \gamma^k. \tag{3.2.13}$$

The homology Mackey functors are

$$\begin{array}{cccccc}
0 & 1 & 2 & 3 & & 2n \\
\blacktriangledown & 0 & \blacktriangledown & 0 & & \underline{H}_{2n} \\
\begin{array}{c}\mathbf{Z}/p\\ \downarrow\uparrow\\ 0\end{array} & \begin{array}{c}0\\ \downarrow\uparrow\\ 0\end{array} & \begin{array}{c}\mathbf{Z}/p\\ \downarrow\uparrow\\ 0\end{array} & \begin{array}{c}0\\ \downarrow\uparrow\\ 0\end{array} & \begin{array}{c}\cdots\\ \\ \cdots\end{array} & \begin{array}{c}\underline{H}_n(G/G)\\ {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla}\\ \mathbf{Z}[G]/({}_{2n})\end{array}
\end{array} \tag{3.2.14}$$

where

$$\underline{H}_{-n}(G/G) = \begin{cases} \mathbf{Z} & \textit{for } n \textit{ even} \\ 0 & \textit{for } n \geqslant 1 \textit{ and } n \textit{ odd} \end{cases} \quad \textit{and} \quad \underline{H}_{-n} = \begin{cases} \square & \textit{for } n \textit{ even} \\ \overline{\square} & \textit{for } n \geqslant 1 \textit{ and odd} \end{cases}$$

For negative n*, the chain complex of* $S^{n\overline{\lambda}}$ *has the form*

$$\begin{tikzcd}[row sep=small]
0 & -1 & -2 & -3 & & -2n \\
\mathbf{Z} \arrow[r, "1"] \arrow[d, bend right, "1"'] & \mathbf{Z} \arrow[r, "0"] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[r, "p"] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[r, "0"] \arrow[d, bend right, "\Delta"'] & \cdots \arrow[r, "0"] & \mathbf{Z} \arrow[d, bend right, "\Delta"'] \\
\mathbf{Z} \arrow[r, "\nabla"] \arrow[u, bend right, "p"'] & \mathbf{Z}[C_p] \arrow[r, "\overline{\gamma}_{-2}"] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[C_p] \arrow[r, "\overline{\gamma}_{-3}"] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[C_p] \arrow[r, "\overline{\gamma}_{-4}"] \arrow[u, bend right, "\nabla"'] & \cdots \arrow[r, "\overline{\gamma}_{-2n}"] & \mathbf{Z}[C_p] \arrow[u, bend right, "\nabla"']
\end{tikzcd} \tag{3.2.15}$$

where

$$\overline{\gamma}_{-2i} = 1 - \gamma \quad \textit{and} \quad \overline{\gamma}_{-2i+1} = \sum_{k=0}^{p-1} \gamma^k. \tag{3.2.16}$$

Passing to homology we get

$$\begin{tikzcd}[row sep=small]
0 & -1 & -2 & -3 & & -2n \\
0 & 0 & 0 & \blacktriangledown & & \text{◩} \\
0 \arrow[d, bend right] & 0 \arrow[d, bend right] & 0 \arrow[d, bend right] & \mathbf{Z}/p \arrow[d, bend right] & \cdots & \mathbf{Z} \arrow[d, bend right, "p"'] \\
0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & \cdots & \mathbf{Z} \arrow[u, bend right, "1"']
\end{tikzcd} \tag{3.2.17}$$

If H is a subgroup of G, then a G-representation sphere is an H-representation sphere once applied the forgetful functor i_H^G. The isomorphism (3.2.2) establishes a way with which partial information of the G-Mackey functor homology of G-representation spheres can be inferred from the H-Mackey functor homology of H-

representation spheres. But we need to investigate the cellular structure carefully in our case where $G = D_{2p}$, since D_{2p} is not a commutative group.

The isomorphism (3.2.2) implies that the information given by C_2-Mackey functor homotopy and C_p-Mackey functor homotopy of $H\underline{\mathbf{Z}}$ as a C_2-spectrum and a C_p-spectrum will determine the G-Mackey functor $\underline{H}_*(S^V;\underline{\mathbf{Z}})$ to a large extent, and actually, completely.

For the reader's convenience, we repost Figure 2.1 on next page.

3.3 Mackey Functor Homology of $G_+ \wedge_H i_H^G S^V$

To prepare for further computation with spectral sequences, we need the Mackey functor homology of $G_+ \underset{C_2}{\wedge} S^{n\sigma}$ and of $G_+ \underset{C_p}{\wedge} S^{m\lambda}$ for all integers n and m. The reader can compare with (3.2.6), (3.2.9), (3.2.12) and (3.2.15) for the results.

The case when $H = C_2$

Proposition 3.3.1. *The chain complex of $G_+ \underset{C_2}{\wedge} S^{n\sigma}$ is given by the tensor product of the chain complexes of ((3.3.2)) or ((3.3.4)) with $\mathbf{Z}[G]$ over $Z[C_p]$, depending on whether m is positive or negative.*

Proof. $G_+ \underset{C_2}{\wedge} (-)$ is the left adjoint of the forgetful functor $i_{C_2}^G(-)$. The chain complex functor passes the smash product of pointed G-spaces to the tensor product of graded G-chain complexes. In this case,the graded chain complex of G/C_2 is $\mathbf{Z}[G/C_2]$, concentrated in degree 0. It is also known that for an H-module M

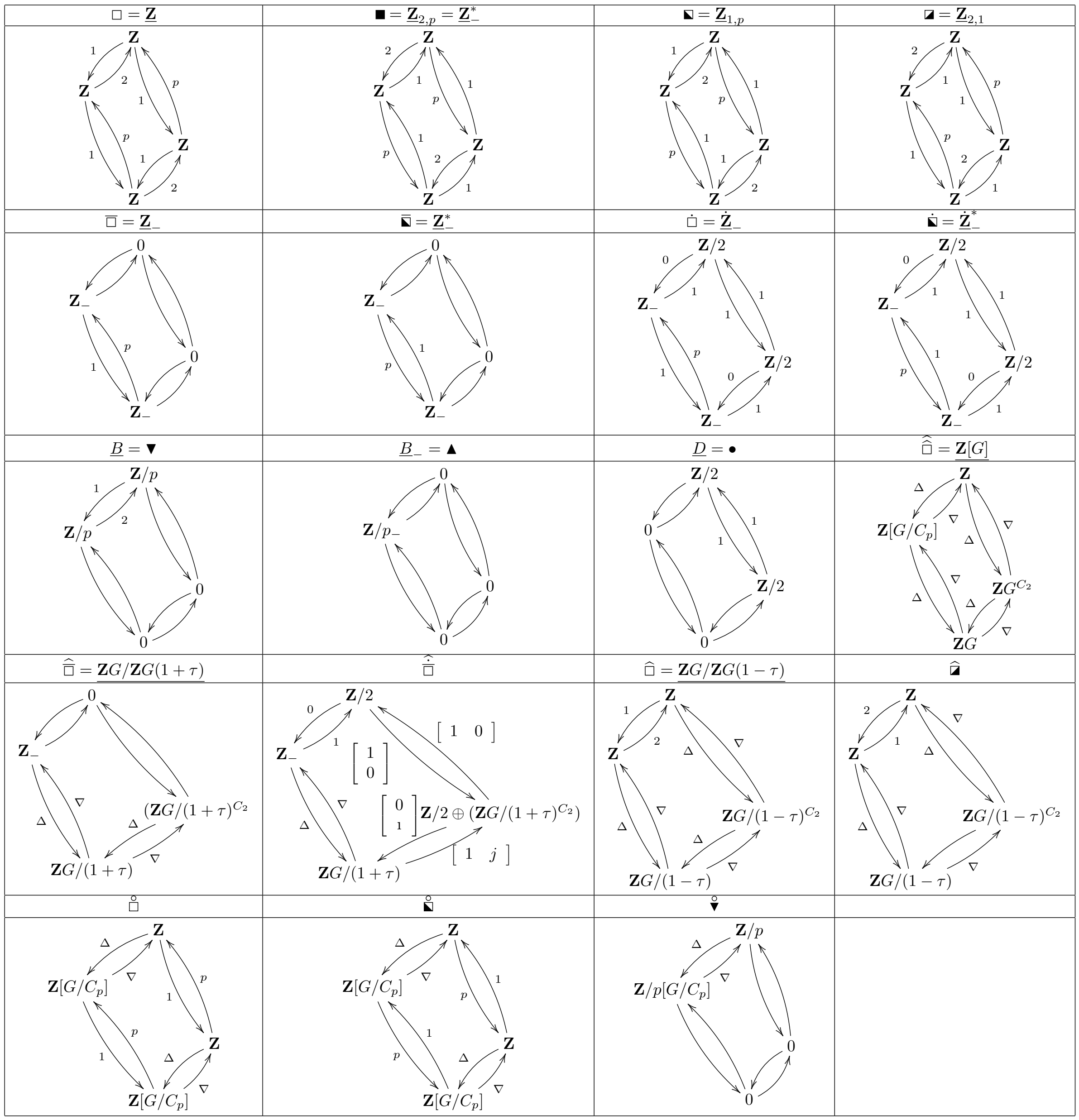


Figure 3.1: Figures for D_{2p} Mackey Functors

where H is a subgroup of G, there is a natural isomorphism

$$M \otimes_{\mathbf{Z}} \mathbf{Z}[G/H] \cong M \otimes_{\mathbf{Z}[H]} \mathbf{Z}[G]$$

of $\mathbf{Z}$-modules. Taking H to be C_2 and taking M to be the graded C_2-chain complex of $S^{n\sigma}$ gives us the statement. $\square$

Therefore we have the chain complex for $G_+ \wedge_{C_2} S^{b\sigma}$:

$$\begin{array}{ccccccccccc} 0 & & 1 & & 2 & & 3 & & & & b \\ \widehat{\square} & \xleftarrow{\nabla} & \widehat{\widehat{\square}} & \xleftarrow{\tau_2} & \widehat{\widehat{\square}} & \xleftarrow{\tau_3} & \widehat{\widehat{\square}} & \xleftarrow{\tau_4} & \cdots & \xleftarrow{\tau_b} & \widehat{\widehat{\square}} \end{array} \tag{3.3.2}$$

for a positive integer n, where

$$\tau_{2i}(x) = x \cdot (1 - \tau) \quad and \quad \tau_{2i+1}(x) = x \cdot (1 + \tau) \tag{3.3.3}$$

for $x \in G_+ \wedge_{C_2} C^{b\sigma}(G/e)$ with the corresponding degrees.

Passing to homology we obtain

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & b \\ \bullet & 0 & \bullet & 0 & & Ker(\tau_n) \end{array}$$

$$Ker(\tau_n) = \begin{cases} \widehat{\overline{\square}} & \text{for } n \text{ odd} \\ \widehat{\square} & \text{for } n \text{ even} \end{cases}$$

For negative n, the chain complex is

$$
\begin{array}{ccccccccccc}
0 & & -1 & & -2 & & -3 & & & & -n \\
\widehat{\square} & \xrightarrow{\Delta} & \widehat{\widehat{\square}} & \xrightarrow{\tau_{-2}} & \widehat{\widehat{\square}} & \xrightarrow{\tau_{-3}} & \widehat{\widehat{\square}} & \xrightarrow{\tau_{-4}} & \cdots & \xrightarrow{\tau_{-n}} & \widehat{\widehat{\square}}
\end{array}
\tag{3.3.4}
$$

The maps τ_i are defined in (3.3.3). By similar argument we get the homology

$$
\begin{array}{cccccccc}
0 & -1 & -2 & -3 & -4 & -5 & & n \\
0 & 0 & 0 & \bullet & 0 & \bullet & \cdots & Coker(\tau_n)
\end{array}
$$

$$
Coker(\tau_n) = \begin{cases} \widehat{\overline{\square}} & \text{for } n = -1 \\ \widehat{\dot{\overline{\square}}} & \text{for } n \leqslant -3 \text{ and odd} \\ \widehat{\squareurblack} & \text{for } n \text{ even} \end{cases}
$$

Therefore we reach the conclusion:

Theorem 3.3.5. *The Mackey functor homology groups of* $G_+ \underset{C_2}{\wedge} S^{n\sigma}$ *are:*
When n *is nonnegative:*

$$
\underline{H}_k = \begin{cases} \widehat{\overline{\square}} & \textit{for } k = n \textit{ and odd} \\ \widehat{\square} & \textit{for } k = n \textit{ and even} \\ \bullet & \textit{for } k \textit{ even and } 0 \leqslant k < n \\ 0 & \textit{else} \end{cases}
$$

When n is negative:

$$\underline{H}_k = \begin{cases} \widehat{\overline{\square}} & \textit{for } k = n = -1 \\ \widehat{\dot{\overline{\square}}} & \textit{for } k = n \leqslant -3 \textit{ and odd} \\ \widehat{\text{◪}} & \textit{for } k = n \textit{ and even} \\ \bullet & \textit{for } n < k \leqslant -3 \textit{ and odd} \\ 0 & \textit{else.} \end{cases}$$

The case when $H = C_p$

For $H = C_p$, we have the following parallel lemmas and theorems:

Proposition 3.3.6. *The chain complex of $G_+ \underset{C_p}{\wedge} S^{m\overline{\lambda}}$ is given by the tensor product of the chain complexes of* (3.3.7) *or* (3.3.9) *with $\mathbf{Z}[G]$ as C_p-modules, depending whether m is positive or negative.*

The chain complex $G_+ \wedge_{C_p} C^{m\overline{\lambda}}$ is

$$\begin{array}{ccccccccc} 0 & & 1 & & 2 & & 3 & & 2m \\ \overset{\circ}{\square} & \xleftarrow{\nabla} & \widehat{\overline{\square}} & \xleftarrow{\gamma_2} & \widehat{\overline{\square}} & \xleftarrow{\gamma_3} & \widehat{\overline{\square}} & \xleftarrow{\gamma_4} \cdots \xleftarrow{\gamma_{2m}} & \widehat{\overline{\square}} \end{array} \tag{3.3.7}$$

for a positive integer m. The boundary maps for $G_+ \wedge_{C_p} C^{m\overline{\lambda}}$

$$\gamma_{2i} = (1 - \gamma) \quad \textit{and} \quad \gamma_{2i+1} = \sum_{k=0}^{p-1} \gamma^k \tag{3.3.8}$$

The Mackey functors homology is

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & 2m-1 & 2m \\ \overset{\circ}{\blacktriangledown} & 0 & \overset{\circ}{\blacktriangledown} & 0 & & 0 & \overset{\circ}{\square} \end{array}$$

For negative m, the chain complex is

$$
\begin{array}{ccccccccc}
0 & & -1 & & -2 & & -3 & & & & -2m \\
\widehat{\square} & \xrightarrow{\Delta} & \widehat{\widehat{\square}} & \xrightarrow{\gamma_{-2}} & \widehat{\widehat{\square}} & \xrightarrow{\gamma_{-3}} & \widehat{\widehat{\square}} & \xrightarrow{\gamma_{-4}} & \cdots & \xrightarrow{\gamma_{-2m}} & \widehat{\widehat{\square}}
\end{array}
\tag{3.3.9}
$$

with γ_i defined in (3.3.8).

The Mackey functor homology is

$$
\begin{array}{cccccccccc}
0 & -1 & -2 & -3 & -4 & & 2-2m & 1-2m & -2m \\
0 & 0 & 0 & \mathring{\blacktriangledown} & 0 & \cdots & 0 & \mathring{\blacktriangledown} & \mathring{\boxbslash}
\end{array}
$$

Therefore we have the conclusion:

Theorem 3.3.10. *The Mackey functor homology groups of $G_+ \underset{C_p}{\wedge} S^{m\overline{\lambda}}$ are:*
When m is nonnegative:

$$
\underline{H}_k = \begin{cases} \mathring{\square} & \textit{for } k{=}2m \\ \mathring{\blacktriangledown} & \textit{for } k = 2l \textit{ where } 0 \leqslant l < m \\ 0 & \textit{else.} \end{cases}
$$

When m is negative:

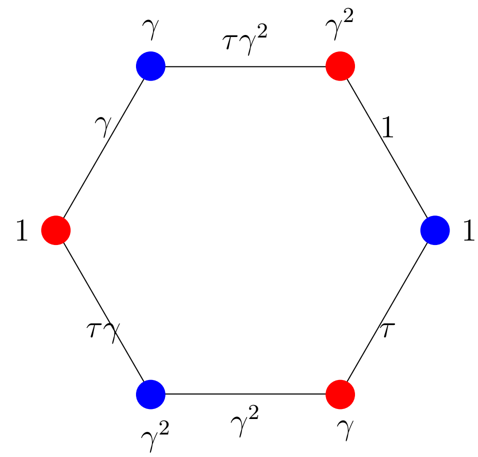
$$
\underline{H}_k = \begin{cases} \mathring{\boxbslash} & \textit{for } k = 2m \\ \mathring{\blacktriangledown} & \textit{for } k = 2j-1 \textit{ where } m \leqslant j < 0. \\ 0 & \textit{else.} \end{cases}
$$

3.4 Computation for Actual Representations

3.4.A Computing $\underline{H}_\star S^{m\lambda}$

We start with the basic case: $V = \lambda$. For simplicity we give the example of S^λ for $p = 3$ i.e. $G = S_3$. The skeleton of S^λ for $p = 3$ is shown in Figure 3.2: The picture above is the equator of S^λ. Each vertex represents an 1-dimension

Figure 3.2: Cellular structure for S^λ



cell, and each side a two dimension cell. We can see the 1-cells of each color are indexed by G/C_2. The 2-cells are indexed by G.

Figure 3.2 displays the equator of S^λ. Each vertex represents an 1-dimensional cell, and each side represents a 2-dimensional cell. We can see the 1-cells of each color are indexed by G/C_2. The 2-cells are indexed by G. The chain complex used to compute its Mackey functor homology is

$$\underline{\mathbf{Z}} \xleftarrow{\ \nabla\ } 2\underline{\mathbf{Z}}[G]/(1-\tau) \xleftarrow{\begin{pmatrix}-\gamma\\ 1\end{pmatrix}} \underline{\mathbf{Z}}[G] \tag{3.4.1}$$

Here $\underline{\mathbf{Z}}[G]/(1-\tau)$ and $\underline{\mathbf{Z}}[G]/(1+\tau)$ are abbreviation for the left $\underline{\mathbf{Z}}[G]$-modules $\underline{\mathbf{Z}}[G]/\underline{\mathbf{Z}}[G](1-\tau)$ and $\underline{\mathbf{Z}}[G]/\underline{\mathbf{Z}}[G](1+\tau)$. ∇ denotes the augmentation map from $\underline{\mathbf{Z}}[G]/(1-\tau)(G/e)$ to $\underline{\mathbf{Z}}(G/e)$. These maps are between fixed point Mackey functors from which we can deduce the result for each G/H.

Iteratively tensoring the chain complex (3.4.1) with itself will give the chain complex $C^{m\lambda}$. However, the result is much too complicated, since graded tensor products are hard to write down, and the differentials are not easy to compute. So we would like to apply the method of equivariant Serre spectral sequence of Mackey functors.

$$\begin{tikzcd}
S^0 \arrow[r] & \overline{S^\lambda} \arrow[r] \arrow[d] & S^\lambda \arrow[d] \\
 & 2G_+ \underset{C_2}{\wedge} S^1 & G_+ \wedge S^2
\end{tikzcd}$$

Every horizontal arrow followed by a vertical is a cofibration. Therefore for a representation V, we smash every space with S^V to get the filtration

$$\begin{tikzcd}
S^V \arrow[r] \arrow[dd, equal] & S^V \wedge \overline{S^\lambda} \arrow[r] \arrow[d] & S^{\lambda+V} \arrow[d] \\
 & 2G_+ \underset{C_2}{\wedge} S^{1+i^G_{C_2}(V)} \arrow[d, equal] & G_+ \wedge S^{2+|V|} \arrow[d, equal] \\
X_0 & X_1 & X_2
\end{tikzcd}$$

Then we have the equivariant Serre spectral sequence of Mackey functors with $E^1_{s,t-s} = \underline{H}_s(X_t)$ that converges to $\underline{H}_s S^{\lambda+V}$.

Specifically, iteratively smashing S^λ with the filtration (3.4.A) gives us a filtration for $S^{m\lambda}$.

$$\begin{tikzcd}
S^0 \arrow[r] \arrow[dd, equal] & \overline{S^\lambda} \arrow[r] \arrow[d] & S^\lambda \arrow[r] \arrow[d] & \cdots \arrow[r] & S^{(m-1)\lambda} \wedge \overline{S^\lambda} \arrow[r] \arrow[d] & S^{m\lambda} \arrow[d] \\
 & 2G_+ \underset{C_2}{\wedge} S^1 \arrow[d, equal] & G_+ \wedge S^2 \arrow[d, equal] & & 2G_+ \wedge_{C_2} S^{m+(m-1)\sigma} \arrow[d, equal] & G_+ \wedge S^{2m} \arrow[d, equal] \\
X_0 & X_1 & X_2 & & X_{2m-1} & X_{2m}
\end{tikzcd} \tag{3.4.2}$$

The filtration (3.4.2) will lead to a first quadrant spectral sequence of $\underline{\mathbf{Z}}G$-modules with E^1 page

$$E^1_{s,t-s} = \underline{H}_{t-s}(X_t).$$

and the spectral sequence will converge to $\underline{H}_{t-s}(S^{m\lambda})$. The homology of all the cofibers X_i is computed in §3.3.

We can see from the Figure 3.3 that d_1 is only allowed in the 0-th row $E^1_{0,t-s}$. To determine d_1 we should mention that the first row is a chain complex of left $\underline{\mathbf{Z}}G$-modules. Furthermore, if we focus on the (G/e) values, we obtain a spectral sequence which converges to $H_*(S^{2m})$. Since all terms above the 0-th row have trivial (G/e) values, they all vanish in the spectral sequence $\underline{E}_{s,t-s}(G/E)$. Therefore, the 0-th row gives us a chain complex of $\tilde{H}_* S^{2m}$, which has only one nonzero term. All Mackey functors appearing in the 0-th row are fixed point Mackey functors. We will show later that it is sufficient to compute all d_1's.

Proposition 3.4.3.

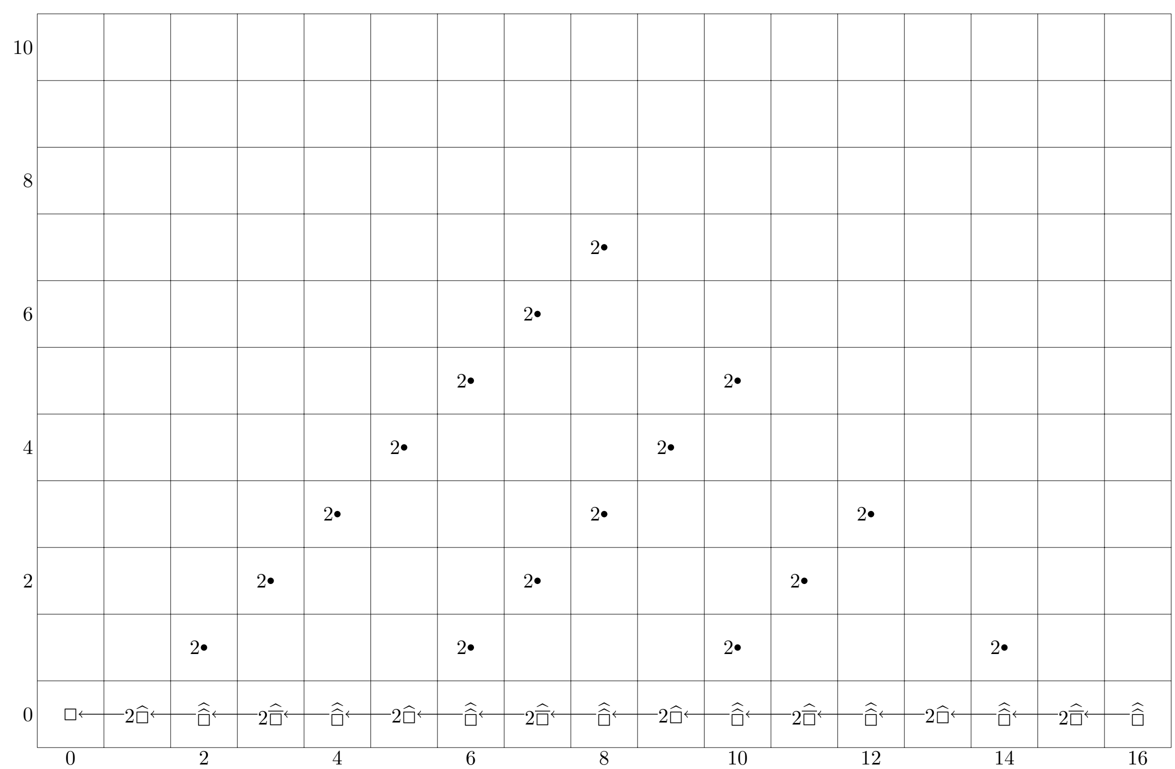
$$d_1 : E^1_{0,3} = 2\widehat{\widehat{\square}} \to E^1_{0,2} = \widehat{\widehat{\square}}$$

maps both generators of $\widehat{\widehat{\square}}(G/e) = \mathbf{Z}G/(1+\tau)$ *to* $(1-\tau)(\sum_{k=0}^{p-1} \gamma^k)$

Proof. $\underline{H}_2(S^\lambda)$ is the fixed point Mackey functor $\mathbf{Z}_-$ generated by $(1-\tau)(\sum_{k=0}^{p-1}\gamma^k)$. But $\underline{H}_2(S^{2\lambda})(G/e) = 0$. This forces d_2 to map the generators of both $\widehat{\overline{\Box}}(G/e) \cong \mathbf{Z}G/\mathbf{Z}G(1+\tau)$ to $(1-\tau)(\sum_{k=0}^{p-1}\gamma^k)$. It is easy to check that it is indeed a morphism of $\mathbf{Z}G$-modules. $\Box$

$$\Box \xleftarrow{\nabla} 2\widehat{\Box} \xleftarrow{\begin{pmatrix} -\gamma \\ 1 \end{pmatrix}} \widehat{\widehat{\Box}} \xleftarrow{\alpha} 2\widehat{\overline{\Box}} \xleftarrow{\begin{pmatrix} -\gamma \\ 1 \end{pmatrix}} \widehat{\widehat{\Box}}$$

Figure 3.3: E^1 page of the spectral sequence associated with the filtration (3.4.2) for $m = 8$



Proposition 3.4.4.

$$d_1 : E^1_{0,4} = \widehat{\widehat{\square}} \to E^1_{0,3} = 2\widehat{\widehat{\square}}$$

maps the generator of $\widehat{\widehat{\square}}(G/e) \cong \mathbf{Z}G$ *to* $(-\gamma, 1)$.

Proof. The differential is completely determined by the boundary map from the 2-cell to the 1-cells of S^λ because we construct the filtration of $S^{2\lambda}$ in the way in (3.4.2). So we use the same name for the differential from the 4-cell to the 3-cells. $\square$

Proposition 3.4.5.

$$d_1 : E^1_{0,5} = 2\widehat{\square} \to E^1_{0,4} = \widehat{\widehat{\square}}$$

maps both generators of $\widehat{\square}(G/e) \cong \mathbf{Z}G/(1-\tau)$ *to* $(1+\tau)(\sum_{k=0}^{p-1}\gamma^k)$.

Proof. $\underline{H}_4(S^{2\lambda})(G/e)$ is generated by $(1+\tau)(\sum_{k=0}^{p-1}\gamma^k)$.

From the fact that $\underline{H}_4(S^{3\lambda}) = 0$ we see $Im(d_1(E^1_{0,5})) = \underline{H}_4(S^{2\lambda})(G/e)$. $\square$

Proposition 3.4.5 tells us that the 0-th row is 4-periodic. Each d_1 on the right of $E^1_{1,5}$ has the same behavior as the d_1 4 units to the left.

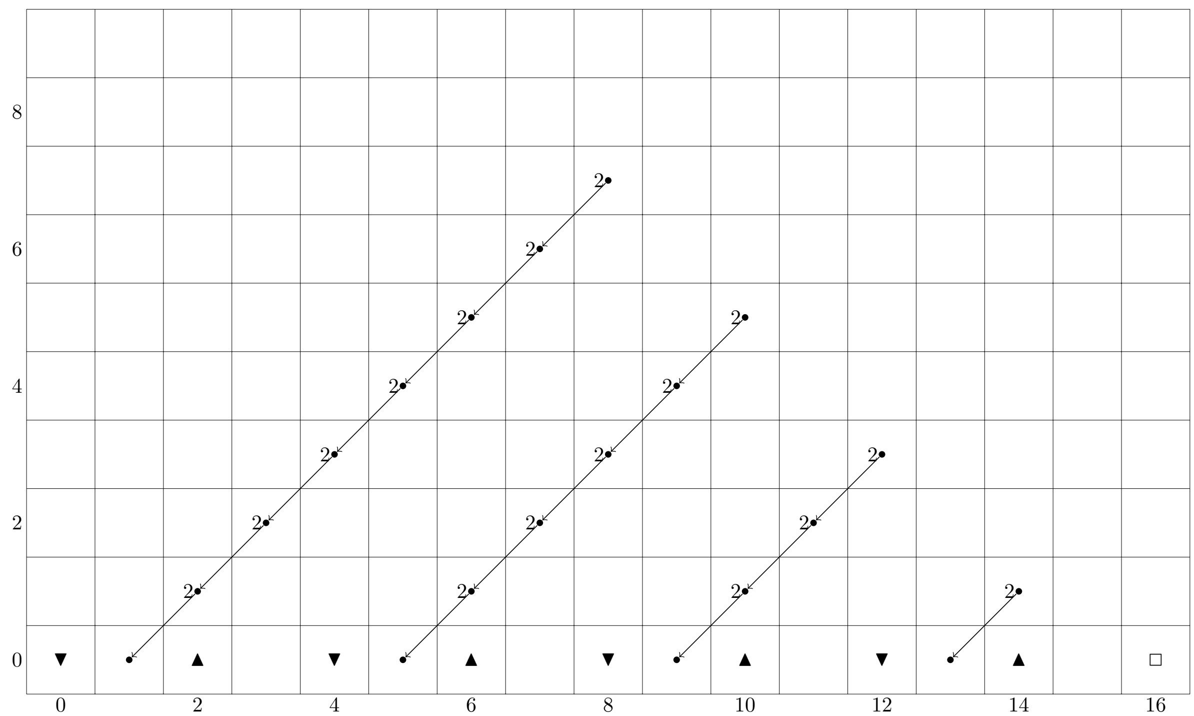
Proposition 3.4.6. *All* $d_1 : E^1_{0,k+5} \to E^1_{0,k+4}$ *are the same as* $d_1 : E^1_{0,k+1} \to E^1_{0,k}$ *for* $k > 0$.

Proof. $\underline{H}_k S^{m\lambda}(G/e) = 0$ for $0 < k < 2m$. So the differentials are completely determined by the maps on (G/e)-values. $\square$

Thus we have the E^1 page in Figure 3.3.

From E^2-page it is enough to know $\underline{H}_{2m}(S^{m\lambda})$

Figure 3.4: E^2 page of the spectral sequence associated to the filtration (3.4.2) for $m = 8$



Theorem 3.4.7. $\underline{H}_{2m}(S^{m\lambda}) = \begin{cases} \square & \textit{for } m \textit{ even and positive} \\ \overline{\square} & \textit{for } m \textit{ odd and positive} \end{cases}$

The differential d_2 is also easy to track.

Proposition 3.4.8.

$$d_2 : E^2_{1,4j+2} \to E^2_{0,4j+1}$$

is ∇ *for* $j \geqslant 0$ *whenever both terms are nontrivial.*

Proof. We restrict to the case $c = 2$.(3.2.2) tells us there is only one copy of $\mathbf{Z}/2$ for the (G/C_p) values, which is $\underline{H}_2(S^{2\lambda})(G/C_p)$. This implies that d_2 kills the $\bullet$

at $(1,2)$ and the kernel of d_2 is also a $\bullet$. Since the two copies of $\bullet$ at $(1,2)$ are given by the two copies of $G_+ \underset{C_2}{\wedge} (S^{2+\sigma})$, by symmetry it has to be ∇. $\square$

Proposition 3.4.9.

$$d_2 : E^2_{2,4j+3} \to E^2_{1,4j+2}$$

is Δ on both summands for $j \geqslant 0$.

Proof. Similar to Proposition 3.4.8 we restrict to the case $m = 3$. By applying the forgetful functor $i^G_{C_2}$ we get $S^{3+3\sigma}$. Hence we can see that when there is a $\bullet$ in the homology groups, there can be no more than one direct summand. So d_2 is nontrivial. $\square$

We have the following corollary with similar argument:

Corollary 3.4.10.

$$d_2 : E^2_{2m+1,4j+2m+2} \to E^2_{2m,4j+2m+1}$$

is ∇ on both summands for $c > 0$.

$$d_2 : E^2_{2m+2,4j+2m+3} \to E^2_{2m+1,4j+2m+2}$$

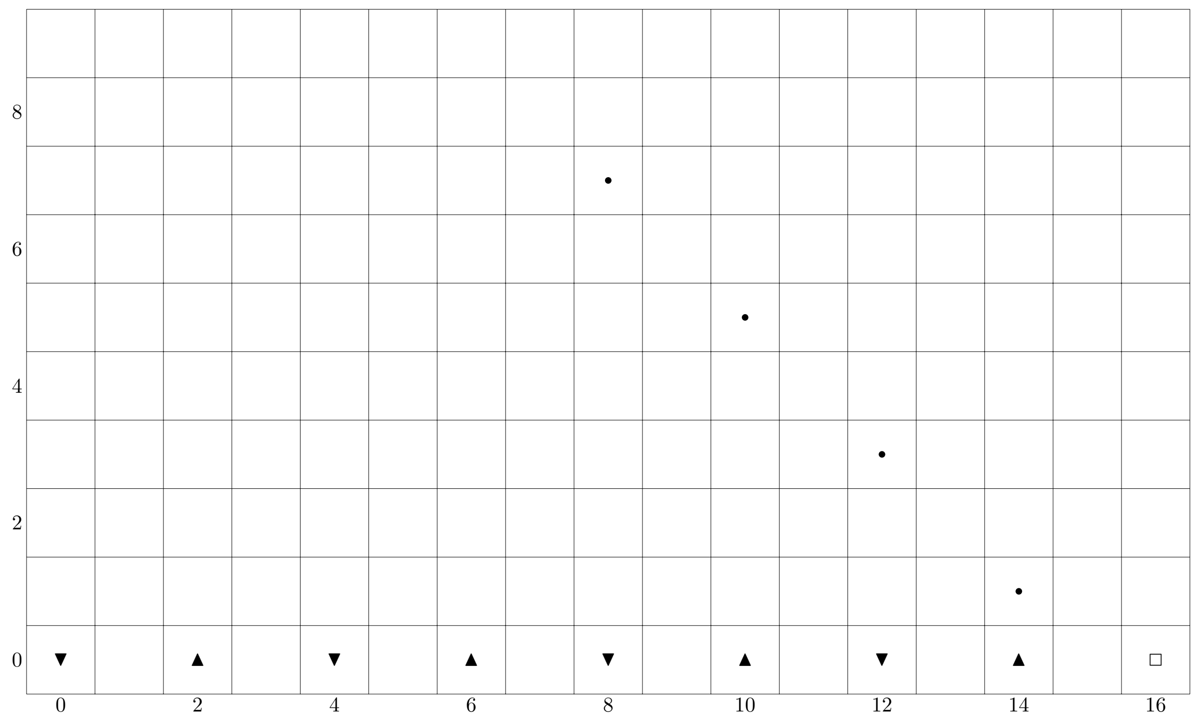
is Δ on both summands for $c > 0$ and $j \geqslant 0$.

Therefore we can see the E^3 page.

It is easy to see that there is no room for d_j for $j \geq 2$. So $E^3 = E^\infty$

Luckily, we don't need to worry about extension problems.

Figure 3.5: E^3 page of the spectral sequence associated to the filtration (3.4.2) for $m = 8$



Proposition 3.4.11. *There is no extension problem for Mackey functor homology groups of $\underline{H}_*(S^{m\lambda})$ when m is positive.*

Proof. From (3.2.2) we know that the only undetermined terms of Mackey functor homology groups are the (G/G) values. But the E^3 page has given the information that is needed. $\square$

3.4.B Computing $\underline{H}_\star S^{b\sigma}$

The computation for $C^{b\sigma}$ is easier. Because $i^G_{C_p} S^{b\sigma} \cong S^n$ as C_P spectra, we can take the computation in (3.2.6) with replacement of Mackey functor notations.

$$C^{b\sigma}_k = \begin{cases} \underline{\mathbf{Z}} & \text{for } n = 0 \\ \underline{\mathbf{Z}}[G/C_p] & \text{for } 0 < n \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.12}$$

The boundary maps are determined by:

$$\partial_k = \begin{cases} \nabla & \text{for } n = 1 \\ 1 - \tau & \text{for } k \text{ even and } 2 \leq k \leq n \\ 1 + \tau & \text{for } k \text{ odd and } 2 \leq k \leq n \\ 0 & \text{otherwise,} \end{cases}$$

The homology Mackey functors are

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & & n \\ \bullet & 0 & \bullet & 0 & \cdots & \underline{H}_n \end{array} \tag{3.4.13}$$

where

$$\underline{H}_n(G/G) = \begin{cases} \mathbf{Z} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases} \quad \text{and} \quad \underline{H}_n = \begin{cases} \square & \text{for } n \text{ even} \\ \overline{\square} & \text{for } n \text{ odd} \end{cases}$$

3.4.C Computing $\underline{H}_\star S^{-V}$

Computing $\underline{H}_\star S^{-m\lambda}$ for $m > 0$

We start from $m = 1$. There is a cellular filtration for $S^{-\lambda}$:

We have the following filtration for $S^{-\lambda}$:

$$\begin{tikzcd}
S^{-\lambda} \arrow[rr] & & \overline{S^{\lambda}} \wedge S^{-\lambda} \arrow[r] & S^0 \arrow[dd, equal] \\
2G_+ \underset{C_2}{\wedge} S^{-1-\sigma} \arrow[u] \arrow[d, equal] & & G_+ \wedge S^{-1} \arrow[u] \arrow[d, equal] & \\
X_{-2} & & X_{-1} & X_0
\end{tikzcd} \tag{3.4.14}$$

$\underline{H}_{-1} G_+ \wedge S^{-1} = \underline{\mathbf{Z}}G = \widehat{\widehat{\square}}$ and its other Mackey functor homology vanishes. $\underline{H}_{-2} G_+ \underset{C_2}{\wedge} S^{-1-\sigma} = \widehat{\widehat{\square}}$ and its other Mackey functor homology also vanishes.

Thus $\underline{H}_* S^{-\lambda}$ can be computed via the chain complex (which is the dual of (3.4.1)):

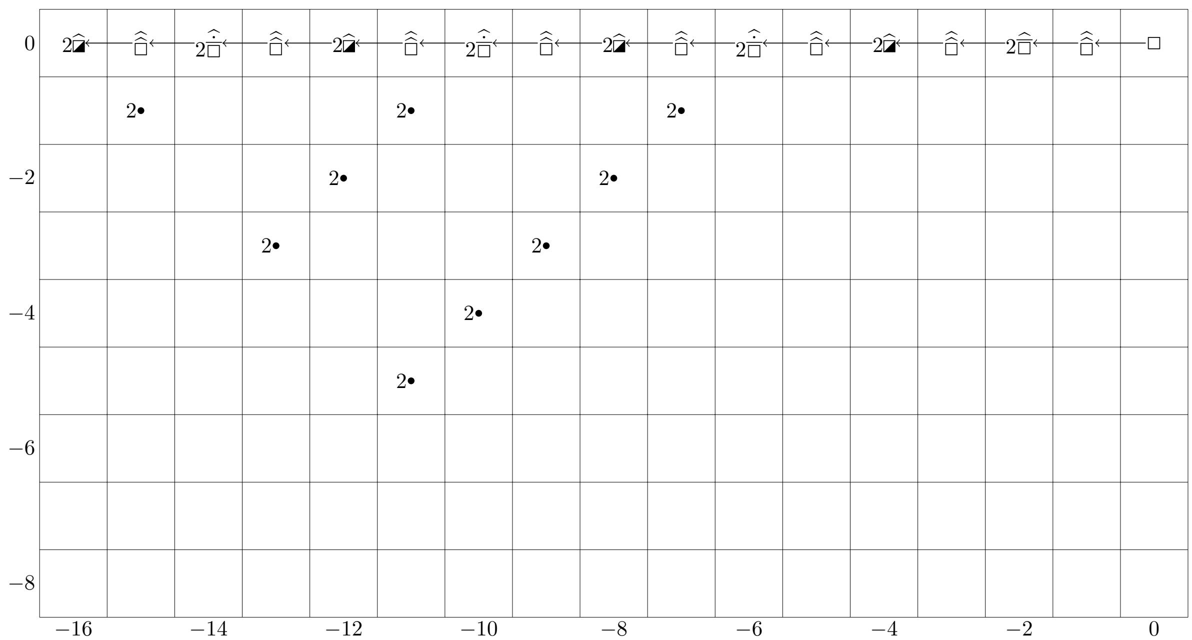
$$2\underline{\mathbf{Z}}[G]/(1+\tau) \xleftarrow{\begin{pmatrix} -\gamma \\ 1 \end{pmatrix}} \underline{\mathbf{Z}}[G] \xleftarrow{\Delta} \underline{\mathbf{Z}} \tag{3.4.15}$$

We have the following filtration for $S^{-m\lambda}$:

$$\begin{tikzcd}
S^{-m\lambda} \arrow[r] & \overline{S^{\lambda}} \wedge S^{-m\lambda} \arrow[r] & \cdots \arrow[r] & S^{-\lambda} \arrow[r] & S^{-\lambda} \wedge \overline{S^{\lambda}} \arrow[r] & S^0 \arrow[ddd, equal] \\
2G_+ \underset{C_2}{\wedge} S^{-m(1+\sigma)} \arrow[u] \arrow[dd, equal] & & & 2G_+ \underset{C_2}{\wedge} S^{-1-\sigma} \arrow[u] \arrow[dd, equal] & & \\
 & G_+ \wedge S^{1-2m} \arrow[uu] \arrow[d, equal] & & & G_+ \wedge S^{-1} \arrow[uu] \arrow[d, equal] & \\
X_{-2m} & X_{-2m+1} & & X_{-2} & X_{-1} & X_0
\end{tikzcd}$$

(3.4.16)

Figure 3.6: E^1 page of the spectral sequence associated to the filtration (3.4.16) for $m = 8$.



Every horizontal arrow followed by a vertical is a fibration.

There is a fourth quadrant spectral sequence with E^1-page

$$E^1_{s,t-s} = \underline{H}_{t-s}(X_t) \Rightarrow \underline{H}_{t-s}(S^{-m\lambda}).$$

The picture is shown in Figure 3.6.

The differentials are dual to the differentials of those shown in Figure 3.3 and Figure 3.4. Therefore we have the following dual propositions:

Proposition 3.4.17.

$$d_1 : E^1_{0,0} = \square \to E^1_{0,-1} = 2\widehat{\widehat{\square}}$$

is Δ

Proposition 3.4.18.

$$d_1 : E^1_{0,-1} = \widehat{\widehat{\square}} \to E^1_{0,-2} = 2\,\widehat{\overline{\square}}$$

maps the generator of $E^1_{0,-1}(G/e) = \underline{\mathbf{Z}}G(G/e) = \mathbf{Z}G$ *to* $(-\gamma, 1)$.

Proof of both Proposition 3.4.17 and Proposition 3.4.18. The chain complex

$$E^1_{0,0} \leftarrow E^1_{0,0} \leftarrow E^1_{0,0}$$

is the desuspension of the chain complex of S^λ in (3.4.15). So the two differentials are determined by the corresponding boundary maps. $\square$

Proposition 3.4.19.

$$d_1 : E^1_{0,-2} = 2\,\widehat{\overline{\square}} \to E^1_{0,-3} = \widehat{\widehat{\square}}$$

maps $(1,0)$ *to* $(1+\tau)\gamma^{-1}$ *and maps* $(0,1)$ *to* $1-\tau$.

Proof. Dual to Proposition 3.4.4 $\square$

Proposition 3.4.20.

$$d_1 : E^1_{0,-3} = \widehat{\widehat{\square}} \to E^1_{0,-4} = 2\,\widehat{\text{◪}}$$

maps the generator of $\underline{\mathbf{Z}}G(G/e)$ *to* $(\sum_{k=0}^{p-1}\gamma^k, \sum_{k=0}^{p-1}\gamma^k)$.

Proof. Dual to Proposition 3.4.5 $\square$

Proposition 3.4.21.

$$d_1 : E^1_{0,-4} = 2\widehat{◪} \to E^1_{0,-5} = \widehat{\widehat{\square}}$$

maps $(1,0) \in 2\underline{\mathbf{Z}}G/(1-\tau))(G/e)$ *to* $-\gamma^{-1}(1+\tau)$ *and maps (0,1) to* $1+\tau$*.*

Proposition 3.4.22.

$$d_1 : E^1_{0,-5} = \widehat{\widehat{\square}} \to E^1_{0,-6} = 2\,\widehat{\dot{\overline{\square}}}$$

maps the generator of $E^1_{0,-5}(G/e) = \underline{\mathbf{Z}}G(G/e) = \mathbf{Z}G$ *to* $(\sum_{k=0}^{p-1}\gamma^k, \sum_{k=0}^{p-1}\gamma^k)$

Proof. The evaluation on (G/e) values is the same as in Proposition 3.4.18. $\square$

Proposition 3.4.23. *Let* $k \leq -2$*. If* $E^1_{0,k-4}$ *and* $E^1_{0,k-5}$ *are both nontrivial, then* $d_1 : E^1_{0,k-4} \to E^1_{0,k-5}$ *is the same as* $d_1 : E^1_{0,k} \to E^1_{0,k-1}$*.*

Proof. Dual to Proposition 3.4.6. It is known that $E^1_{0,k-4}$ and $E^1_{0,k}$ are isomorphic if $-2m < k-4 < 0$. The chain complex is 4-periodic. $\square$

Proposition 3.4.17 to Proposition 3.4.21 gives E^2 page of the spectral sequence and $\underline{H}_{-2m}(S^{-m\lambda})$.

Theorem 3.4.24. $\underline{H}_{-2m}(S^{-m\lambda}) = \begin{cases} \blacksquare & \textit{for } m \textit{ even} \\ \overline{\square} & \textit{for } m = -1 \\ \dot{\overline{\square}} & \textit{for } m \textit{ odd and less than } -1 \end{cases}$

for a positive integer m*.*

All d_2's are determined by the d_2's from the 0th row to the -1st row by (3.2.2). This argument is similar to Proposition 3.4.8.

Figure 3.7: E^2 page of the spectral sequence associated to the filtration (3.4.16) for $m = 8$.

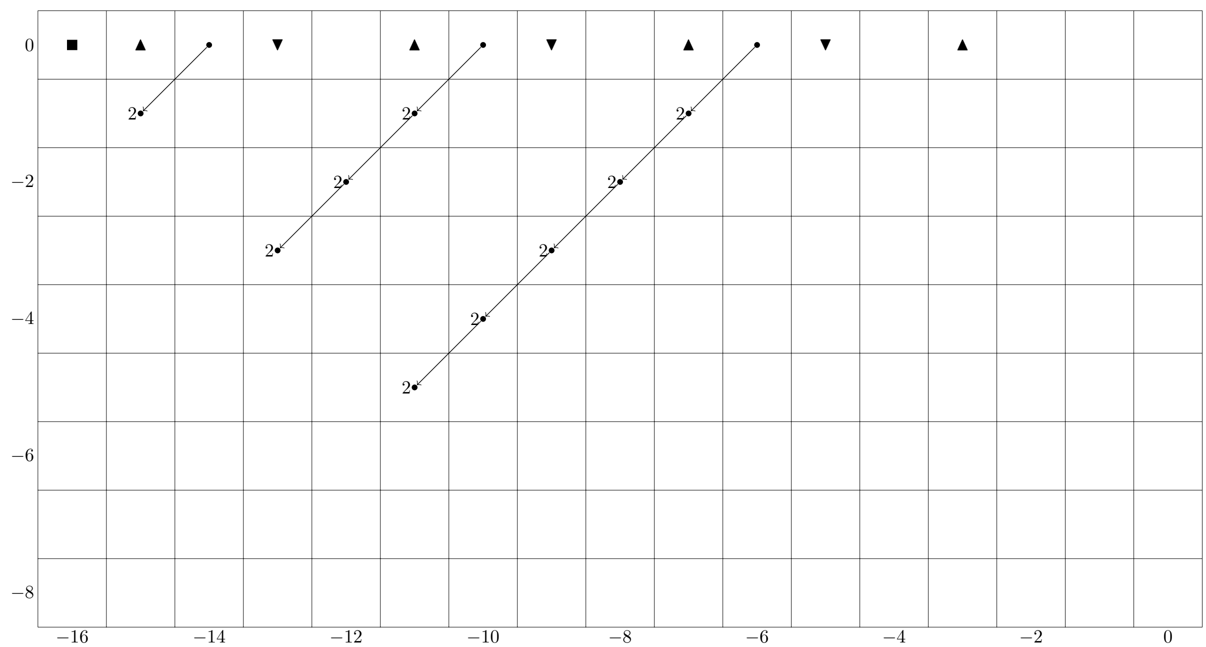
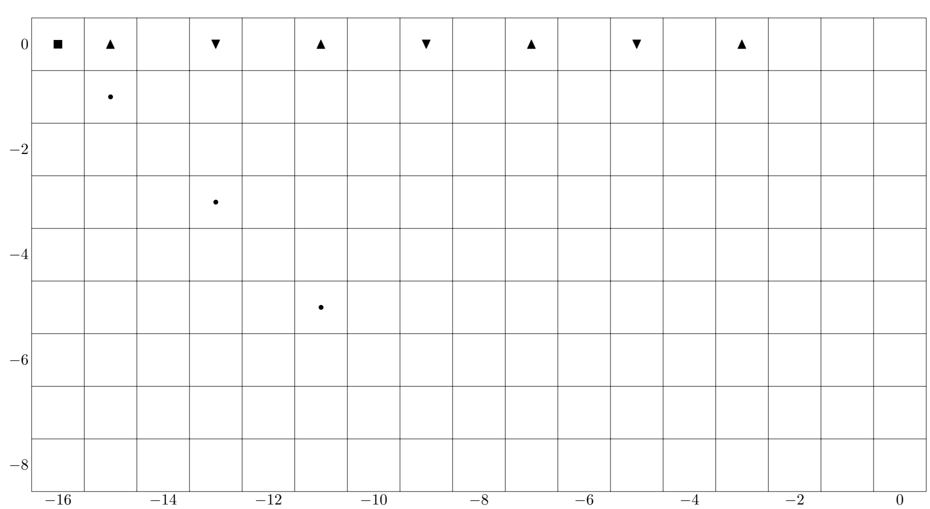


Figure 3.8: E^3 page of the spectral sequence associated to the filtration (3.4.2) for $m = 8$.



Proposition 3.4.25.

$$d_2 : E^2_{0,-5-4j} \to E^2_{-1,-6-4j}$$

is $\Delta : \bullet \to 2\bullet$ *and it is trivial for the summand* $\blacktriangle$*, for all* $j \geqslant 0$*.*

Proposition 3.4.26.

$$d_2 : E^2_{2,4j+3} \to E^2_{1,4j+2}$$

is ∇ *for* $j \geqslant 0$ *on both summands.*

Corollary 3.4.27.

$$d_2 : E^2_{2m+1,4j+2m+2} \to E^2_{2m,4j+2m+1}$$

is ∇ *on both summands for* $m > 0$ *and* $j \geqslant 0$*.*

$$d_2 : E^2_{2m+2,4j+2m+3} \to E^2_{2m+1,4j+2m+2}$$

is Δ *for* $m > 0$ *and* $j \geqslant 0$*.*

Now we have the E^3-page.

There is no room for higher differentials. So $E^3 = E^\infty$.

Computing $\underline{H}_\star S^{-b\sigma}$ for $n > 0$

The good part of $C^{-b\sigma}$ is that the computation can also be done with a simple chain complex, which is the dual of (3.3.2):

$$C_k^{-b\sigma} = \begin{cases} \underline{\mathbf{Z}} & \text{for } k = 0 \\ \underline{\mathbf{Z}}[G/C_p] & \text{for } -n \leq k \leq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.28}$$

The boundary maps are determined by:

$$\partial_k = \begin{cases} \nabla & \text{for } k = 0 \\ 1 - \tau & \text{for } k \text{ odd and } -n \leq k < 0 \\ 1 + \tau & \text{for } k \text{ even and } -n \leq k < -1 \\ 0 & \text{otherwise,} \end{cases}$$

so the homology is similar to (3.2.10)

$$\begin{array}{cccccc} 0 & -1 & -2 & -3 & & -n \\ 0 & 0 & 0 & \bullet & & \underline{H}_{-n} \end{array} \tag{3.4.29}$$

where

$$\underline{H}_{-n} = \begin{cases} \text{◪} & \text{for } n \text{ even} \\ \overline{\square} & \text{for } n = 1 \\ \dot{\square} & \text{for } n \geqslant 1 \text{ and odd} \end{cases}$$

3.5 Computation for Virtual Representations

The hard part of computation is the case that $V = a + b\sigma + m\lambda$, where n and m have opposite signs. It can be done with tensoring the corresponding graded chain complexes. However, the computation results show that we can actually infer $\underline{H}_i S^V(G/G)$, together with the structure maps Res_H^G and Tr_H^G as well, from $Res_{\{e\}}^H$ and $Tr_{\{e\}}^H$ for the two nontrivial proper subgroups $H = C_2$ and $H = C_p$. The inference is done with the properties of Mackey functors:

a. $Res_{\{e\}}^H Res_H^G = Res_{\{e\}}^G$

b. $Tr_{\{e\}}^H Tr_H^G = Tr_{\{e\}}^G$

c. $Tr_{\{e\}}^{C_p} Res_{\{e\}}^{C_2} = Res_{C_p}^G Tr_{C_2}^G$

d. $Res_{C_p}^G Tr_{C_2}^G = Tr_{\{e\}}^{C_p} Res_{\{e\}}^{C_2}$

e. $Tr_{\{e\}}^H Res_{\{e\}}^H = |H|$

f. $Tr_H^G Res_H^G = |G/H|$.

(a) to (d) are from the definition of Mackey functors, (e) and (f) are the properties of cohomological Mackey functors.

Below are a few results for cohomological Mackey functors:

Lemma 3.5.1. *Suppose $\underline{M}$ is a cohomological G-Mackey functor. If $\underline{M}(G/C_p) = \mathbf{Z}_-$ and $\underline{M}(G/C_2) = 0$, then $\underline{M}(G/G) = 0$.*

Proof. The composition $Tr_{C_2}^G \circ Res_{C_2}^G = p$. But $\underline{M}(G/C_2) = 0$ implies the composition is 0. So $\underline{M}(G/G)$ only contains p-torsions.

On the other hand, $Tr_{C_p}^G \circ Res_{C_p}^G = 2$, but $Res_{C_p}^G$ is a zero map since $\underline{M}(G/C_p) = \mathbf{Z}_-$. So the composition $Tr_{C_p}^G \circ Res_{C_p}^G$ is also 0. Therefore $\underline{M}(G/G)$ only contains 2-torsions. So $\underline{M}(G/G)$ must be 0. □

Proposition 3.5.2. *If the cohomological Mackey functor $\underline{M}$satisfies the condition that $\underline{M}(G/C_2)$ and $\underline{M}(G/C_p)$ are both trivial, then $\underline{M}(G/G)$ is trivial.*

Proof. By the same argument we just used, we know $p\underline{M}(G/G) = 0$ and $2\underline{M}(G/G) = 0$. So $\underline{M}(G/G)$ is trivial. $\square$

For S^σ we have the following cofibration sequence:

$$S^0 \longrightarrow S^\sigma \longrightarrow G_+ \underset{C_2}{\wedge} S^1$$

Smashing the cofibration with S^V, we have the cofibration.

$$S^V \xrightarrow{i} S^{V+\sigma} \xrightarrow{p} G_+ \underset{C_2}{\wedge} i^G_{C_p}(S^{1+V}) \tag{3.5.3}$$

The cofibration gives us a long exact sequence of Mackey functors:

$$\cdots \xrightarrow{p_*} \underline{H}_{k+1}(G_+ \underset{C_p}{\wedge} i^G_{C_p} S^{1+V}) \xrightarrow{\partial} \underline{H}_k(S^V) \xrightarrow{i_*} \underline{H}_k(S^{V+\sigma}) \xrightarrow{p_*} \underline{H}_k(G_+ \underset{C_p}{\wedge} i^G_{C_p} S^{1+V}) \to \cdots \tag{3.5.4}$$

In the computation for $S^{b\sigma}$ we didn't use the method, because the chain complex was easy. But for mixed virtual representation, the method will help us simplify computation.

3.5.A Torsion free part

Suppose $V = b\sigma + m\lambda$.

Theorem 3.5.5. *For $\underline{H}_{|V|}S^V$, we have the following conclusions:*

(i) If V is oriented, that is, $m+n$ is even. Then $\underline{H}_{2m+n}(S^{n\sigma+m\lambda})$ it a $\mathbf{Z}$-valued Mackey functor, then $\underline{H}_{2m+n}(S^{n\sigma+m\lambda})(G/H)=\mathbf{Z}$ for all $H\leq G$.

If m is nonnegative then $Res_e^{C_p}=1$, otherwise it is p. If $m+n$ is nonnegative then $Res_e^{C_2}=1$, otherwise it is 2.

(ii) If V is not oriented, i.e. $m+n+c$ is odd, then

$$\underline{H}_{2m+n}(S^{n\sigma+m\lambda})(G/C_p)=\underline{H}_{2m+n}(S^{n\sigma+m\lambda})(G/e)=\mathbf{Z}_{-}.$$

$$\underline{H}_{2m+n}(S^{n\sigma+m\lambda})(G/G)=\underline{H}_{2m+n}(S^{n\sigma+m\lambda})(G/C_2).$$

Furthermore if $m+n\leq -3$, then $\underline{H}_{2m+n}(S^{n\sigma+m\lambda})(G/G)=\mathbf{Z}$, otherwise it is trivial. If m is nonnegative then $Res_e^{C_p}=1$, otherwise it is p.

Proof. Proof of (a) uses the combination of Lemma 2.3.8 and (3.2.2), which is direct. Proof of (b) uses the long exact sequence (3.5.4). If $S^{b\sigma+m\lambda}$ is not oriented, then $S^{(n-1)\sigma+m\lambda}$ is oriented. We have the following exact sequences to consider:

a. when m is positive, the first few terms of the long exact sequence will be

$$0=\underline{H}_{2m+n}(S^{(n-1)\sigma+m\lambda})\to\underline{H}_{2m+n}(S^{n\sigma+m\lambda})\xrightarrow{p_*}\underline{H}_{2m+n}(G_+\underset{C_p}{\wedge}i^G_{C_p}S^{n+m\lambda})$$

$$\xrightarrow{\partial}\underline{H}_{2m+n-1}(S^{m\lambda+(n-1)\sigma})\xrightarrow{i_*}\underline{H}_{2m+n-1}(S^{n\sigma+m\lambda})\to\cdots$$

The first three terms are the first three terms of one of the following exact sequence:

(1) $0\to\overline{\square}\to\mathring{\square}\to\square\to\bullet\to 0$

(2) $0\to\square\to\mathring{\square}\to\overline{\square}\to 0$

b. when m is negative, the last few terms of the long exact sequence will be

$$\cdots \to \underline{H}_{2m+n+1}(S^{(n+1)\sigma+\lambda}) \xrightarrow{p_*} \underline{H}_{2m+n+1}(G_+ \underset{C_p}{\wedge} i^G_{C_p} S^{1+n+m\lambda})$$

$$\xrightarrow{\partial} \underline{H}_{2m+n}(S^{n\sigma+m\lambda}) \xrightarrow{i_*} \underline{H}_{2m+n}(S^{(n+1)\sigma+m\lambda}) \to \underline{H}_{2m+n}(G_+ \underset{C_p}{\wedge} i^G_{C_p} S^{1+n+m\lambda}) = 0$$

The last four terms of the exact sequence are the last four terms of one of the following exact sequence

(1) $\cdots \to \blacksquare \to \overset{\circ}{\boxbslash} \to \dot{\boxbslash} \to 0 \to 0$

(2) $\cdots \to \boxbslash \to \overset{\circ}{\boxbslash} \to \overline{\square} \to 0 \to 0$

(3) $\cdots \to \boxbslash \to \overset{\circ}{\boxbslash} \to \dot{\boxbslash} \to \bullet \to 0$

All maps are maps of $\underline{\mathbf{Z}}G$-modules, hence are uniquely defined. In all of the exact sequences, the conclusions of part(b) hold. $\square$

3.5.B Torsion part

At first, we need to clarify what "torsion" means. "Torsion" refers to nontrivial Mackey functors with trivial $\underline{M}(G/G)$ values. In the case of homology group Mackey functors of spheres, torsion may only show up in $\underline{H}_k$ if k does not equal to $2m+n$.

We shall recall $\underline{H}_k(G_+ \underset{C_p}{\wedge} S^{m\overline{\lambda}})$ in (Theorem 3.3.10):

When m is nonnegative,

$$\underline{H}_k(G_+ \underset{C_p}{\wedge} S^{m\overline{\lambda}}) = \begin{cases} \overset{\circ}{\square} & \text{for k=2m} \\ \overset{\circ}{\blacktriangledown} & \text{for } k = 2l \text{ where } 0 \leqslant l < m \\ 0 & else. \end{cases}$$

When m is negative

$$\underline{H}_k(G_+ \underset{C_p}{\wedge} S^{m\overline{\lambda}}) = \begin{cases} \overset{\circ}{\boxslash} & \text{for } k = 2m \\ \overset{\circ}{\blacktriangledown} & \text{for } k = 2j-1 \text{ where } m \leqslant j < 0. \\ 0 & else. \end{cases}$$

Case 1: $m > 0$**:**

For any integer n, we have a five-term exact sequence

$$\begin{aligned} 0 = \underline{H}_{2j+n+1}(G_+ \underset{C_p}{\wedge} S^{n+m\overline{\lambda}}) &\to \underline{H}_{2j+n}(S^{m\lambda+(n-1)\sigma}) \to \underline{H}_{2j+n}(S^{n\sigma+m\lambda}) \\ \to \underline{H}_{2j+n}(G_+ \underset{C_p}{\wedge} S^{n+m\overline{\lambda}}) &= \overset{\circ}{\blacktriangledown} \to \underline{H}_{2j+n-1}(S^{m\lambda+(n-1)\sigma}) \\ \to \underline{H}_{2j+n-1}(S^{n\sigma+m\lambda}) &\to \underline{H}_{2j+n-1}(G_+ \underset{C_p}{\wedge} S^{n+m\overline{\lambda}}) = 0 \end{aligned} \tag{3.5.6}$$

for $0 < j < m$ and isomorphisms

$$\underline{H}_k S^{n\sigma+m\lambda} \cong \underline{H}_k S^{m\lambda+(n-1)\sigma} \tag{3.5.7}$$

for $k < n$ or $k > 2m + n$.

We see from this exact sequence (3.5.6) that for $0 < j < m$:

(i) $\underline{H}_{2j+n+1}(S^{m\lambda+(n-1)\sigma}) \xrightarrow{i_*} \underline{H}_{2j+n}(S^{n\sigma+m\lambda})$ is an injection.

(ii) $\underline{H}_{2j+n}(S^{m\lambda+(n-1)\sigma}) \xrightarrow{i_*} \underline{H}_{2j}(S^{n\sigma+m\lambda})$ is a surjection.

(iii) if one of $\underline{H}_{2j+n}(S^{n\sigma+m\lambda})$ and $\underline{H}_{2j+n-1}(S^{m\lambda+(n-1)\sigma})$ has a direct summand $\blacktriangledown$, then the other has the direct summand $\blacktriangle$.

(iv) if $\underline{H}_{2j+n}(S^{n\sigma+m\lambda})$ contains a summand $\bullet$, so does $\underline{H}_{2j+n-1}(S^{m\lambda+(n-1)\sigma})$.

(v) if $\underline{H}_{2j+n-1}(S^{m\lambda+(n-1)\sigma})$ contains a direct summand $\bullet$, so does $\underline{H}_{2j+n-1}(S^{n\sigma+m\lambda})$.

Statement (iii) is not obvious. It comes from the fact that

$$0 \to \blacktriangledown \to \overset{\circ}{\blacktriangledown} \to \blacktriangle \to 0$$

is a split exact sequence and $\underline{H}_{2j}(S^{m\lambda})$ is either $\blacktriangledown$ or $\blacktriangle$ and induction on n. We leave the proof to the reader.

Statement (iv) and (v) are deductions of (i) and (ii).

(3.2.2) tells us $\underline{H}_k(S^V)(G/C_p)$ can only contain at most one copy of $Z/3$.

From Figure 3.4 and Figure 3.8 we know

a. $\underline{H}_{2j-1}(S^{m\lambda}) = \bullet$ if m is odd and $m < 2j-1 < 2m$.

b. $\underline{H}_{2j-2}(S^{m\lambda})$ has a summand $\bullet$ if m is even and $m < 2j-2 < 2m$.

c. $\underline{H}_{2j-2}(S^{m\lambda})$ has a summand $\blacktriangledown$ if j is odd and $1 \leqslant j \leqslant m$

d. $\underline{H}_{2j-2}(S^{m\lambda})$ has a summand $\blacktriangle$ if j is even and $2 \leqslant j \leqslant m$

Combining the four equations with the five-term exact sequence gives Mackey functor homology $\underline{H}_*(S^{m\lambda+\sigma})$.

Corollary 3.5.8. *a.* $\underline{H}_{2j-1}(S^{m\lambda+\sigma})$ *has a summand* $\bullet$ *if* m *is odd and* $m < 2j-1 < 2m$.

b. $\underline{H}_{2j-1}(S^{m\lambda+\sigma})$ *has a summand* $\blacktriangle$ *if* j *is odd and* $1 \leqslant j \leqslant m$

c. $\underline{H}_{2j-1}(S^{m\lambda+\sigma})$ *has a summand* $\blacktriangledown$ *if* j *is even and* $2 \leqslant j \leqslant m$

d. $\underline{H}_{2j-2}(S^{m\lambda+\sigma})$ *is* $\bullet$ *if* m *is even and* $m < 2j-2 < 2m$.

Induction on n gives all the homology groups:

Theorem 3.5.9. *Assuming $m > 0$,*

(i) If $n+m>0$, then $\underline{H}_k(S^{n\sigma+m\lambda})$ has a summand $\bullet$ if $m \leq k < 2m+n$ and $k-m$ is even.

(ii) If $n+m<0$, then $\underline{H}_k(S^{n\sigma+m\lambda})$ has a summand $\bullet$ if $2m+n < k \leqslant m-3$ and $k-m$ is odd.

(iii) $\underline{H}_k(S^{n\sigma+m\lambda})$ has a summand $\blacktriangledown$ if k equals $n+4l+1+(-1)^{n+1}$ for some nonnegative integer l and $k<2m+n$.

(iv) $\underline{H}_k(S^{n\sigma+m\lambda})$ has a summand $\blacktriangle$ if k equals $n+4l+1+(-1)^{n}$ for some nonnegative integer l and $k<2m+n$.

Remark 3.5.10. *In part (iii) and part (iv) of either Theorem 3.5.9 or Theorem 3.5.14, k is implicitly assumed to be greater than n.*

Case 2: $m<0$**:**

There are three cases to be considered: $m=-1$,$m=-2$ and $m \leqslant -3$.

Subcase (i) $m=-1$

We have a short exact sequence

$$0 \to \underline{H}_{n-2}(S^{-\lambda+n\sigma}) \to \underline{H}_{n-2}(G_+ \underset{C_p}{\wedge} S^{n-\overline{\lambda}}) = \overset{\circ}{\blacksquare} \to \underline{H}_{n-3}(S^{-\lambda+(n-1)\sigma}) = 0 \quad (3.5.11)$$

and isomorphisms $\underline{H}_k(S^{-\lambda+(n-1)\sigma}) \cong \underline{H}_k(S^{-\lambda+n\sigma})$ for all $k \geqslant n-1$ or $k \leqslant n-4$.

In the exact sequence (3.5.11), all terms are torsion free. Therefore we can compute the torsion sub-Mackey functors by applying forgetful functor $i_{C_2}^G$ to $S^{-\lambda+n\sigma}$: when $\underline{H}_k(S^{-\lambda+n\sigma})(G/C_2) = \mathbf{Z}/2$, there is a summand $\bullet$ in $\underline{H}_k(S^{-\lambda+n\sigma})$. Combining this with Proposition 3.2.5 gives us the complete result.

Subcase (ii) $m = -2$

We have an exact sequence

$$
\begin{aligned}
0 &\to \underline{H}_{n-3}(S^{-2\lambda+n\sigma}) \to \underline{H}_{n-3}(G_+ \underset{C_p}{\wedge} S^{n-2\overline{\lambda}}) = \mathring{\blacktriangledown} \to \underline{H}_{n-4}(S^{-2\lambda+(n-1)\sigma}) \\
&\to \underline{H}_{n-4}(S^{-2\lambda+n\sigma}) \to \underline{H}_{n-4}(G_+ \underset{C_p}{\wedge} S^{n-2\overline{\lambda}}) = \mathring{\boxslash} \to \underline{H}_{n-5}(S^{-2\lambda+(n-1)\sigma}) \to 0
\end{aligned}
\tag{3.5.12}
$$

and isomorphisms $\underline{H}_k(S^{-2\lambda+(n-1)\sigma}) \cong \underline{H}_k(S^{-2\lambda+n\sigma})$ for $k \geqslant n-2$ or $k \leqslant n-5$.

The only torsion subgroup Mackey functor that we need to compute is $\underline{H}_{n-3}(S^{-2\lambda+n\sigma})$ in the exact sequence. Induction on n (in the negative direction) shows that it is $\underline{B}$ if n is odd and it is $\underline{B}_-$ if n is even. $\underline{H}_{n-3}(S^{-2\lambda+n\sigma})(G/C_2) = 0$, so by Proposition 3.5.2 there is no summand $\bullet$.

Subcase (iii) $m \leqslant -3$

We have a five-term exact sequence

$$
\begin{aligned}
0 &= \underline{H}_{2j+n+1}(G_+ \underset{C_p}{\wedge} S^{n+m\overline{\lambda}}) \to \underline{H}_{2j+n}(S^{m\lambda+(n-1)\sigma}) \to \underline{H}_{2j+n}(S^{n\sigma+m\lambda}) \\
&\to \underline{H}_{2j+n}(G_+ \underset{C_p}{\wedge} S^{n+m\overline{\lambda}}) = \mathring{\blacktriangledown} \to \underline{H}_{2j+n-1}(S^{m\lambda+(n-1)\sigma}) \to \underline{H}_{2j+n-1}(S^{n\sigma+m\lambda}) \\
&\to \underline{H}_{2j+n-1}(G_+ \underset{C_p}{\wedge} S^{n+m\overline{\lambda}}) = 0
\end{aligned}
\tag{3.5.13}
$$

for $1-m \leqslant j \leqslant -2$, and isomorphisms $\underline{H}_k(S^{m\lambda+(n-1)\sigma}) \cong \underline{H}_k(S^{n\sigma+m\lambda})$ for

$k \geqslant n-2$ or $k \leqslant 2m+n+1$.

Similar argument about positive m gives the Mackey functor homology: if there is a summand $\bullet$ in $\underline{H}_{2j+n}(S^{m\lambda+(n-1)\sigma})$(resp. $\underline{H}_{2j+n-1}(S^{m\lambda+(n-1)\sigma})$), then there is a $\bullet$ in $\underline{H}_{2j+n}(S^{n\sigma+m\lambda})$(resp. $\underline{H}_{2j+n-1}(S^{n\sigma+m\lambda})$.).

Therefore we obtain the following results:

Theorem 3.5.14. *Assuming $m < 0$,*

(i) If $n+m > 0$, then $\underline{H}_k(S^{n\sigma+m\lambda})$ has a summand $\bullet$ if $m \leq k < 2m+n$ and $k-m$ is even.

(ii) If $n+m < 0$, then $\underline{H}_k(S^{n\sigma+m\lambda})$ has a summand $\bullet$ if $2m+n < k \leq m-3$ and $k-m$ is odd.

(iii) $\underline{H}_k(S^{n\sigma+m\lambda})$ has a summand $\blacktriangledown$ if k equals $n-4l-4+(-1)^{b+1}$ for some nonnegative integer l and $2m+n < k$.

(iv) $\underline{H}_k(S^{n\sigma+m\lambda})$ has a summand $\blacktriangle$ if k equals $n-4l-4+(-1)^{n}$ for some nonnegative integer l and $2m+n < k$.

To summarize, we will split the results from the chain complex as follows:

Theorem 3.5.15.

(i) If V is orientable, i.e., if $n+m$ is even (see the beginning of this section), then $\underline{H}_{|V|}(S^V)$ it a $\mathbf{Z}$-valued Mackey functor, i.e. $\underline{H}_V(S^V)(G/H) = \mathbf{Z}$ for all $H \leq G$.

If m is nonnegative then $Res_e^{C_p} = 1$, otherwise it is p. If $b+c$ is nonnegative then $Res_e^{C_2} = 1$, otherwise it is 2.

(i) *If V is not orientable, i.e. $n+m$ is odd, then*

$$\underline{H}_{|V|}(S^V)(G/C_p) = \underline{H}(S^V)(G/e) = \mathbf{Z}_-.$$

$$\underline{H}_{|V|}(S^V)(G/G) = \underline{H}_{|V|}(S^V)(G/C_2).$$

Furthermore if $b+c \leq -3$, then

$$\underline{H}_{|V|}(S^V)(G/G) = \mathbf{Z}/2,$$

otherwise it is trivial. If m is nonnegative then $Res_e^{C_p} = 1$, otherwise it is p.

Theorem 3.5.16. *We have the following conclusion for $\underline{H}_k S^V$ for $V = m\lambda + n\sigma$ and $k \neq |V|$:*

(i) *If $n+m > 0$, then $\underline{H}_k(S^V;\underline{\mathbf{Z}})$ has a summand $\underline{D}$ if $m \leq k < |V|$ and $k-m$ is even.*

(ii) *If $n+m < 0$, then $\underline{H}_k(S^V;\underline{\mathbf{Z}})$ has a summand $\underline{D}$ if $|V| < k \leq m-3$ and $k-m$ is odd.*

(iii) *If $m > 0$, $\underline{H}_k(S^V;\underline{\mathbf{Z}})$ has a summand $\underline{B}$ if $k = n+4l+1+(-1)^{n+1}$ for some nonnegative integer l and $k < |V|$.*

(iv) *If $m > 0$, $\underline{H}_k(S^V;\underline{\mathbf{Z}})$ has a summand $\underline{B}_-$ if $k = n+4l+1+(-1)^{n}$ for some nonnegative integer l and $k < |V|$.*

(v) *If $m < 0$, $\underline{H}_k(S^V;\underline{\mathbf{Z}})$ has a summand $\underline{B}$ if k equals $n-4(l+1)+(-1)^{n+1}$ for some nonnegative integer l and $|V| < k$.*

(vi) If $m < 0$, $\underline{H}_k(S^V;\underline{\mathbf{Z}})$ has a summand $\underline{B}_-$ if k equals $n - 4(l+1) + (-1)^n$ for some nonnegative integer l and $|V| < k$.

3.6 Ring Structure of $\underline{\pi}_\star H\underline{\mathbf{Z}}$

In §3.4 and §3.5, we have obtained $\underline{\pi}_\star H\underline{\mathbf{Z}}$ with description in Mackey functors. In this section, we introduce some families of elements in $\underline{\pi}_\star H\underline{\mathbf{Z}}$, which are defined analogously to Definition 3.4 of [HHRc]. We let V be an actual representation of G with isotropy group G_V.

Definition 3.6.1. Four elements in $\underline{\pi}^G_\star(H\mathbf{Z})$.

(i) We use a_V to denote the composition of the equivariant inclusion of $S^0 \to S^V$ with the map $S^V \to S^V \wedge H\underline{\mathbf{Z}}$ induced by the smash product $S^0 \to S^0 \wedge H\underline{\mathbf{Z}} \approx H\underline{\mathbf{Z}}$. Therefore a_V is an element in $\underline{\pi}_{-V} H\underline{\mathbf{Z}}(G/G)$.

(ii) If λ is a summand of V, the computation in the previous section shows that

$$\underline{\pi}_{|V^{C_2}|-V} H\underline{\mathbf{Z}}(G/C_2) \cong \mathbf{Z}/2.$$

We use b_V to denote the generator of $\underline{\pi}_{|V^{C_2}|-V} H\underline{\mathbf{Z}}(G/G)$.

(iii) If W is an oriented representation of G (we do not require that $W^G = 0$), then $\underline{\pi}_{|W|-W} H\underline{\mathbf{Z}} = \underline{\mathbf{Z}}$. We let u_W denote the generator of $\underline{\pi}_{|W|-W} H\underline{\mathbf{Z}}(G/G)$. For nonoriented W, $\underline{\pi}_{|W|-W} H\underline{\mathbf{Z}} = \underline{\mathbf{Z}}_-$. We let u_W denote the generator of $\underline{\pi}_{|W|-W} H\underline{\mathbf{Z}}(G/C_p)$.

(iv) The underlying equivalence $S^V \to S^{|V|}$ defines an element e_V in

$$\underline{\pi}_V S^{|V|}(G/G_V) = \underline{\pi}_{V-|V|} S^0(G/G_V)$$

There is one thing we should point out in advance: each element mentioned above in $\underline{\pi}_\star H\underline{\mathbf{Z}}$ has a corresponding fixed point Mackey functor generated by it. When discussing the properties, we should link the properties of the elements with the associated Mackey functors.

Before discussing the ring structure, we should point out that the ring structure is compatible with the properties of box products and Green functors in §2.3, and our computation relies on such properties heavily.

The following lemma is the D_{2p} version of Lemma 3.6 of [HHRc]:

Lemma 3.6.2. *Properties of a_V, b_V, e_V and u_V*

The elements $a_V \in \underline{\pi}_{-V} H\mathbf{Z}(G/G)$, $b_v \in \underline{\pi}_{|V^{C_2}|-V} H\underline{\mathbf{Z}}(G/G)$,$e_V \in \underline{\pi}_{V-|V|} H\mathbf{Z}(G/G_V)$ and $u_W \in \underline{\pi}_{|W|-W} H\mathbf{Z}(G/G)$ for W oriented as in Definition 3.6.1 satisfy the following:

(i) $a_{V+W} = a_V a_W$ *and* $u_{V+W} = u_V u_W$.

(ii) $a_{m\sigma}$ *and* b_W *(if it can be defined) have order 2.* $a_{n\lambda}$ *has order* p.

(iii) For oriented V, $\mathrm{Tr}_{G_V}^{G}(e_V)$ *and* $\mathrm{Tr}_{G_V}^{G'}(e_{V+\sigma})$ *have infinite order, while* $\mathrm{Tr}_{G_V}^{G}(e_{V+\sigma})$ *has order 2 if* $|V| > 0$ *and* $\mathrm{Tr}_{G_V}^{G}(e_\sigma) = \mathrm{Tr}_{G'}^{G}(e_\sigma) = 0$.

(iv) For $V = b\sigma$ and W containing a copy of λ, we have $a_V b_W = b_{V+W}$.

(v) For oriented V and $G_V \subseteq H \subseteq G$

$$\mathrm{Tr}_{G_V}^{G}(e_V)u_V = |G/G_V| \in \underline{\pi}_0 H\underline{\mathbf{Z}}(G/G) = \mathbf{Z}$$

$$\textit{and} \qquad \mathrm{Tr}_{G_V}^{G'}(e_{V+\sigma})u_{V+\sigma} = |G'/G_V| \in \underline{\pi}_0 H\underline{\mathbf{Z}}(G/G') = \mathbf{Z} \qquad \textit{for } |V| > 0.$$

(vi) $a_{V+W}\mathrm{Tr}_{G_V}^{G}(e_{V+W}) = 0$ *if* $|V| > 0$.

(vii) For V and W oriented, $u_W \mathrm{Tr}_{G_V}^G(e_{V+W}) = |G_V/G_{V+W}|\mathrm{Tr}_{G_V}^G(e_V)$.

For nonoriented W similar statements hold in $\underline{\pi}_\star H\mathbf{Z}(G/G')$. $2W$ is oriented and u_{2W} is defined in $\underline{\pi}_{2|W|-2W}H\underline{\mathbf{Z}}(G/G)$ with $\mathrm{Res}_{G'}^G(u_{2W}) = u_W^2$.

If we look at the Mackey functor $\underline{H}_n S^V \cong \underline{\pi}_{n-V}H\underline{\mathbf{Z}}$ to which the element in $\underline{\pi}_V H\underline{\mathbf{Z}}(G/H)$ corresponds, the lemma is straightforward.

Proof. Left as exercises. □

We first describe the negative representation graded subring $\underline{\pi}_\star^- H\underline{\mathbf{Z}}(G/G)$.

Theorem 3.6.3.

$$\underline{\pi}_\star^- H\underline{\mathbf{Z}}(G/G) = \mathbf{Z}[u_{2\sigma}, u_{2\lambda}, u_{\lambda+\sigma}, a_\sigma, a_\lambda, b_\lambda]/(u_{\lambda+\sigma}^2 - u_{2\sigma}u_{2\lambda}, 2a_\sigma, 2b_\lambda, pa_\lambda)$$

Another way to describe the free part of $\underline{\pi}_\star^- H\underline{\mathbf{Z}}(G/G)$ as the subring of even degree polynomials in $\mathbf{Z}[u_\lambda, u_\sigma]$.

Note that $a_0 = u_0 = e_0 = 1 \in \underline{\pi}_0 H\underline{\mathbf{Z}}(G/G)$, we can define invertibility in $\underline{\pi}_\star H\underline{\mathbf{Z}}$. We say that $x \in \underline{\pi}_{n-V}H\underline{\mathbf{Z}}(G/H)$ is invertible, if there exists $y \in \underline{\pi}_{V-|V|}H\underline{\mathbf{Z}}(G/H)$ such that $xTr_H^G y = u_0 = 1$.

More generally, we can define divisibility similarly: $x \in \underline{\pi}_{n-V}H\underline{\mathbf{Z}}(G/H)$ is divisible by $y \in \underline{\pi}_{m-W}H\underline{\mathbf{Z}}(G/H)$, if there exists $z \in \underline{\pi}_{n-m+V-W}H\underline{\mathbf{Z}}(G/H)$ such that $yz = x$. In this way, invertible elements can be considered as elements divisible by $1 \in \underline{\pi}_0 H\underline{\mathbf{Z}}(G/H)$.

Regarding the divisibility by u_V, we have the following corollary of Lemma 3.6.2.

Corollary 3.6.4. *For some elements in* $\underline{\pi}_{V-|V|}H\underline{\mathbf{Z}}(G/G)$ *or* $\underline{\pi}_{V-|V|}H\underline{\mathbf{Z}}(G/G')$ *we have the following divisibility:*

(i) $u_{2\sigma}Tr_{G'}^{G}(e_{2\sigma}) = Tr_{G'}^{G}(Res_{G'}^{G}u_{2\sigma}e_{2\sigma}) = 2$. *So 2 is divisible by* $u_{2\sigma}$, *i.e.* ${2u_{2\sigma}}^{-1}$ *is well defined.*

(ii) $u_{2\lambda}Tr_{e}^{G}(e_{2\lambda}) = 2p = u_{\lambda+\sigma}Tr_{e}^{G}(e_{\lambda+\sigma})$. *So* $2p$ *is divisible by both* $u_{2\lambda}$ *and* $u_{\lambda+\sigma}$.

(iii) $u_{\lambda}Tr_{e}^{C_p}(e_{\lambda}) = p$. *So* p *is divisible by* u_{λ}, *i.e.* ${pu_{\lambda}}^{-1}$ *is well defined.*

(iv) $u_{\sigma}e_{\sigma} = 1$. *So* u_{σ} *is invertible.*

3.7 Some Eilenberg-Mac Lane Spectra

Proposition 3.7.1. *There is an element* $u_{\lambda-\sigma}$ *in* $\underline{\pi}_{\star}H\underline{\mathbf{Z}}(G/G)$, *such that* $Res_{H}^{G}u_{\lambda-\sigma} = u_{\lambda}/u_{\sigma}$.

Proof. By Theorem 3.5.15,

$$\underline{\pi}_{1+\sigma-\lambda}H\underline{\mathbf{Z}} \cong \underline{H}_0(S^{\lambda-\sigma}) = \underline{\mathbf{Z}}.$$

$Res_{C_p}^{G} = 1$. So u_{λ}/u_{σ} is in the image of the restriction map from $\underline{\pi}_{1+\sigma-\lambda}H\underline{\mathbf{Z}}$, which is an isomorphism. $\square$

Computation of some suspensions of $H\underline{\mathbf{Z}}$ are actually Eilenberg-Mac Lane spectra.

Theorem 3.7.2. *We have the following Eilenberg-Mac Lane spectra which are suspensions of* $H\underline{\mathbf{Z}}$:

(i) $H\underline{\mathbf{Z}}_{1,p} \cong S^{1+\sigma-\lambda} \wedge H\underline{\mathbf{Z}}$;

(ii) $H\underline{\mathbf{Z}}_{2,1} \cong S^{2-2\sigma} \wedge H\underline{\mathbf{Z}}$;

(iii) $H\underline{\mathbf{Z}}_{2,p} \cong S^{3-\sigma-\lambda} \wedge H\underline{\mathbf{Z}}$;

(iv) $H\underline{\mathbf{Z}}_{-}^{*} \cong S^{2-\lambda} \wedge H\underline{\mathbf{Z}}$;

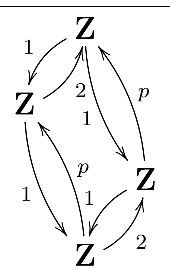
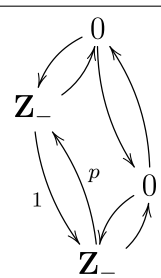
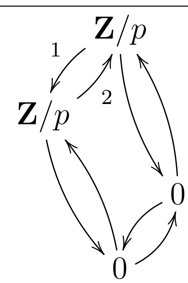
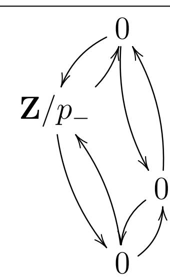
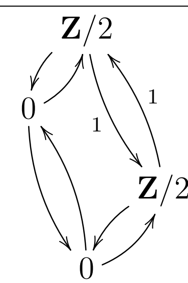
(v) $H\underline{\mathbf{Z}}_{-} \cong S^{1-\sigma} \wedge H\underline{\mathbf{Z}}$.

The notions of the Mackey functorS can be found in Equation 2.2.1.

Proof. Compute the Mackey functor homotopy using the results in §3.5. ☐

We recall Table 1.1 here

Table 3.3: Some G-Mackey functors

Symbol	$\underline{\mathbf{Z}}$	$\underline{\mathbf{Z}}_{-}$	$\underline{B}$	$\underline{B}_{-}$	$\underline{D}$
Lewis diagram					

For the Mackey functor $\underline{B}$,$\underline{B}_{-}$ and $\underline{D}$, we can construct the Eilenberg-Mac Lane spectra leveraging Theorem 3.5.15 Corollary 3.6.4 and Proposition 3.7.1:

Proposition 3.7.3. *Applying the Eilenberg-Mac Lane spectrum functors to the exact sequences of Mackey functors whose meaning can be found in Equation 2.2.1:*

(i) $\underline{\mathbf{Z}}_{1,p} \to \underline{\mathbf{Z}} \to \underline{B}$;

(ii) $\underline{\mathbf{Z}}_{-}^{*} \to \underline{\mathbf{Z}}_{-} \to \underline{B}_{-}$

(iii) $\underline{\mathbf{Z}}_{2,1} \to \underline{\mathbf{Z}} \to \underline{D}$

We have the cofiber sequences

(i) $H\underline{\mathbf{Z}}_{1,p} \xrightarrow{u_{\lambda-\sigma}} H\underline{\mathbf{Z}} \to H\underline{B}$;

(ii) $H\underline{\mathbf{Z}}_{-}^{*} \xrightarrow{u_{\lambda-\sigma}} H\underline{\mathbf{Z}}_{-} \to H\underline{B}_{-}$

(iii) $H\underline{\mathbf{Z}}_{2,1} \xrightarrow{u_{2\sigma}} H\underline{\mathbf{Z}} \to H\underline{D}$

Proof. The only thing that needs to be proved is that $u_{\lambda-\sigma}$ and $u_{2\sigma}$ exists and the restrictions to $(G/(\{e\}))$ induce isomorphisms. This is the conclusion of Proposition 3.7.1 and Definition 3.6.1(iii). □

In the case that $G = C_2$ and $\bullet$ is the Mackey functormentioned in Table 2.1, let σ denote the sign representation of C_2, then $\Sigma^1 H\bullet \cong \Sigma^{\rho_{C_2}} H\bullet$ and $\Sigma^{\sigma} H\bullet \cong H\bullet$ (ref. [Hil12]). A similar result applies in the case that $G = C_p$ and $\blacktriangledown$ the Mackey functor in Table 3.2, then $\Sigma^1 H\blacktriangledown \cong \Sigma^{\rho_{C_p}} H\bullet$ and $\Sigma^{\overline{\lambda}} H\blacktriangledown \cong H\blacktriangledown$.

We have analogous results for D_{2p}.

Theorem 3.7.4. *We have the following equivalences of Eilenberg − Mac Lane spectra if we suspend $H\underline{B}$, $H\underline{B}_{-}$ and $H\underline{D}$ with representation spheres of G:*

(i) $\Sigma^1 H\underline{B} \cong \Sigma^{1+\lambda} H\underline{B} \cong \Sigma^{1+(p-1)/2\lambda} H\underline{B}$, $\Sigma^{\sigma} H\underline{B} \cong \Sigma^1 H\underline{B}_{-}$, $\Sigma^{\lambda} H\underline{B} = H\underline{B}$;

(ii) $\Sigma^1 H\underline{B}_{-} \cong \Sigma^{1+\lambda} H\underline{B}_{-} \cong \Sigma^{1+(p-1)/2\lambda} H\underline{B}_{-}$, $\Sigma^{\sigma} H\underline{B}_{-} \cong \Sigma^1 H\underline{B}$, $\Sigma^{\lambda} H\underline{B} = H\underline{B}$;

(iii) $\Sigma^1 H\underline{D} \cong \Sigma^{1+\sigma} H\underline{B} \cong \Sigma^{\lambda} H\underline{B}$.

Proof. Smash product preserves cofibration sequences. So the computation can be done by smashing the cofibration sequences in Proposition 3.7.3 with the representation spheres. □

As a corollary, we investigate the suspension of $H\underline{B}$, $H\underline{B}_-$ and $H\underline{D}$ with the regular representation sphere.

Corollary 3.7.5. *Suspension with regular representation spheres will give the following equivalences:*

(i) $\Sigma^{\rho_G} H\underline{B} \cong \Sigma^2 H\underline{B}_-$;

(ii) $\Sigma^{\rho_G} H\underline{B}_- \cong \Sigma^2 H\underline{B}$;

(iii) $\Sigma^{\rho_G} H\underline{D} \cong \Sigma^p H\underline{D}$.

The results will be used in §4.4.

Chapter 4

Slice Spectral Sequences

The slice tower is an analog of the Postnikov tower in classical stable homotopy theory. The associated spectral sequence is called the slice spectral sequence. In this chapter, we will describe the slice spectral sequence of $RO(G)$-graded suspensions of $H\underline{\mathbf{Z}}$.

4.1 Definition of the Slice filtration and the Algebraic Slice Filtration

The original version of the (topological) slice spectral sequence was first introduced in [HHR16]. In Ullman's thesis [Ull13], he introduced a variant of the slice spectral sequence, called the regular slice spectral sequence, which was convenient in both definition and application. Moreover, the definition given by Ullman is equivalent to the old one in [HHR16] up to suspension. In this paper, when we say "slice spectral sequence", we refer to the regular slice spectral sequence defined in [Ull13].

Definition 4.1.1. *A full subcategory τ of $\mathcal{S}p^G$, the category of genuine G spectra is called a* ***localizing subcategory****, if* *(i) When X is in τ, any object weakly equivalent to X is also in τ.*

(ii) τ is closed under cofibration and extensions. That means, if

$$X \to Y \to Z$$

is a cofiber sequence and $X \in \tau$, then Z is in τ if and only if Y is in τ.

(iii) τ is closed under coproducts.

Definition 4.1.2. *The **slice spheres** of dimension k are defined to be the spectra of the form $G_+ \underset{H}{\wedge} S^{n\rho_H}$, where $n|H| = k$. The category τ_k^G is defined to be the localizing category generated by the slice spheres of dimension $\geq k$.*

Definition 4.1.3. *A G-spectrum X is called **slice** $(n-1)$**-connected**, if $X \in \tau_n^G$. We also write $X \geq n$. A G-spectrum Y is called **slice** n**-coconnected**, if for every slice sphere $\widehat{S}$ of dimension $\geq n$, $[\widehat{S}, Y]$ is the trivial Mackey functor $\emptyset$. We also write $Y < n$.*

Since τ_n^G is a localizing subcategory, there is a unique functorial fiber sequence

$$P_n X \to X \to P^{n-1} X$$

where $P_n X \in \tau_n^G$ and $P^{n-1} X < n$.

Since $\tau_{n+1}^G \subset \tau_n^G$, $\{P_n X\}$ is a decreasing filtration, and hence $\{P^n X\}$ is an increasing filtration. We have natural maps $P_{n+1} X \to P_n X$ and $P^n X \to P^{n-1} X$.

Definition 4.1.4. *We denote the cofiber of the map $P_{n+1} X \to P_n X$ by $P_n^n X$. It is called the n**-th slice** of X.*

Proposition 4.1.5. *$P_n^n X$ is equivalent to the fiber of the map $P_n X \to P_{n-1} X$.*

Proof. We have the following commutative diagram of cofibration sequences:

$$\begin{tikzcd}
P_{n+1}X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & P^nX \arrow[d] \\
P_nX \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & P^{n-1}X \arrow[d] \\
P_n^nX \arrow[r] & * \arrow[r] & \Sigma F
\end{tikzcd} \tag{4.1.6}$$

where F is the fiber of the map $P_nX \to P_{n-1}X$. Each row and each column isa a cofibration sequence. So $F \cong P_n^nX$. $\square$

Thus we have the following corollary as a criterion of n-slices:

Corollary 4.1.7. *If $X \geq n$ and $X < n+1$, then X is a G-n-slice.*

Let $\mathcal{P}$ be the family of all proper subgroups of G and let $E\mathcal{P}$ be the universal $\mathcal{P}$-space. That means $E\mathcal{P}$ is determined up to equivariant homotopy equivalence by the following property:

$$(E\mathcal{P}_+)^H = \begin{cases} * & H = G \\ S^0 & H \neq G. \end{cases}$$

Let $\tilde{E}\mathcal{P}$ be the mapping cone of $\mathcal{P} \to *$. Similarly $\tilde{E}\mathcal{P}$ is determined up to equivariant homotopy equivalence by the following property:

$$(\tilde{E}\mathcal{P})^H = \begin{cases} S^0 & H = G \\ * & H \neq G. \end{cases}$$

With the definitions above we can define the geometric fixed point spectrum, which plays an important role in equivariant stable homotopy theory:

Definition 4.1.8. *Let X be a G-spectrum. The **geometric fixed point of** X, denoted as $\Phi^G X$, is the fixed point spectrum $\left((\tilde{E}\mathcal{P} \wedge X)_f\right)^G$, where the subscript f denotes the fibrant replacement. Moreover, for a subgroup H of G, $\Phi^H X$ is defined to be $\Phi^H(i_H^G X)$.*

The homotopical characterization of $\Phi^G X$ is simple: $\pi_*^G \Phi^G X = \pi_*^G X = \underline{\pi}_* X(G/G)$ and $\pi_*^H \Phi^H X = \underline{\pi}_* X(G/H) = 0$ if $H \neq G$. A computationally convenient criteria of τ_n^G is a theorem by Hill and Yarnall [HY18]:

Theorem 4.1.9. *A G-spectrum X is in τ_n^G if and only if*

$$\pi_k \Phi^H X = 0$$

for all $H \leq G$ and $k < n/|H|$.

Applying the geometric fixed point argument in Theorem 4.1.9, we can derive the following results for τ_n^G:

Proposition 4.1.10. *Let V be a representation of degree d and let n be an integer such that the following equation holds for all $H \subseteq G$:*

$$\left\lceil \frac{n}{|H|} \right\rceil + dim V^H = \left\lceil \frac{n+d}{|H|} \right\rceil .$$

Then the functor $S^V \wedge (-) : \tau_n^G \to \tau_{n+d}^G$ is an equivalence of categories, with inverse $S^{-V} \wedge (-)$.

As a result, we have the following equivalences between the localizing subcategory τ_n^G:

Theorem 4.1.11. *Smashing spheres induce equivalences for τ_n^G:*

(i) $S^{\rho_G} \wedge (-)$ *induces the equivalence* $\tau_n^G \to \tau_{n+|2p|}^G$.

(ii) $S^1 \wedge (-)$ *induces the equivalence* $\tau_0^G \to \tau_1^G$.

(iii) $S^\lambda \wedge (-)$ *induces the following equivalences of categories*

$$\tau_1^G \to \tau_3^G \to \cdots \to \tau_p^G$$

$$\tau_{p+1}^G \to \tau_{p+3}^G \to \cdots \to \tau_{2p}^G$$

$$\tau_2^G \to \tau_4^G \to \cdots \to \tau_{p-1}^G$$

$$\tau_{p+2}^G \to \tau_{p+4}^G \to \cdots \to \tau_{2p-1}^G.$$

(iv) $S^\sigma \wedge (-)$ *induces the equivalence* $\tau_p^G \to \tau_{p+1}^G$.

Proof. Notice that $dim\rho_G^H = [G : H]$, $dim\lambda_2^C = 1$, $dim\lambda^{C_p} = dim\lambda^G = 0$, $dim\sigma^{C_2} = dim\sigma^G = 0$, and $dim\sigma^{C_p} = 1$. Then apply Proposition 4.1.10. □

Corollary 4.1.12. *There are three equivalence classes of categories* $\{\tau_n^G\}$. *Since* $\tau_n^G \simeq \tau_{n+2p}^G$, *we only need to determine the equivalence class for* $0 \leq n \leq 2p$.

(i) $\tau_0^G \simeq \tau_1^G \simeq \tau_3^G \simeq \cdots \simeq \tau_p^G \simeq \tau_{p+1}^G \simeq \tau_{p+3}^G \simeq \cdots \simeq \tau_{2p}^G$

(ii) $\tau_2^G \simeq \tau_4^G \simeq \cdots \to \tau_{p-1}^G$

(iii) $\tau_{p+2}^G \simeq \tau_{p+4}^G \simeq \cdots \simeq \tau_{2p-1}^G$

Corollary 4.1.13. *Smashing an n-slice with* $S^{k\rho_G}$ *will result in an* $(n + k|G|)$*-slice. Here* k *can be any integer. Specifically,* $S^{k\rho_G} \wedge H\underline{\mathbf{Z}}$ *is a* $k|G|$ *slice for each* k.

Proof. Smash product preserves cofibration. Then use Theorem 4.1.11(i). □

With the slice tower defined, we want to define a spectral sequence. The slice tower is an analog of the Postnikov tower, and because of the way we define it, we can expect the spectral sequence to converge to the Mackey functor homotopy of the G-spectrum. The spectral sequence is called the slice spectral sequence, though originally the words "slice spectral sequence" refers to a slight different spectral sequence, firstly used in [HHR16]. The version we use is called regular slice spectral sequencein [Ull13]. Besides, we need to find computational methods for the slice spectral sequence.

Definition 4.1.14. *([Ull13])We define the* ***slice filtration on the homotopy groups of*** X *by*

$$F^s\underline{\pi}_t X := im(\underline{\pi}_t P_{s+t} X \to \underline{\pi}_t X) = ker(\underline{\pi}_t X \to \underline{\pi}_t P^{s+t-1} X).$$

The slice spectral sequence converges to

$$\underline{E}_{\infty}^{s,t} X = \underline{\pi}_{t-s} P_t^t X \cong F^s \underline{\pi}_{t-s} X / F^{s+1} \underline{\pi}_{t-s} X.$$

In this way, computation with slice spectral sequences associated with X is converted to computation with the slice filtration $\{F^s\}$ on the homotopy groups of X.

Definition 4.1.15. *A slice is called a* ***spherical slice*** *if it is a slice of the form* $\Sigma^V H\underline{\mathbf{Z}} = S^V \wedge H\underline{\mathbf{Z}}$*, where* V *is a representation of* G*.*

Remark 4.1.16. *The reader should be careful that a spherical slice is NOT a slice sphere.*

The good part of spherical slices is that computation done in Chapter 3 helps us figure out when $\Sigma^V H\underline{\mathbf{Z}}$ is a slice. We will discuss it later in §4.4.

Definition 4.1.17. *Let c be a real number and let $\underline{M}$ be a Mackey functor. Define i_c^* to be the map between Mackey functors such that $i_c^*\underline{M}(G/H) = 0$ if $|H| > c$ and $i_c^*\underline{M}(G/H) = \underline{M}(G/H)$ otherwise. The restrictions and transfers of $i_c^*\underline{M}(G/H)$ are inherited from $\underline{M}$.*

Definition 4.1.18. *([Ull13, I.8]) the* ***algebraic slice filtration on a Mackey functor*** *is defined by the following:*

$$\mathscr{F}^k\underline{M}(G/H) = \{x \in \underline{M}(G/H) | i_{|J|}^* x = 0, \forall J \subset H, |J| \leq k\},$$

and we define $\mathscr{F}_k$ to be the sub-Mackey functor generated by $\underline{M}(G/H)$ for $H \leq k$ by restrictions and transfers between G-sets. Here we only require k to be a real number

Example 4.1.19. *Let M be the Mackey functor $\underline{B}$. If $k = 3.5$, then $\mathscr{F}^k\underline{B} = \emptyset$ and $\mathscr{F}_k\underline{B} = \underline{B}$. If $k = 2.5$, then $\mathscr{F}^k\underline{M} = \underline{B}$ and $\mathscr{F}_k\underline{B} = \emptyset$.*

The relations between the slice filtrations on the homotopy groups (Definition 4.1.14) and the algebraic slice filtrations (Definition 4.1.18) are established in the following theorems:

Theorem 4.1.20. *[Ull13, Proposition 8.4] If $n < 0$, then for a $(n-1)$-connected spectrum X, we have the identity*

$$F^s\underline{\pi}_n X = \mathscr{F}_{(s+n)/n}\underline{\pi}_n X.$$

Theorem 4.1.21. *[Ull13, Corollary 8.6] If $n > 0$, then for a $(n+1)$-coconnected spectrum X, we have the identity*

$$F^s \underline{\pi}_n X = \mathscr{F}^{(s+n-1)/n} \underline{\pi}_n X.$$

We omit the proofs of the two theorems.

4.2 Slices for prime order cyclic groups

Let us start with the easiest case: which trivial suspensions of $H\underline{\mathbf{Z}}$ will be slices? The answer can be found at the end of [Hil12], which can be stated as the following theorem:

Theorem 4.2.1. *For cyclic groups of prime orders, we have the following results:*

(i) if $G = C_2$, then $\Sigma^n H\underline{\mathbf{Z}}$ is an n-slice for $0 \leq n \leq 6$;

(ii) If $G = C_3$, then $\Sigma^n H\underline{\mathbf{Z}}$ is an n-slice for $0 \leq n \leq 4$;

(iii) If $G = C_p$ where $p \geq 5$, $\Sigma^n H\underline{\mathbf{Z}}$ is an n-slice for $0 \leq n \leq 3$.

Proof. Computation for $\underline{H}_* S^{n-k\rho_H}$ for $H = C_2$ and $H = C_p$ can be found in (3.2.10) and (3.2.17).

If $H = C_2$, then

$$[S^{k\rho_{C_2}}, S^n \wedge H\underline{\mathbf{Z}}]^{C_2} \cong \underline{H}_{k-n}(S^{-k\sigma}; \underline{\mathbf{Z}})(C_2/C_2).$$

by (3.2.9) we know that it is nontrivial if $k - n$ is a negative odd integer and less than or equal to -3. Combine with the requirement that $2k > n$, we find that $n \geq 7$. So when $0 \leq n \leq 6$, $S^n \wedge H\underline{\mathbf{Z}}$ will be a slice.

If $H = C_p$, then

$$[S^{k\rho_{C_p}}, S^n \wedge H\underline{\mathbf{Z}}] \cong \underline{H}_{k-n}(S^{-\frac{1}{2}(p-1)k\overline{\lambda}}; \underline{\mathbf{Z}}).$$

by (3.2.9) we know that it is nontrivial if $k-n$ is a negative odd integer and less than or equal to -3. Combine with the requirement that $pk > n$, we need to solve the inequality $pk > n \geq k+3$. So when $p = 3$, we get $k \geq 2$ and $n \geq 5$. Therefore for $0 \leq n \leq 4$, $S^n \wedge H\underline{\mathbf{Z}}$ will be a slice. When $p \geq 5$, we get $k \geq 1$ and $n \geq 4$. Therefore for $0 \leq n \leq 3$, $S^n \wedge H\underline{\mathbf{Z}}$ will be a slice. □

Theorem 4.2.2. *Let G be a cyclic group of prime order, and V be a representation of G. $S^V \wedge H\underline{\mathbf{Z}}$ is a spherical slice, if the following conditions are satisfied:*

(i) If $G = C_2$ and $V = a + b\sigma$ where σ is the sign representation, then

$$b \leq a \leq b + 6.$$

(ii) If $G = C_3$ and $V = a + c\overline{\lambda}$ where where $\overline{\lambda}$ is the 2-dimensional irreducible representation, then $c \leq a \leq c+4$;

(iii) If $G = C_p$ and $V = a + c\overline{\lambda}$, where $\overline{\lambda}$ is the 2-dimensional irreducible representation if and only if

$$|V|/p = (a+2c)/p \leq a \leq (|V|+4)/p,$$

or equivalently,

$$2c/(p-1) \leq a \leq (2c+4)/(p-1).$$

Proof. First, since $\underline{\pi}_{|V|}S^V \wedge H\underline{\mathbf{Z}}(G/e) = \mathbf{Z}$ and $\underline{\pi}_k S^V \wedge H\underline{\mathbf{Z}}(G/e) = 0$ for $k \neq |V|$, $S^V \wedge H\underline{\mathbf{Z}}$ has to be a $|V|$-slice if it is a slice.

If $H = C_2$, we can use Corollary 4.1.13 and the result obtained in Theorem 4.2.1: a representation sphere $S^V \wedge H\underline{\mathbf{Z}}$ is a $|V|$-slice if and only if V is a suspension of $S^n \wedge H\underline{\mathbf{Z}}$ by $S^{k\rho_{C_2}}$ where $0 \leq n \leq 6$.

If $H = C_3$, the same technique shows $S^V \wedge H\underline{\mathbf{Z}}$ is a $|V|$-slice if and only if V is a suspension of $S^n \wedge H\underline{\mathbf{Z}}$ by $S^{k\rho_{C_3}}$, where $0 \leq n \leq 4$.

When $p \geq 5$, the previous method do not apply. (As an example, the reader can check that $S^{3+\overline{\lambda}}$ is a 5-slice.) We need to do hard-core computation:

$$[S^{k\rho_{C_p}}, S^{a+c\overline{\lambda}} \wedge H\underline{\mathbf{Z}}]^{(C_p/C_p)} \cong \underline{H}_{k-a}(S^{(c-(p-1)k/2)\overline{\lambda}}; \underline{\mathbf{Z}})(C_p/C_p).$$

If $S^V \wedge H\underline{\mathbf{Z}}$ is a slice, then $[S^{k\rho_{C_p}}, S^{a+c\overline{\lambda}} \wedge H\underline{\mathbf{Z}}] = \emptyset$ for all $k > |V|/p$.

$\underline{H}_{k-a}(S^{(c-(p-1)k/2)}\overline{\lambda}(G/e)$ is nonzero only if $k-a$ and $c-(p-1)k/2$ are either both negative or nonnegative. We will discuss the following cases.

Case 1:

If $k - a \geq 0$ and $c - (p-1)k/2 \geq 0$, then $c \geq (p-1)k/2 \geq a(p-1)/2$. Then we have $a \leq (a+2c)/p$, or equivalently, $pa \leq |V|$.

If $a = (a+2c)/p$, then $V = a\rho_{C_p}$, and $S^V \wedge H\underline{\mathbf{Z}}$ is a slice. Otherwise $pa < |V|$. The smallest integer k such that $\underline{\pi}_k S^V \wedge H\underline{\mathbf{Z}} \neq \emptyset$ is a. By Theorem 4.1.9, $S^V \wedge H\underline{\mathbf{Z}}$ is not in $\tau_{pa}^{C_p}$. So it cannot be in in $\tau_{|V|}^{C_p}$, therefore not a slice.

Case 2: If $k - a < 0$ and $c - (p-1)k/2 < 0$, then $c < (p-1)k/2 < a(p-1)/2$ that is, $a > |V|/p$. Transforming the inequalities, we obtain

$$2c/(p-1) < k < a$$

that is, $2c/(p-1)-a < k-a < 0$. From (3.2.17) we know that $\underline{H}_{k-a}(S^{(c-(p-1)k/2)\overline{\lambda}}(C_p/C_p) \neq 0$ if and only if $c-(p-1)k/2 \leq -2$ and $k-a$ is a negative odd integer less than or equal to -3. So there exists k such that $\underline{H}_{k-a}(S^{(c-(p-1)k/2)\overline{\lambda}}(C_p/C_p) \neq 0$ if and only if $a > (2c+4)/(p-1)$. Therefore $S^V \wedge H\underline{\mathbf{Z}}$ is a slice if and only if $a \leq (2c+4)/(p-1)$.

$\square$

4.3 Slices of Eilenberg-Mac Lane spectra

We mention two theorems which were first proved in [Hil12]:

Theorem 4.3.1. *A spectrum is a* 0*-slice, if and only if it is an Eilenberg-Mac Lane spectrum* $H\underline{M}$ *for some Mackey functor* $\underline{M}$*.*

A spectrum is a 1*-slice, if and only if it is* $\Sigma\underline{M}$ *for a Mackey functor* $\underline{M}$ *whose restrictions are all injective maps.*

For the cases of cyclic groups of prime orders, there are good results for $\Sigma^n H\underline{M}$ when $\underline{M}(G/G) = 0$. Remember that Corollary 4.1.13 works for arbitrary finite groups. We can apply it for cyclic groups of prime orders.

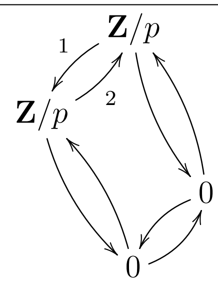
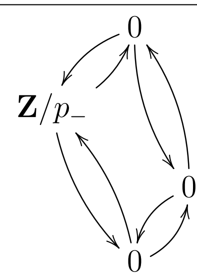
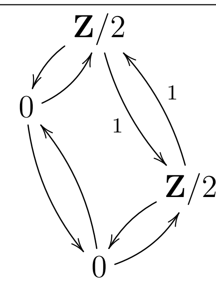
If $G = C_2$ and $\bullet$ is the Mackey functorin Table 2.1, let σ denote the sign representation of C_2, then $\Sigma^n H\bullet \cong \Sigma^{n\rho_{C_2}} H\bullet$ is a $2n$-slice.

If $G = C_p$ and $\blacktriangledown$ the Mackey functor in Table 3.2, then $\Sigma^n H\blacktriangledown \cong \Sigma^{n\rho_{C_p}} H\bullet$ is an pn-slice.

Let me remind the reader the meaning of $H\underline{B}$, $H\underline{B}_-$ or $H\underline{D}$, by Table 4.1:

Notice that $i^G_{C_p} H\underline{B} = H\underline{B}_- = H\blacktriangledown$ and $i^G_{C_2} H\underline{D} = H\bullet$, A natural question is: will the suspensions of $H\underline{B}$, $H\underline{B}_-$ or $H\underline{D}$ be slices? If they are, what will

Table 4.1: Some G-Mackey functors with $\underline{M}(G/e) = 0$

Symbol	$\underline{B}$	$\underline{B}_-$	$\underline{D}$
Lewis diagram			

their slice filtration be? The answer is surprisingly YES. Furthermore, their slice filtration levels remains the same as the cases of cyclic groups.

Theorem 4.3.2. *$\Sigma^n H\underline{B}$ is a pn-slice. $\Sigma^n H\underline{B}_-$ is a pn-slice. $\Sigma^n H\underline{D}$ is a $2n$-slice.*

Proof. For the cases that $n > 0$, by Theorem 4.1.21, $F^{s-1}\underline{\pi}_n\Sigma^n H\underline{B} \neq F^s\underline{\pi}_n\Sigma^n H\underline{B}$ if and only if $\mathscr{F}^{(s+n-1)/n}\underline{B} \neq \mathscr{F}^{(s+n)/n}\underline{B}$. This can only happen when $(s+n-1)/n < p$ and $(s+n)/n \geq p$.

So the only possible integer s is n(p-1). In this case, $F^{s-1}\underline{\pi}_n\Sigma^n H\underline{B} = \emptyset$ and $F^s\underline{\pi}_n\Sigma^n H\underline{B} = \underline{B}_-$. So the only nontrivial slice of $\Sigma^n H\underline{B}$ is the np slice $\Sigma^n H\underline{B}$.

For the cases that $n < 0$, by Theorem 4.1.20, $F_{s-1}\underline{\pi}_n\Sigma^n H\underline{B} \neq F_s\underline{\pi}_n\Sigma^n H\underline{B}$ will only happen when $(s+n-1)/n \geq p$ and $(s+n)/n < p$, where $F^{s-1}\underline{\pi}_n\Sigma^n H\underline{B} = \emptyset$ and $F^s\underline{\pi}_n\Sigma^n H\underline{B} = \underline{B}_-$. So the only possible integer s is again n(p-1). The only nontrivial slice of $\Sigma^n H\underline{B}$ is the np slice $\Sigma^n H\underline{B}$.

Similar proofs apply to $H\underline{B}_-$ and $H\underline{D}$. $\square$

4.4 Slice spectral sequences for $\Sigma^V H\underline{\mathbf{Z}}$

In this section we describe the slice spectral sequences for $\Sigma^V H\underline{\mathbf{Z}}$ leveraging Theorem 4.3.2 and the results we have obtained in Chapter 3. In this section, we assume the representation V is of the form $V = a + b\sigma + c\lambda$. We use d to denote $a + b + 2c$, the degree of V.

First recall Proposition 3.5.2

Proposition 4.4.1. *Suppose $\underline{M}$ is a D_{2p}-cohomological Mackey functor. If*

$$\underline{M}(G/C_2) = 0 = \underline{M}(G/C_p),$$

then $\underline{M}$ is the trivial Mackey functor $\emptyset$.

Proof. Suppose $x \in M(G/G)$. We have

$$px = Tr^G_{C_2} Res^G_{C_2} x = 0.$$

$$2x = Tr^G_{C_p} Res^G_{C_p} x = 0.$$

Then x is both a 2-torsion and a p-torsion. Hence $x = 0$. $\square$

Lemma 4.4.2. *Let G be a finite group and let H be a subgroup of G. If a spectrum E is an G-n-slice, then $i^G_H E$ is an H-n-slice for H.*

Proof. See Proposition 4.13 of [HHR16]. $\square$

For the special case that $G = D_{2p}$, we have the converse of Lemma 4.4.2.

Theorem 4.4.3. *Let X be a G-spectrum all of whose Mackey functor homotopy are cohomological. If it is both a C_2-n-slice and a C_p-n-slice after applying the forgetful functors $i^G_{C_2}$ and $i^G_{C_p}$, then it is a G-n-slice.*

Proof. We only need to check $[S^{k\rho_G}, X]^G = \underline{\pi}_0(S^{-k\rho_G} \wedge X)(G/G) = 0$ for $k|G| > n$. If the Mackey functor $\underline{\pi}_0(S^{-k\rho_G} \wedge X)(G/H) = 0$ for both $H = C_p$ and $H = C_2$, then $\underline{\pi}_0(S^{-k\rho_G} \wedge X)(G/H) = 0$ by Proposition 3.5.2. $i^G_H \rho_G \cong |G/H|\rho_H$, thus the condition is satisfied because X is both a C_p-slice and a C_2-slice with proper forgetful functor applied on it. □

Thus we have the following detection theorem for spherical slices as a corollary of Theorem 4.4.3: if $i^G_{C_p} S^V \wedge H\underline{\mathbf{Z}}$ and $i^G_{C_2} S^V \wedge H\underline{\mathbf{Z}}$ are both d-slices for the subgroups. Then $S^V \wedge H\underline{\mathbf{Z}}$ is an G-d-slice. We know that $i^G_{C_2} S^V \cong S^{(a+c)+(b+c)\sigma}$, $i^G_{C_p} S^V = S^{a+b+c\bar{\lambda}}$. Applying Theorem 4.2.2, we obtain the relations that a,b and c have to follow:

Corollary 4.4.4. *$S^V \wedge H\underline{\mathbf{Z}}$ is a G-d-slice if:*

(i) $b + c \leq a + c \leq b + c + 6$*, or equivalently,* $b \leq a \leq b + 6$*;*

(ii) *if* $p = 3$*, then* $c \leq a + b \leq c + 4$*,*
if $p \geq 5$*, then* $2c/(p-1) \leq a + b \leq (2c + 4)/(p-1)$*.*

The main theorem of the whole article is the following one:

Theorem 4.4.5. *Let $V - W \in RO(G)$, where both V and W are representations of G. Then there exists $e(V-W) \in RO(G)$, such that $S^{V-W} \wedge H\underline{\mathbf{Z}}$ has a spherical $(|V| - |W|)$-slice $S^{e(V-W)} \wedge H\underline{\mathbf{Z}}$. The other slices are suspensions of $H\underline{B}$, $H\underline{B}_-$ or $H\underline{D}$, or wedges of them.*

Assuming that Proposition 4.4.7, Proposition 4.4.9, Proposition 4.4.10 and Proposition 4.4.11 are all correct, we give a proof of the theorem.

Proof. For a virtual representation $V-W$ in $RO(G)$, we can always find an integer k such that $V-W+k\rho_G$ is a representation of G. By Theorem 4.1.11, smashing with $S^{k\rho_G}$ induces an equivalences of slices. So we only need to work with actual representations.

We first assume that $S^{e(V)} \wedge H\underline{\mathbf{Z}}$ exists for each $S^V \wedge H\underline{\mathbf{Z}}$. We will figure out what conditions $S^{e(V)} \wedge H\underline{\mathbf{Z}}$ should satisfy.

Carolyn Yarnall proved that if G is a cyclic p-group, then $S^n \wedge H\underline{\mathbf{Z}}$ has a spherical n-slice. As in the naive case, the theorem holds for $G = C_p$ and C_2. A similar computation proves that for $G = C_q$ where q is a prime and for V a representation of C_q, $S^V \wedge H\underline{\mathbf{Z}}$ also has a spherical slice.

Let d be the degree of V. From Lemma 4.4.2, the slice filtration commutes with the forgetful functor i_H^G. Therefore $i_{C_2}^G(P_d^d S^V \wedge H\underline{\mathbf{Z}})$ is a spherical C_2-d-slice and $i_{C_p}^G(P_d^d S^V \wedge H\underline{\mathbf{Z}})$is a spherical C_p-d-slice.

Noticing that when $G = C_q$ where q is a prime and V is a representation of C_p, then either $S^V H\underline{\mathbf{Z}} \geq |V|$ or $S^V H\underline{\mathbf{Z}} \leq |V|$ holds. In either case, $S^V H\underline{\mathbf{Z}}$ has a C_q-V-slice (however, it can be either the top slice or the bottom slice).

Combining the results we obtained in Proposition 4.4.7, Proposition 4.4.9, Proposition 4.4.10 and Proposition 4.4.11 will give us the result. Furthermore, the proofs of the four propositions give explicit construction of how the slices are obtained.

$\square$

The map $u_{\lambda-\sigma}$ we introduced in Proposition 3.7.1 plays a crucial in this section, because the cofiber of $u_{l(\lambda-\sigma)}$ has a good slice filtration:

Theorem 4.4.6. *The cofiber of the map $u_{l(\lambda-\sigma)} : S^V \wedge H\underline{\mathbf{Z}} \to S^{V+l(\lambda-1-\sigma)} \wedge H\underline{\mathbf{Z}}$ is a wedge of suspension of $H\underline{B}$ and $H\underline{B}_-$. Thus the cofiber only has kp-slices for some integers k.*

Proof. $V + l(\lambda - 1 - \sigma) = a - l + (b - l)\sigma + (c + l)\lambda$. The cofiber has nontrivial Mackey functor homotopy in dimension $c - 2$, $c - 4$,...,$c - 2l$, with value either $\underline{B}$ or $\underline{B}_-$ appearing altenatively in such dimensions. So the cofiber has $(c - 2)p$, $(c - 4)p$,...,$(c - 2l)p$ slices by Theorem 4.3.2. □

As an example, consider the cofiber sequence

$$S^8 \wedge H\underline{\mathbf{Z}} \xrightarrow{u_{2(\lambda-\sigma)}} S^{6-2\sigma+2\lambda} \wedge H\underline{\mathbf{Z}} \longrightarrow X.$$

Then $\underline{\pi}_4 X = \underline{B}$, $\underline{\pi}_6 X = \underline{B}_-$. Other Mackey functor homotopy of X is trivial. So X has a $4p$-slice $\Sigma^4 H\underline{B}$ and a $6p$-slice $\Sigma^6 H\underline{B}_-$.

We shall remind the readers that in Chapter 3 we proved that $i^G_{C_2} S^\lambda = S^{1+\sigma}$, $i^G_{C_p} S^\sigma = S^1$. Thus $i^G_{C_2} S^V \cong S^{(a+c)+(b+c)\sigma}$, $i^G_{C_p} S^V = S^{a+b+c\overline{\lambda}}$.

Proposition 4.4.7. *If $i^G_{C_2}(S^V \wedge H\underline{\mathbf{Z}}) \leq d$ and $i^G_{C_p}(S^V \wedge H\underline{\mathbf{Z}}) \leq d$ both hold, then $S^V \wedge H\underline{\mathbf{Z}}$ has a spherical G-d-slice. For $0 \leq k < d$, the k-slices of $S^V \wedge H\underline{\mathbf{Z}}$ is either (i)$\Sigma^{k/p} H\underline{B}$, or(ii) $\Sigma^{k/p} H\underline{B}$, or (iii)$\Sigma^{k/2} H\underline{D}$, or (iv) the wedge of (i) and (iii), or (iv) the wedge of (ii) and (iii).*

Proof. By Theorem 4.1.9, $i^G_{C_2}(S^V \wedge H\underline{\mathbf{Z}}) \leq d$ implies $a + c \leq d/2 = (a + b + 2c)/2$, so $a < b$. $i^G_{C_p}(S^V \wedge H\underline{\mathbf{Z}}) \leq d$ implies $a + b \leq d/p$. If we can find a spherical d-slice together with a map to $S^V \wedge H\underline{\mathbf{Z}}$ such that the cofiber is $\leq d-1$, then we are done.

Let $S^{e(V)} \wedge H\underline{\mathbf{Z}}$ denote the desired spherical slice. Let $e(V) = a(V) + b(V)\sigma + c(V)\lambda$. Then the following conditions which we obtain from Theorem 4.2.2 are required if such a spherical slice exists:

(i) (C_2-slice condition) $b(V) \leq a(V) \leq b(V) + 6$

(ii) (C_p-slice condition) if $p = 3$, then $0 \leq a(V) + b(V) - c(V) \leq 4$
if $p \geq 5$, then $2c(V)/(p-1) \leq a(V) + b(V) \leq (2c(V)+4)/(p-1)$

(iii) (requirement on orientation). $b(V) + c(V) = b + c$, i.e $e(V)$ has the same orientability as V.

Condition (iii) guarantees that we can use the map $u_{\lambda-\sigma}$ in the proof.

If such an $e(V)$ exists, then we have the cofiber sequence

$$S^{e(V)} \wedge H\underline{\mathbf{Z}} \xrightarrow{S}{}^{V} \wedge H\underline{\mathbf{Z}} \to P^{d-1}X. \tag{4.4.8}$$

The desired representation $e(V)$ can be found in two steps:

First, we take a look at $i^G_{C_p} S^V \wedge H\underline{\mathbf{Z}} \cong S^{(a+b)+c\bar{\lambda}} \wedge H\underline{\mathbf{Z}}$. The condition $i^G_{C_p}(S^V \wedge H\underline{\mathbf{Z}}) \leq d$ implies that $a+b \leq d/p$. So there exists a degree representation on $U = u + v\bar{\lambda}$ of C_p, such that $S^U \wedge H\underline{\mathbf{Z}}$ is the d-slice for $i^G_{C_p}(S^V \wedge H\underline{\mathbf{Z}})$ (See [Yar17]). Therefore we have the equation $a + b + 2c = u + 2v$, hence $c - v = (a + b - u)/2$. Let f be $c - v$. Then we have a map

$$u_{f(\lambda-\sigma)} : S^{V-f(\lambda-1-\sigma)} \wedge H\underline{\mathbf{Z}} \longrightarrow S^V \wedge H\underline{\mathbf{Z}}$$

with cofiber Cof. According to Theorem 4.4.6, the cofiber Cof is wedge of suspensions of $H\underline{B}$ and $H\underline{B}_-$. Since

$$\begin{aligned} i_{C_p}^G(V - f(\lambda - 1 - \sigma)) =& a + b + c\overline{\lambda} - f(\overline{\lambda} - 2) \\ =& (a + b - 2f) + (c - f)\overline{\lambda} \\ =& u + v\overline{\lambda}, \end{aligned}$$

we have $i_{C_p}^G Cof < n$, thus it has slices with filtration $< d$.

Second, we consider $i_{C_2}^G \S^U \wedge H\underline{\mathbf{Z}}$. There is a positive integer g such that

$$u_{2g\sigma} : S^{U-2g\sigma} \wedge H\underline{\mathbf{Z}} \to S^U \wedge H\underline{\mathbf{Z}}$$

with cofiber Cof_1 such that $i_{C_2}^G Cof_1 < d$. Similarly, Cof_1 is a wedge sum of suspensions of $H\underline{D}$ with slice filtration $< d$.

Notice that $S^{V-f(\lambda-\sigma-1)-2g\sigma} \wedge H\underline{\mathbf{Z}}$ is a H-d-slice for both $H = C_2$ and $H = C_p$, according to Corollary 4.4.4 it is a G-d-slice. So $e(V) = V - f(\lambda - \sigma - 1) - 2g\sigma$.

We shall point out that we cannot exchange the two steps, because the first step uses the fact that $i_{C_2}^G \lambda = \rho_{C_2}$ implicitly, so $i_{C_2}^G S^{V-f(\lambda-1-\sigma)} \simeq i_{C_2}^C S^V$, therefore the operation does not affect the slices of $i_{C_2}^G S^V \wedge H\underline{\mathbf{Z}}$.

Therefore we have obtained all the slices of $S^V \wedge H\underline{\mathbf{Z}}$: the G-d-slice is

$$S^{V-f(\lambda-\sigma-1)-2g\sigma} \wedge H\underline{\mathbf{Z}} = S^{(a+f)+(b+f-2g)\sigma+(c-f)\lambda} \wedge H\underline{\mathbf{Z}},$$

and other slices are computed as before. $\square$

Proposition 4.4.9. *If $i_{C_2}^G(S^V \wedge H\underline{\mathbf{Z}}) \geq d$ and $i_{C_p}^G(S^V \wedge H\underline{\mathbf{Z}}) \geq d$, then $S^V \wedge H\underline{\mathbf{Z}}$ has a spherical G-d-slice, and $\underline{\pi}_k P_{d+1}(S^V \wedge H\underline{\mathbf{Z}})(G/e)$ is trivial for all k.*

Proof. The proof is similar to that of Proposition 4.4.7. The only difference is that this time we need to find the bottom slice $S^{e(V)} \wedge H\underline{\mathbf{Z}}$ and the map $u_{f(\lambda-\sigma)+2g\sigma}$ to $S^{e(V)} \wedge H\underline{\mathbf{Z}}$. $\square$

Proposition 4.4.10. *If $i^G_{C_2}(S^V \wedge H\underline{\mathbf{Z}}) \leq d$ and $i^G_{C_p}(S^V \wedge H\underline{\mathbf{Z}}) > d$, then $S^V \wedge H\underline{\mathbf{Z}}$ has a spherical G-d-slice.*

Proof. Proposition 3.7.1 proved that there is a map

$$u_{\lambda-\sigma} : S^{1+\sigma-\lambda} \wedge H\underline{\mathbf{Z}} \to H\underline{\mathbf{Z}}.$$

$i^G_{C_2} S^{1+\sigma-\lambda} \cong S^0$ as a C_2-spectrum. So smashing $S^V \wedge H\underline{\mathbf{Z}}$ with $S^{1+\sigma-\lambda}$ induces an equivalence between C_2-spectra. We choose l to be the lease positive integer such that $i^G_{C_p}(S^{V+l\lambda-l-l\sigma} \wedge H\underline{\mathbf{Z}}) \leq d$. There is a map from $S^V \wedge H\underline{\mathbf{Z}}$ to $S^{V+l\lambda-l-l\sigma} \wedge H\underline{\mathbf{Z}}$, induced by multiplication with $u_{l(\lambda-\sigma)}$. $S^{V+l\lambda-l-l\sigma} \wedge H\underline{\mathbf{Z}}$ satisfied the condition in Proposition 4.4.7, so we can apply Proposition 4.4.7 to it. Since $i^G_{C_p}(S^V \wedge H\underline{\mathbf{Z}}) > d$ and $i^G_{C_p}(S^{V+l\lambda-l-l\sigma} \wedge H\underline{\mathbf{Z}}) \leq d$, the fiber of $u_{l(\lambda-\sigma)}$, which is a wedge of trivial suspensions of $H\underline{B}$ and $H\underline{B}_-$, is slice d-connected. $S^{V+l\lambda-l-l\sigma} \wedge H\underline{\mathbf{Z}}$ is slice $d+1$-coconnected since $i^G_{C_2}(S^{V+l\lambda-l-l\sigma} \wedge H\underline{\mathbf{Z}})$ and $i^G_{C_p}(S^{V+l\lambda-l-l\sigma} \wedge H\underline{\mathbf{Z}})$ are both slice $d+1$-coconnected. So it has slices if slice filtration d and less. Thus we get the slice filtration for $S^V \wedge H\underline{\mathbf{Z}}$ $\square$

Proposition 4.4.11. *If $i^G_{C_2}(S^V \wedge H\underline{\mathbf{Z}}) > d$ and $i^G_{C_p}(S^V \wedge H\underline{\mathbf{Z}}) \leq d$, then $S^V \wedge H\underline{\mathbf{Z}}$ has a spherical G-d-slice.*

Proof. Similar to the proof Proposition 4.4.10. Let l be the least integer such that $S^{V+l(1+\sigma-\lambda)} \wedge H\underline{\mathbf{Z}} \geq d$. Then the cofiber of

$$u_{l(\lambda-\sigma)} : S^{V+l(1+\sigma-\lambda)} \wedge H\underline{\mathbf{Z}} \to S^V \wedge H\underline{\mathbf{Z}}$$

is slice d-coconnected because of the choice of l. Then we can apply Proposition 4.4.7 to $S^{V+l(1+\sigma-\lambda)} \wedge H\underline{\mathbf{Z}}$ for its slices of slice filtration d and more. $\square$

4.5 Examples of Slice Spectral Sequences

In this section, we will work on specific examples of slice towers (and therefore, the slice spectral sequences) for $S^V \wedge H\underline{\mathbf{Z}}$.

4.5.A $S^n \wedge H\underline{\mathbf{Z}}$

The simplest example is $S^n \wedge H\underline{\mathbf{Z}}$. We will give the slice tower of $\Sigma^{16} H\underline{\mathbf{Z}}$ for $G = D_6$ and $G = D_{10}$. In both cases, the condition in Proposition 4.4.9 are satisfied, so we apply the proposition to obtain the slices.

Example 4.5.1. *We want to find the triple* (a, b, c) *such that:*

(i) $b + c \leq a + c \leq b + c + 6$*, or equivalently,* $b \leq a \leq b + 6$*;*

(ii) $c \leq a + b \leq c + 4$.

$(6, 2, 4)$ *turns out to be the only solution. Following the steps in Proposition 4.4.9,* *the Slice tower of* $\Sigma^{16}H\underline{\mathbf{Z}}$ *for* $G = D_6$ *is as follows:*

$$\begin{tikzcd}
39-slice & \Sigma^{13}H\underline{B}_- \arrow[r] & S^{16}\wedge H\underline{\mathbf{Z}} \arrow[d, "u_{\lambda-\sigma}"] \\
33-slice & \Sigma^{11}H\underline{B} \arrow[r] & S^{15-\sigma+\lambda}\wedge H\underline{\mathbf{Z}} \arrow[d, "u_{\lambda-\sigma}"] \\
27-slice & \Sigma^{9}H\underline{B}_- \arrow[r] & S^{14-2\sigma+2\lambda}\wedge H\underline{\mathbf{Z}} \arrow[d, "u_{\lambda-\sigma}"] \\
26-slice & \Sigma^{13}H\underline{D} \arrow[r] & S^{13-3\sigma+3\lambda}\wedge H\underline{\mathbf{Z}} \arrow[d, "u_{2\sigma}"] \\
22-slice & \Sigma^{11}H\underline{D} \arrow[r] & S^{11-\sigma+3\lambda}\wedge H\underline{\mathbf{Z}} \arrow[d, "u_{2\sigma}"] \\
21-slice & \Sigma^{7}H\underline{B} \arrow[r] & S^{9+\sigma+3\lambda}\wedge H\underline{\mathbf{Z}} \arrow[d, "u_{\lambda-\sigma}"] \\
18-slice & \Sigma^{9}H\underline{D} \arrow[r] & S^{8+4\lambda}\wedge H\underline{\mathbf{Z}} \arrow[d, "u_{2\sigma}"] \\
16-slice & & S^{6+2\sigma+4\lambda}\wedge H\underline{\mathbf{Z}}
\end{tikzcd}$$

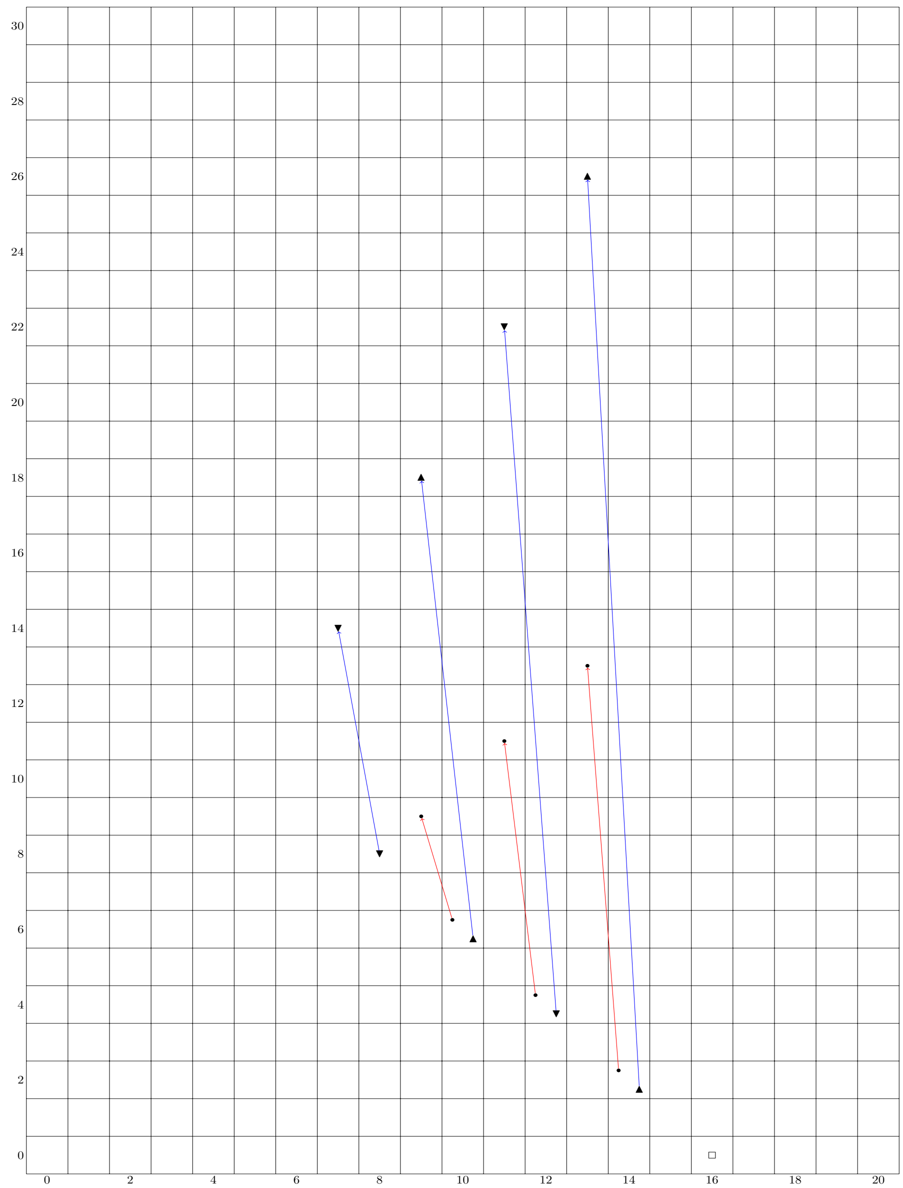
Example 4.5.2. *When* $G = D_{10}$, *we want to find another triple* (a, b, c) *such that*

(i) $b + c \leq a + c \leq b + c + 6$, *or equivalently,* $b \leq a \leq b + 6$;

(ii) $2c/(p-1) \leq a + b \leq (2c+4)/(p-1)$.

Coincidentally, $(6, 2, 4)$ *is the solution again. The* 16*-slice for* $S^{16}\wedge H\underline{\mathbf{Z}}$ *is* $S^{6+2\sigma+4\lambda}\wedge H\underline{\mathbf{Z}}$ *as well. However, the slice tower (and the slices) are quite different from the slice tower (and the slices) for* $G = S_3$, *which is shown as follows: The*

Figure 4.1: Slice Spectral Sequence of $\Sigma^{16} \wedge H\underline{\mathbf{Z}}$ for $G = S_3$



Slice tower of $\Sigma^{16}H\underline{\mathbf{Z}}$ for $G = D_{10}$ is:

$$\begin{tikzcd}
65-slice & \Sigma^{13}H\underline{B}_{-} \arrow[r] & S^{16}\wedge H\underline{\mathbf{Z}} \arrow[d, "u_{\lambda-\sigma}"] \\
55-slice & \Sigma^{11}H\underline{B} \arrow[r] & S^{15-\sigma+\lambda}\wedge H\underline{\mathbf{Z}} \arrow[d, "u_{\lambda-\sigma}"] \\
45-slice & \Sigma^{9}H\underline{B}_{-} \arrow[r] & S^{14-2\sigma+2\lambda}\wedge H\underline{\mathbf{Z}} \arrow[d, "u_{\lambda-\sigma}"] \\
35-slice & \Sigma^{7}H\underline{B} \arrow[r] & S^{13-3\sigma+3\lambda}\wedge H\underline{\mathbf{Z}} \arrow[d, "u_{\lambda-\sigma}"] \\
26-slice & \Sigma^{13}H\underline{D} \arrow[r] & S^{12-4\sigma+4\lambda}\wedge H\underline{\mathbf{Z}} \arrow[d, "u_{2\sigma}"] \\
22-slice & \Sigma^{11}H\underline{D} \arrow[r] & S^{10-2\sigma+4\lambda}\wedge H\underline{\mathbf{Z}} \arrow[d, "u_{2\sigma}"] \\
18-slice & \Sigma^{9}H\underline{D} \arrow[r] & S^{8+4\lambda}\wedge H\underline{\mathbf{Z}} \arrow[d, "u_{2\sigma}"] \\
16-slice & & S^{6+2\sigma+4\lambda}\wedge H\underline{\mathbf{Z}}
\end{tikzcd}$$

4.5.B $S^{n\sigma} \wedge H\underline{\mathbf{Z}}$

We will give the slice tower of $S^{11\sigma} \wedge H\underline{\mathbf{Z}}$ for $G = D_{10}$.

Applying the forgetful functor argument, we get

$$i^G_{C_2}(S^{11\sigma} \wedge H\underline{\mathbf{Z}}) \simeq S^{11\sigma} \wedge H\underline{\mathbf{Z}}$$

and

$$i^G_{C_5} S^{11\sigma} \wedge H\underline{\mathbf{Z}} \simeq S^{11} \wedge H\underline{\mathbf{Z}}.$$

$i^G_{C_2} S^{11\sigma} \wedge H\underline{\mathbf{Z}} \leq 11$, but $i^G_{C_5} S^{11\sigma} \wedge H\underline{\mathbf{Z}} \geq 11$. So the condition in Proposition 4.4.10

is satisfied.

Example 4.5.3. *We choose l to be the lease positive integer such that*

$$i_{C_p}^{G}(S^{11\sigma+l\lambda-l-l\sigma} \wedge H\underline{\mathbf{Z}}) \simeq S^{11-2l+l\lambda} \leq 11.$$

The answer is that $l=3$. So

$$P^{11}(S^{11} \wedge H\underline{\mathbf{Z}}) = S^{11\sigma+3(\lambda-1-\sigma)} \wedge H\underline{\mathbf{Z}} = S^{-3+8\sigma+3\lambda} \wedge H\underline{\mathbf{Z}}$$

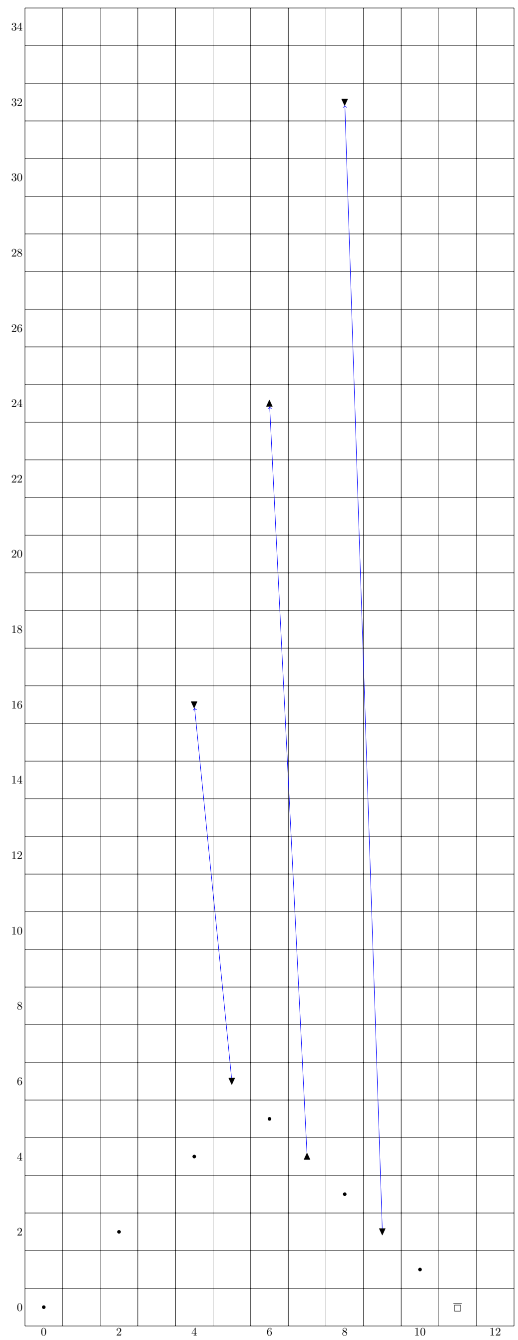
The Slice tower of $S^{11\sigma} \wedge H\underline{\mathbf{Z}}$ for $G = D_{10}$ is:

$$\begin{tikzcd}
40-slice & \Sigma^8 H\underline{B} \arrow[r] & S^{11\sigma} \wedge H\underline{\mathbf{Z}} \arrow[d, "u_{\lambda-\sigma}"] \\
30-slice & \Sigma^6 H\underline{B}_- \arrow[r] & S^{-1+10\sigma+\lambda} \wedge H\underline{\mathbf{Z}} \arrow[d, "u_{\lambda-\sigma}"] \\
20-slice & \Sigma^4 H\underline{B} \arrow[r] & S^{-2+9\sigma+2\lambda} \wedge H\underline{\mathbf{Z}} \arrow[d, "u_{\lambda-\sigma}"] \\
11-slice & S^{3+2\sigma+3\lambda} \wedge H\underline{\mathbf{Z}} \arrow[r] & S^{-3+8\sigma+3\lambda} \wedge H\underline{\mathbf{Z}} \arrow[d] \\
8-slice & \Sigma^4 H\underline{D} \arrow[r] & \Sigma^4 H\underline{D} \wedge \Sigma^2 H\underline{D} \wedge H\underline{D} \arrow[d] \\
4-slice & \Sigma^2 H\underline{D} \arrow[r] & \Sigma^2 H\underline{D} \wedge H\underline{D} \arrow[d] \\
0-slice & & H\underline{D}
\end{tikzcd}$$

4.5.C $S^{m\lambda} \wedge H\underline{\mathbf{Z}}$

We will deal with a comparatively simple case: $S^{-6\lambda} \wedge H\underline{\mathbf{Z}}$ while $G = D_6$.

Figure 4.2: Slice Spectral Sequence of $\Sigma^{11\sigma} \wedge H\underline{\mathbf{Z}}$ for $G = D_{10}$



First we smash $S^{-6\lambda} \wedge H\underline{\mathbf{Z}}$ with $S^{3\rho_G}$, so the slices are preserved (Corollary 4.1.13). Since $\rho_G = 1 + \sigma + 2\lambda$, we get $S^{3+3\sigma} \wedge H\underline{\mathbf{Z}}$.

The cofiber sequence

$$\Sigma^3 H\underline{B} \to S^{3+3\sigma} \wedge H\underline{\mathbf{Z}} \to S^{2+2\sigma+\lambda} \wedge H\underline{\mathbf{Z}}$$

already gives the slices of $S^{3+3\sigma} \wedge H\underline{\mathbf{Z}}$: $\Sigma^3 H\underline{B}$ is a 9-slice, and $S^{2+2\sigma+\lambda} \wedge H\underline{\mathbf{Z}}$ is a 6-slice. Then we smash the whole cofiber sequence with $S^{-3\rho_G}$ to get the slice spectral sequence for $S^{-6\lambda} \wedge H\underline{\mathbf{Z}}$. We get

$$S^{-3\rho_G} \wedge \Sigma^3 H\underline{B} \longrightarrow S^{-6\lambda} \wedge H\underline{\mathbf{Z}} \longrightarrow S^{-3\rho_G} \wedge S^{2+2\sigma+\lambda} \wedge H\underline{\mathbf{Z}}$$

Taking advantage of Corollary 3.7.5, we get

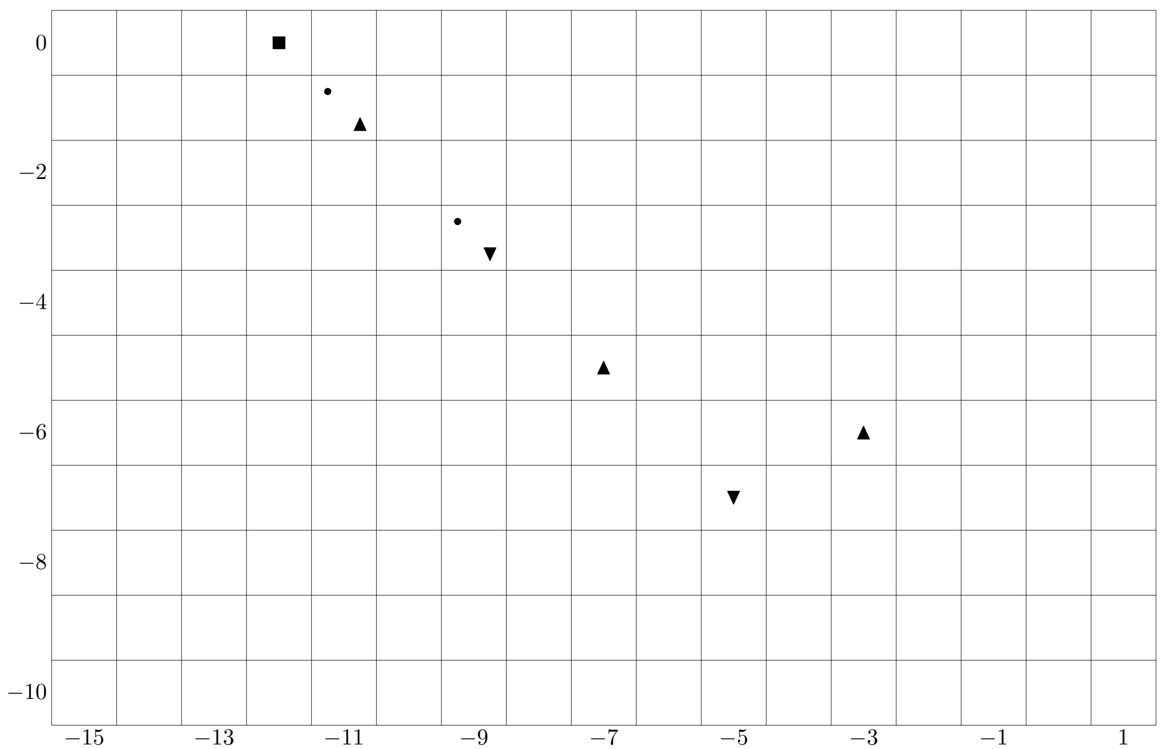
$$S^{-3\rho_G} \wedge \Sigma^3 H\underline{B} = \Sigma^{-3} H\underline{B}_-$$

to be the -9-slice and

$$S^{-3\rho_G} \wedge S^{2+2\sigma+\lambda} \wedge H\underline{\mathbf{Z}} = S^{-1-\sigma-5\lambda} \wedge H\underline{\mathbf{Z}}$$

to be the -12-slice.

Figure 4.3: Slice Spectral Sequence of $\Sigma^{-6\lambda} \wedge H\underline{\mathbf{Z}}$ for $G = S_3$



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