$$
\begin{cases}
h_1 = a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} h_{1,n} \quad (M_{15} = 9s + 18) \\
h_2 = a_n^{p-5} h_{n,0} h_{5,0} h_{n-2,2} h_{n-3,3} b_{n-4,3} \quad (M_{16} = 4np - 4n - 8p - 9) \\
h_3 = a_n^{p-5} h_{n,0} h_{5,0} h_{n-2,2} h_{n-4,4} b_{n-3,2} \quad (M_{17} = 4np - 4n - 6p - 11) \\
h_4 = a_n^{p-5} h_{n,0} h_{5,0} h_{n-3,3} h_{n-4,4} b_{n-2,1} \quad (M_{18} = 4np - 4n - 4p - 13) \\
\left.\begin{array}{c}
h_5 = a_n^{p-5} h_{n,0} h_{1,0} h_{n-1,1} h_{n-2,2} h_{n-3,3} h_{1,4} \\
h_6^{(i)} = a_n^{p-5} h_{n,0} h_{i,0} h_{n-i,i} h_{n-2,2} h_{n-3,3} h_{1,4} (4 \le i \le n-1) \\
h_7^{(i)} = a_n^{p-5} h_{n,0} h_{i,0} h_{n-i,i} h_{n-2,2} h_{2,3} h_{n-4,4} (1 \le i \le n-1, i \ne 2, 4) \\
h_8^{(i)} = a_n^{p-5} h_{n,0} h_{i,0} h_{n-i,i} h_{3,2} h_{n-3,3} h_{n-4,4} (1 \le i \le n-1, i \ne 3, 4) \\
h_9 = a_n^{p-6} a_1 h_{n-1,1} h_{n,0} h_{5,0} h_{n-2,2} h_{n-3,3} h_{n-4,4} \\
h_{10} = a_n^{p-5} h_{1,0} h_{n-1,1} h_{5,0} h_{n-2,2} h_{n-3,3} h_{n-4,4} \\
h_{11}^{(i)} = a_n^{p-5} h_{i,0} h_{n-i,i} h_{5,0} h_{n-2,2} h_{n-3,3} h_{n-4,4} (6 \le i \le n-1) \\
h_{12}^{(i)} = a_n^{p-5} h_{n,0} h_{i,0} h_{5-i,i} h_{n-2,2} h_{n-3,3} h_{n-4,4} (1 \le i \le 4) \\
h_{13}^{(i)} = a_n^{p-5} h_{n,0} h_{5,0} h_{i,2} h_{n-i-2,i+2} h_{n-3,3} h_{n-4,4} (3 \le i \le n-3) \\
h_{14}^{(i)} = a_n^{p-5} h_{n,0} h_{5,0} h_{n-2,2} h_{i,3} h_{n-i-3,i+3} h_{n-4,4} (2 \le i \le n-4) \\
h_{15}^{(i)} = a_n^{p-5} h_{n,0} h_{5,0} h_{n-2,2} h_{n-3,3} h_{i,4} h_{n-i-4,i+4} (1 \le i \le n-5)
\end{array}\right\} (M_{19})
\end{cases}
$$

where $M_{19} = 2_{np} - 2_n +_p -19$.

In what follows, we give the proof of Theorem 3.1 case-by-case when r and n take different values.

Proof. For $t = s + 2 + (s + 2)p + (s + 3)p^2 + (s + 4)p^3 + p^n$, we see that

when $n = 1$, there exists the sequence $\overline{S}_1 = (s + 2, s + 3, s + 3, s + 4)$;

when $n = 2$, there exists the sequence $\overline{S}_2 = (s + 2, s + 2, s + 4, s + 4)$;

when $n = 3$, there exists the sequence $\overline{S}_3 = (s + 2, s + 2, s + 3, s + 5)$;

when $n \geq 4$, there exists the sequence with form

$$\overline{S} = (\overline{c}_0, \cdots, \overline{c}_n) = (s + 2, s + 2, s + 3, s + 4, 0, \cdots, 0, 1).$$

For a given generator $h \in E_1^{s-r+7, tq+(s-r+1),*}$ with $1 \leq r \leq s + 7$, we have

$$\dim(h) = s - r + 7 \text{ and } \deg(h) = tq + (s - r + 1).$$

If $s + 1 < r \leq s + 7$, then there is $s - r + 1 < 0$. This implies that the number of a_i in h is $s - r + 1 + q$. This is impossible due to the reason of dimension. Thus, we can now assume $2 \leq r \leq s + 1$ which follows $s - r + 1 \geq 0$.

Since $s - r + 7 < s - r + 1 + q$, we see that the number of x_i in h is $s - r + 1$. According to the reason of dimension, we list all the possibilities of h as

$$
\begin{cases}
x_1 \cdots x_{s-r+1} z_1 z_2 z_3 \\
x_1 \cdots x_{s-r+1} y_1 y_2 z_1 z_2 \\
x_1 \cdots x_{s-r+1} y_1 y_2 y_3 y_4 z_1 \\
x_1 \cdots x_{s-r+1} y_1 y_2 y_3 y_4 y_5 y_6.
\end{cases}
$$

When $1 \leq n \leq 3$, the sequence $K = (k_1, \cdots, k_n)$ becomes $(0, \cdots, 0)$ and thus the corresponding $(c_0, \cdots, c_n)$ equals to $\overline{S}_n = (\overline{c}_0, \cdots, \overline{c}_n)$. When $n = 4$, we have $(k_1, \cdots, k_4) = (0, \cdots, 0)$ and then

$$(c_0, \cdots, c_n) = \overline{S} = (s + 2, s + 2, s + 3, s + 4, 1)$$

When $n \geq 5$, all the possibilities of $K = (k_1, \cdots, k_n)$ are listed as

$$
\begin{cases}
K_1 = (0, \cdots, 0) \\
K_i = (0, 0, 0, 0, \cdots, 0, 1^{(i)}, \cdots, 1) \quad (5 \leq i \leq n)
\end{cases}
$$

In Table 1, for the generator with May filtration M_2, since its first May differential is nonzero and does not contain $a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,n-1}$, this implies that $E_r^{s+6,tq+s,M_2} = 0$ for $r \geq 2$, and hence,

$$a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,n-1} \notin d_r(E_r^{s+6,tq+s,M_2}) \text{ for } r \geq 1.$$

For the generators with May filtration M_3, their first May differentials all contain at least one term, which does not lie in the first May differential of any other generators. This implies that all the first May differentials of the generators are linearly independent and thus $E_1^{s+6,tq+s,9s+p+16}$ is trivial. It follows that $E_r^{p+1,t+p-5,2np+p-2n-19} = 0$ for $r \geq 2$, and then

$$a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,n-1} \notin d_r(E_r^{s+6,tq+s,M_3}) \text{ for } r \geq 1.$$

The same argument shows that for the remaining generators, there is

$$a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,n-1} \notin d_r(E_r^{s+6,tq+s,M_i}).$$

for $r \geq 1$ and $4 \leq i \leq 19$.

From the above results, we see that $a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,n-1}$ is not hit by any May differential when $s \leq p - 5$, $n \geq 1$, and $n \neq 5$. It follows that $a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,n-1}$ is a permanent cycle in the MSS and then converges nontrivially to $\tilde{\delta}_{s+4} h_0 b_{n-1} \in \mathrm{Ext}_{\mathcal{A}}^{s+7,tq+s}(\mathbb{Z}/p, \mathbb{Z}/p)$.

Let us then consider the Adams differential $d_r: E_r^{s-r+7,tq+s-r+1} \to E_r^{s,tq+s}$ for $r \geq 2$, which possibly hit $\tilde{\delta}_{s+4} h_0 b_{n-1}$ in the ASS. According to Lemma 4.2, we know that $\mathrm{Ext}_{\mathcal{A}}^{s-r+7,t+s-r+1}(\mathbb{Z}/p, \mathbb{Z}/p) = 0$, which follows $E_r^{s-r+7,tq+s-r+1} = 0$. Hence, the corresponding Adams differential is trivial and then cannot hit $\tilde{\delta}_{s+4} h_0 b_{n-1}$ in the ASS. Thus, $\tilde{\delta}_{s+4} h_0 b_{n-1}$ is nontrivial in the ASS. This finishes our proof.

Remark 4.3. For $s = p - 5$, $r = 1$, and $n = 5$, by Theorem 3.1(e) there is one non-zero generator

$$E_1^{p+1,tq+p-5,11p-29} = \mathbb{Z}/p\{a_5^{p-5} h_{4,1} h_{3,2} h_{2,3} h_{1,4} h_{1,0}\}.$$

By applying the three-filtrated cobar construction, we can show the following differential holds:

$$d_{p-1}(a_5^{p-5} h_{4,1} h_{3,2} h_{2,3} h_{1,4} h_{1,0}) = a_4^{p-5} h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,4}.$$

It follows that $a_4^{p-5} h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,4}$ vanishes in the E_p-term of the MSS. Thus, it cannot converge nontrivially to $\tilde{\delta}_{p-1} h_0 b_4$, and then follows $\tilde{\delta}_{p-1} h_0 b_4 = 0$.

5. Conclusion

This paper applies a new effective computation method to determine the convergence of a product element $\tilde{\delta}_{s+4} h_0 b_{n-1}$ in the classical ASS. The key point of this method is to construct a group of linear equations according to the triple degrees of the representative in the MSS of $\tilde{\delta}_{s+4} h_0 b_{n-1}$, and then compute out the related generators in the corresponding E_1-term of the MSS. What we compute helps us to show that the product element $\tilde{\delta}_{s+4} h_0 b_{n-1}$ is a permanent cycle without being bound in any term of the ASS. This new method admits a wide application range in determining nontrivial elements of the stable homotopy groups of spheres. In the future, we plan to use this method to detect more nontrivial elements in the stable homotopy groups of spheres.

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Conflict of interest

The authors declare that there is no personal or organizational conflict of interest in this work

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