CONGRUENCES OF EISENSTEIN SERIES OF LEVEL $\Gamma_1(N)$ VIA DIEUDONNÉ THEORY OF FORMAL GROUPS

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Abstract. In this paper, we first explain congruences of Eisenstein series of level $\Gamma_1(N)$ and character χ. Our approach is based on Katz's algebro-geometric explanation of p-adic congruences of normalized Eisenstein series E_{2k} of level 1. One crucial step in our argument is to reformulate a Riemann-Hilbert correspondence in Katz's explanation in terms of Dieudonné theory of height 1 formal A-modules and their finite subgroup schemes.

We further connect congruences of modular forms in the Eisenstein subspace $\mathcal{E}_k(\Gamma_1(N),\chi)$ with certain group cohomology involving the Dirichlet character χ. When χ is trivial, this group cohomology computes the image of the J-homomorphism in the stable homotopy groups of spheres. We have therefore connected congruences of Eisenstein series E_{2k} of level 1 to the image of J.

Contents

In [Kat73b], Katz gave an algebro-geometric explanation of the p-adic congruences of normalized Eisenstein series E_{2k} of weight $2k$ and level 1. Using a Riemann-Hilbert type correspondence (Theorem 2.4.1) and a theorem of Igusa, Katz showed:

Theorem. [Kat73b, Corollary 4.4.1] *The followings are equivalent:*

(1) $E_{2k}(q)\equiv 1 \mod p^m$.

(2) The $2k$-th power representation $\mathbb{Z}_p^{\otimes 2k}$ of $\mathbb{Z}_p^\times$ is trivial mod p^m.

The first goal of this paper is to adapt Katz method's to study congruences of modular forms in the Eisenstein subspace

$$\mathcal{E}_k(\Gamma_1(N),\chi)\subseteq M_k(\Gamma_1(N),\chi)\coloneqq M_k(\Gamma_1(N))^{\chi^{-1}},$$

where $\chi:(\mathbb{Z}/N)^\times\to\mathbb{C}^\times$ is a primitive Dirichlet character of conductor N. The strategy is to study a p-adic version of this problem and then assemble the congruence at each prime. As we will be working integrally and p-adically, it is necessary to specify meaning of level structures. Let $\mathcal{M}_{ell}(\mu_N)$ be a stack over $\mathbb{Z}$ whose R points are:

$$\mathcal{M}_{ell}(\mu_N)(R)\coloneqq\left\{(C/R,\eta:\mu_N\hookrightarrow C)\,\middle|\,\begin{array}{c}C\text{ is an elliptic curve over }R,\\ \eta\text{ is an embedding of group schemes}\end{array}\right\}.$$

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When N is invertible in R, a μ_N-level structure on an elliptic curve is (non-canonically) equivalent to a classical $\Gamma_1(N)$-level structure. Write $N = p^v N'$ where $p \nmid N'$. The p-adic version of $\mathcal{M}_{ell}(\mu_N)$ we will consider is $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$, whose R points are

$$\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))(R) = \left\{ (C/R, \eta_p, \eta') \;\middle|\; \begin{array}{l} C \text{ is a } p\text{-ordinary elliptic curve over } R, \\ \eta_p : \mu_{p^v} \xrightarrow{\sim} \widehat{C}[p^v], \quad \eta' : \mathbb{Z}/N' \hookrightarrow C[N] \end{array} \right\}.$$

$\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')) = \mathcal{M}_{ell}(\mu_N)_p^{\wedge}$ when $p \mid N$ and is an open substack otherwise. Now let $\chi : (\mathbb{Z}/N)^{\times} \to \mathbb{C}_p^{\times}$ be a p-adic primitive Dirichlet character of conductor N. Write $\mathbb{Z}_p[\chi] := \mathbb{Z}_p[\operatorname{Im}\chi]$. χ is uniquely factorized as product $\chi = \chi_p \cdot \chi'$ where χ_p and χ' have conductors p^v and N', respectively. Let k be an integer such that $(-1)^k = \chi(-1)$. Let $\mathbb{Z}_p^{\otimes k}[\chi]$ be the $\mathbb{Z}_p^{\times} \times (\mathbb{Z}/N')^{\times}$-representation, whose underlying module is $\mathbb{Z}_p[\chi]$ and where $(a,b) \in \mathbb{Z}_p^{\times} \times (\mathbb{Z}/N')^{\times}$ acts by multiplication by $a^k \cdot \chi_p(a) \cdot \chi'(b)$. The first main result of this paper is:

Theorem (Main Theorem 2.6.1)**.** *Let $\mathcal{I} \trianglelefteq \mathbb{Z}_p[\chi]$ be an ideal. The followings are equivalent:*

(i). *There is an Eisenstein series f in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ with q-expansion $f(q) \in 1 + \mathcal{I}q[\![q]\!]$.*
(v). *The $\mathbb{Z}_p^{\times} \times (\mathbb{Z}/N')^{\times}$-representation $\mathbb{Z}_p^{\otimes k}[\chi]$ is trivial modulo $\mathcal{I}$.*

The proof of the Main Theorem has three major steps:

I. Identify the Dirichlet character χ to the Galois descent data of formal $\mathbb{Z}_p[\chi]$-modules $\widehat{C}^{k,\chi}$ over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. This allows us to translate congruences of Eisenstein series in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ to those of elements in the Dieudonné module $\mathbb{D}(\widehat{C}^{k,\chi})$ of $\widehat{C}^{k,\chi}$.
II. Reformulate a Riemann-Hilbert correspondence in Katz's explanation in terms of the Dieudonné module and the Galois descent data of the formal A-modules:

  Theorem. (2.4.6) *Let R be a flat algebra over $\mathbb{W}\kappa$ for some separable extension κ of $\mathbb{F}_p$, such that R/p is an integrally closed integral domain. Suppose R is formally smooth over κ, so that it admits a lift of Frobenius $\varphi : R \to R$. Let $\widehat{G}$ be a formal A-module over R whose reduction mod p has height/slope 1. Write $\mathbb{D}(\widehat{G}) = (M, F : M \xrightarrow{\sim} \varphi^* M)$ and $\rho : \pi_1^{ét}(R) \to A^{\times}$ for the Dieudonné module and Galois descent data for $\widehat{G}$, respectively. Let $\mathcal{I} \trianglelefteq A$ be an ideal and denote the $\mathcal{I}$-torsion of $\widehat{G}$ by $\widehat{G}[\mathcal{I}]$. Then the followings are equivalent:*
  (a) There is a generator γ of M as an $R \otimes A$-module such that $F\gamma \equiv \gamma \mod \mathcal{I}$.
  (b) $\widehat{G}[\mathcal{I}] \simeq (\widehat{G}_m \otimes A)[\mathcal{I}]$.
  (c) The composition homomorphism $\rho_{\mathcal{I}} : \pi_1^{ét}(R) \xrightarrow{\rho} A^{\times} \twoheadrightarrow (A/\mathcal{I})^{\times}$ is trivial.

  From this, we relate congruences of generators in $\mathbb{D}(\widehat{C}^{k,\chi})$ to those of the Galois representation $[\rho^{k,\chi}]$ attached to $\widehat{C}^{k,\chi}$.
III. Factorize the character $\rho^{k,\chi}$ associated to the Galois representation $[\rho^{k,\chi}]$ and use a relative version of Igusa's theorem to reduce the group to $\mathbb{Z}_p^{\times} \times (\mathbb{Z}/N')^{\times}$.

The main theorem implies the maximal congruence of Eisenstein series in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ is equal to that of the $\mathbb{Z}_p^{\times} \times (\mathbb{Z}/N')^{\times}$-representation $\mathbb{Z}_p^{\otimes k}[\chi]$. The latter is further related to the group cohomology of $\mathbb{Z}_p^{\times} \times (\mathbb{Z}/N')^{\times}$.

Corollary (4.1.8)**.** *The followings are equivalent:*
(1) $\mathcal{I} \trianglelefteq \mathbb{Z}_p[\chi]$ is the maximal congruence of Eisenstein series in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$.
(2) $H_c^1(\mathbb{Z}_p^{\times} \times (\mathbb{Z}/N')^{\times}; \mathbb{Z}_p^{\otimes k}[\chi]) \simeq \mathbb{Z}_p[\chi]/\mathcal{I}$.

The maximal congruences of the $\mathbb{Z}_p^{\times} \times (\mathbb{Z}/N')^{\times}$-representations $\mathbb{Z}_p^{\otimes k}[\chi]$ are easy to compute since the group is topologically finitely generated. The result of this computation is recorded in Theorem 3.1.4. We then want to find explicit formulas of Eisenstein series in $\mathcal{E}_k(p^v,\Gamma_1(N'),\chi)$ that matches the congruence of $\mathbb{Z}_p^{\otimes k}[\chi]$.

Write $\chi=\chi_p\cdot\chi'$ as above. When $|\operatorname{Im}\chi'|$ is *not* a power of p, the maximal congruence of Eisenstein series in $\mathcal{E}_k(p^v,\Gamma_1(N'),\chi)$ is realized by

$$E_{k,\chi}(q)=1-\frac{2k}{B_{k,\chi}}\sum_{n=1}^{\infty}\sigma_{k-1,\chi}(n)q^n,\ \text{ where } \sigma_{m,\chi}(n)=\sum_{0<d|n}\chi(d)d^m.$$

The argument in paper is therefore a cohomological explanation of the *denominator* of $\frac{B_{k,\chi}}{2k}$, whose arithmetic properties were described in [Car59] up to a factor of 2.

When $|\operatorname{Im}\chi'|$ is a power of p greater than 1, the maximal congruence is realized as a linearly combination of $E_{k,\chi}$ with some other basis in the Eisenstein subspace $\mathcal{E}_k(p^v,\Gamma_1(N'),\chi)$. In this case, the group cohomology $H_c^1(\mathbb{Z}_p^\times\times(\mathbb{Z}/N')^\times;\mathbb{Z}_p^{\otimes k}[\chi])$ sheds light on the *numerator* of $\frac{B_{k,\chi}}{2k}$. One such example is:

Corollary (3.2.6 and 4.1.8)**.** *Let $p>2$ be a prime and $\chi:(\mathbb{Z}/\ell)^\times\to\mathbb{C}_p^\times$ be a Dirichlet character of conductor ℓ such that $\ell\neq p$ is a prime number and $|\operatorname{Im}\chi'|=|\operatorname{Im}\chi|$ is a p-power. Denote the maximal ideal of $\mathbb{Z}_p[\chi]$ by* $\mathfrak{m}$. *Assume $(-1)^k=\chi(-1)$, $\frac{B_{k,\chi}}{2k}\in\mathbb{Z}_p[\chi]$ by* [Car59, Theorem 1]. *We then have*

$$\frac{B_{k,\chi}}{2k}\in\mathfrak{m}\iff(p-1)\nmid k.$$

This relation is reflected in the cohomological computation that

$$H_c^1(\mathbb{Z}_p^\times\times(\mathbb{Z}/\ell)^\times;\mathbb{Z}_p^{\otimes k}[\chi^{-1}])=\begin{cases}\mathbb{Z}_p[\chi]/\mathfrak{m}, & (p-1)\mid k;\\ 0, & \text{otherwise.}\end{cases}$$

The continuous group cohomology $H_c^1(\mathbb{Z}_p^\times\times(\mathbb{Z}/N')^\times;\mathbb{Z}_p^{\otimes k}[\chi])$ is on the E_2-page of a spectral sequence to compute the homotopy groups of the Dirichlet $K(1)$-local spheres, introduced in [Zha19]:

$$E_2^{s,2t}=H_c^s(\mathbb{Z}_p^\times\times(\mathbb{Z}/N')^\times;\mathbb{Z}_p^{\otimes t}[\chi^{-1}])\Longrightarrow\pi_{2t-s}\left(S^0_{K(1)}(p^v)^{h\chi}\right).$$

These Dirichlet $K(1)$-local spheres at each prime assemble into the Dirichlet J-spectra $J(N)^{h\chi}$, which are analogs of Dirichlet L-functions in chromatic homotopy theory.

We have therefore connected the homotopy groups of the Dirichlet J-spectra $J(N)^{h\chi}$ with congruences of Eisenstein series in $\mathcal{E}_k(\Gamma_1(N),\chi)$. This explains how homotopy groups of $J(N)^{h\chi}$ are related to the special values of the Dirichlet L-function $L(s;\chi)$. When χ is trivial, our argument gives a new explanation of the relation between congruences of Eisenstein series of level 1 with the image of the J-homomorphism in the stable homotopy groups of spheres.

Notations and conventions.

- Denote the Teichmüller character by the Greek letter ω and denote the sheaf of invariant differentials on various stacks by the boldface version of the same Greek letter $\boldsymbol{\omega}$.
- $\mathbb{C}_p$ is the analytic completion of $\overline{\mathbb{Q}_p}$, the algebraic closure of the rational p-adics.
- We write $\underline{G}$ for the constant G-group scheme.
- $\widehat{G}_m$ is the multiplicative formal groups, respectively. $\widehat{G}_a$ is the additive formal groups. μ_N and α_N are finite subgroup schemes of $\widehat{G}_m$ and $\widehat{G}_a$ of rank N, respectively.
- By a height 1 or slope 1 formal group $\widehat{G}$, we mean $\widehat{G}$ is étale locally isomorphic $\widehat{G}_m^{\oplus d}$, where d is the dimension of $\widehat{G}$.
- Let M be a G-representation in an R-modules and $\chi:G\to R^\times$ be a character. We write M^χ for the χ-eigensubspace of M.
- We will suppress the $\mathbb{Z}_p$ in $M\otimes_{\mathbb{Z}_p}N$ when M and N are both $\mathbb{Z}_p$-modules.

- Let χ be a Dirichlet character of conductor N. Write $N = p^v N'$, where $p \nmid N'$. Then there is a unique decomposition $\chi = \chi_p \chi'$, where the conductors of χ_p and χ' are p^v and N', respectively. We fix the meanings of N, N', v, χ_p, and χ' throughout the paper.

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1. μ_N-level structures on elliptic curves and modular forms

1.1. **The Eisenstein subspace.** Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character of conductor N. We are now going to introduce the Eisenstein series of level $\Gamma_1(N)$ and character χ, following [Hid93, §5.1] and [Ste07, Chapter 5].

Definition 1.1.1. Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. Let $\mathbb{T} \subseteq \mathrm{End}(M_k(\Gamma))$ be the subring generated by the Hecke operators. Then there is decomposition of $\mathbb{T}$-modules:

$$M_k(\Gamma) = \mathcal{E}_k(\Gamma) \oplus \mathcal{S}_k(\Gamma), \tag{1.1.2}$$

where $\mathcal{S}_k(\Gamma)$ is subspace of cusp forms, i.e. modular forms that vanish at all cusps. The subspace $\mathcal{E}_k(\Gamma)$ is the **Eisenstein subspace** of weight k and level Γ.

Example 1.1.3. Below is a family of Eisenstein series in $\mathcal{E}_k(\Gamma_1(N), \chi)$. Let $\chi_1 : (\mathbb{Z}/N_1)^\times \to \mathbb{C}^\times$ and $\chi_2 : (\mathbb{Z}/N_2)^\times \to \mathbb{C}^\times$ be two primitive Dirichlet characters of conductors N_1 and N_2. Define the Eisenstein series:

$$G_{k,\chi_1,\chi_2}(z) := \sum_{(n,m)\neq(0,0)} \frac{\chi_1(m)\chi_2^{-1}(n)}{(mNz+n)^k}$$

G_{k,χ_1,χ_2} is an Eisenstein series of weight k and level $N_1 N_2$.

Theorem 1.1.4. *Let $N > 1$ be a positive integer. $\{G_{k,\chi_1,\chi_2}(tz) \mid (N_1 N_2 t) | N, \chi_2/\chi_1 = \chi\}$ forms a basis of $\mathcal{E}_k(\Gamma_1(N), \chi)$.*

1.2. **μ_N-level structures.** As we will be working integrally and p-adically at levels divisible by p, it is necessary to specify the meaning of $\Gamma_1(N)$-level structures.

Definition 1.2.1. A μ_N-level structure on an elliptic curve C is an embedding of group schemes $\eta : \mu_N \hookrightarrow C$. Denote by $\mathcal{M}_{ell}(\mu_N)$ the moduli stack of elliptic curves with μ_N-level structures. Let R be a ring. The R points of $\mathcal{M}_{ell}(\mu_N)$ are

$$\mathcal{M}_{ell}(\mu_N)(R) = \left\{ (C/R, \eta) \;\middle|\; \begin{array}{c} C \text{ is an elliptic curve over } R \text{ and} \\ \eta : \mu_N \hookrightarrow C \text{ is an embedding of group schemes} \end{array} \right\}.$$

Define the space of modular forms of weight k and level μ_N by

$$M_k(\mu_N) := H^0(\mathcal{M}_{ell}(\mu_N), \boldsymbol{\omega}^{\otimes k}), \quad M_k(\mu_N, \chi) := M_k(\mu_N)^{\chi^{-1}},$$

where χ is a Dirichlet character of conductor N.

Lemma 1.2.2. $M_k(\Gamma_1(N), \chi) = M_k(\mu_N, \chi)$ *over* $\mathbb{C}$.

Proof. This is because $\mathcal{M}_{ell}(\Gamma_1(N))(R) \simeq \mathcal{M}_{ell}(\mu_N)(R)$ when R contains a primitive N-th root of unity. □

Proposition 1.2.3. *When $N \geq 4$, $\mathcal{M}_{ell}(\mu_N)$ is represented by a smooth affine curve over $\mathbb{Z}$.*

Proof. By [KM85, Corollary 4.7.1], it suffices to show:

(1) The forgetful map $\mathcal{M}_{ell}(\mu_N) \to \mathcal{M}_{ell}$ is relatively representable, affine, and étale.
(2) $\mathcal{M}_{ell}(\mu_N)$ is rigid, meaning that there is no non-trivial automorphism of the pair $(C, \eta : \mu_N \hookrightarrow C)$.

(1) is proved in [KM85, Section 4.9, 4.10]. (2) is proved in the [KM85, Corollary 2.7.4] when $N \geq 4$. ☐

1.3. **The q-expansion principle.** Let $\mathcal{M}_{ell}(\Gamma)_R$ be moduli stack of generalized elliptic curves over R-schemes with Γ-level structures.

Definition 1.3.1. A **cusp** in $\mathcal{M}_{ell}(\Gamma)_R$ is an embedding $\operatorname{Spf} R[\![q]\!] \to \mathcal{M}_{ell}(\Gamma)_R$ that classifies a Γ-level structure on the Tate curve $T(q)$. The **q-expansion** of a modular form $f \in H^0(\mathcal{M}_{ell}(\Gamma)_R, \boldsymbol{\omega}^{\otimes k})$ at a cusp is its image under restriction map to the said cusp.

Proposition 1.3.2 (The q-expansion principle). *A modular form $f \in H^0(\mathcal{M}_{ell}(\Gamma)_R, \boldsymbol{\omega}^{\otimes k})$ is zero iff its restriction to all cusps are zero. Furthermore, when $\mathcal{M}_{ell}(\Gamma)_R$ is connected, the restriction map to any cusp is injective.*

It follows that congruences of modular forms are determined by their q-expansions at any cusp when $\mathcal{M}_{ell}(\Gamma)_R$ is connected. By [Con07, Theorem 1.2.1], this is indeed the case when $\Gamma = \Gamma_1(N)$ and $R = \mathbb{Z}$ (so works for any ring R).

Now normalize E_{k,χ_1,χ_2} so that its coefficients are algebraic integers.

Definition 1.3.3 (Normalization of G_{k,χ_1,χ_2})**.** When χ_2 is non-trivial,

$$E_{k,\chi_1,\chi_2}(q) = \sum_{n\geq 1}\left(\sum_{0<d|n} \chi_2(d)\chi_1(n/d)d^{k-1}\right)q^n.$$

When χ_1 is the trivial character χ^0 and $\chi_2 = \chi$, we define $E_{k,\chi}$ and $E_{k,\chi^0,\chi}$ by

$$E_{k,\chi}(q) := 1 - \frac{2k}{B_{k,\chi}}\sum_{n\geq 1}\left(\sum_{0<d|n}\chi(d)d^{k-1}\right)q^n$$

$$E_{k,\chi^0,\chi}(q) := c\cdot E_{k,\chi}(q) = c_0 + c_1\sum_{n\geq 1}\left(\sum_{0<d|n}\chi(d)d^{k-1}\right)q^n, \quad c_0, c_1 \in \mathbb{Z}[\chi] \text{ are coprime and } c_0/c_1 = -\frac{B_{k,\chi}}{2k}.$$

Remark 1.3.4. As $\mathbb{Z}[\chi]$ has non-trivial unit group, the constant c is not unique in general.

Proposition 1.3.5. $E_{k,\chi_1,\chi_2}(q) \in (H^0(\mathcal{M}_{ell}(\mu_N), \boldsymbol{\omega}^{\otimes k}) \otimes_{\mathbb{Z}} \mathbb{Z}[\chi_1,\chi_2])^{\chi_1/\chi_2}$.

Proof. By Lemma 1.2.2, $E_{k,\chi_1,\chi_2} \in M_k(\mu_N)$. It is in the χ_1/χ_2-eigensubspace by Theorem 1.1.4. As the coefficients of $E_{k,\chi_1,\chi_2}(q)$ are all in $\mathbb{Z}[\chi_1,\chi_2]$ by Definition 1.3.3, the q-expansion principle Proposition 1.3.2 implies that

$$E_{k,\chi_1,\chi_2} \in H^0(\mathcal{M}_{ell}(\mu_N) \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[\chi_1,\chi_2], \boldsymbol{\omega}^{\otimes k}).$$

When the conductors of χ_1 and χ_2 are 3, their images are $\{\pm 1\}$ and $\mathbb{Z}[\chi_1,\chi_2] = \mathbb{Z}$. When the conductors of χ_1 and χ_2 are at least 4, the claim follows from Proposition 1.2.3. ☐

1.4. **p-adic modulis.** We will study congruences of Eisenstein series in $\mathcal{E}_k(\mu_N, \chi)$ p-adically.

Definition 1.4.1. An elliptic curve C over a p-complete ring is called (p-)**ordinary** if it has nodal singularity, or its reduction mod p is ordinary, i.e. the formal group $\widehat{C}$ associated to C has height 1 reduction mod p.

Denote the p-completed moduli stack of p-ordinary elliptic curve by $\mathcal{M}_{ell}^{ord}$. This is an open substack of $\mathcal{M}_{ell}$, since it is the non-vanishing locus of the Hasse invariant.

Restricted to $\mathcal{M}_{ell}^{ord}$, the μ_{p^v}-level structures on an elliptic curve C are identified with the corresponding level structures on the height 1 formal group $\widehat{C}$. As formal groups of height 1 are étale locally isomorphic to $\widehat{G}_m$, the multiplicative formal group, there is a tower of stacks:

$$\mathcal{M}_{ell}^{triv} \longrightarrow \cdots \longrightarrow \mathcal{M}_{ell}^{ord}(p^2) \longrightarrow \mathcal{M}_{ell}^{ord}(p) \longrightarrow \mathcal{M}_{ell}^{ord},$$

where $\mathcal{M}_{ell}^{ord}(p^v)$ and $\mathcal{M}_{ell}^{triv}$ are the moduli stacks of the pairs $(C, \eta : \mu_{p^v} \xrightarrow{\sim} \widehat{C}[p^v])$ and $(C, \eta : \widehat{G}_m \xrightarrow{\sim} \widehat{C})$ respectively, where C is an ordinary elliptic curves. The forgetful map $\mathcal{M}_{ell}^{ord}(p^v) \to \mathcal{M}_{ell}^{ord}$ is a $(\mathbb{Z}/p^v)^\times$-torsor and $\mathcal{M}_{ell}^{triv} \to \mathcal{M}_{ell}^{ord}$ is a $\mathbb{Z}_p^\times$-torsor. There is a pullback diagram of towers of stacks:

$$\begin{tikzcd}
\mathcal{M}_{ell}^{triv} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \cdots \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}_{ell}^{ord}(p^2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}_{ell}^{ord}(p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}_{ell}^{ord} \arrow[d] \\
\operatorname{Spf} \mathbb{Z}_p \arrow[r] & \cdots \arrow[r] & B(1+p^2\mathbb{Z}_p) \arrow[r] & B(1+p\mathbb{Z}_p) \arrow[r] & B\mathbb{Z}_p^\times
\end{tikzcd} \tag{1.4.2}$$

Proposition 1.4.3. [Kat75; Beh14] *When $p > 2$ or $p = 2$ and $v > 1$, $\mathcal{M}_{ell}^{ord}(p^v)$ and $\mathcal{M}_{ell}^{triv}$ are affine formal schemes. In particular, $\mathcal{M}_{ell}^{triv} \simeq \operatorname{Spf} D_p$ where D_p is the ring of divided congruences of p-adic modular forms.*

The strategy now is to relate congruences of $E_{k,\chi}$ to finite subgroups of the formal groups and formal A-modules associated to p-ordinary elliptic curves. Below are some facts about needed in the study of formal group of a p-ordinary elliptic curve.

Proposition 1.4.4. *Let C be a p-ordinary elliptic curve over a $\mathbb{Z}_p$-algebra. Denote its formal group by $\widehat{C}$.*

(1) C has a canonical subgroup H of order p, where $H = \widehat{C}[p]$.
(2) The quotient map $\varphi : C \mapsto C/H$ is the relative Frobenius map on $\mathcal{M}_{ell}^{ord}$.
(3) Let $f(q)$ be the q-expansion of a modular form over $\mathcal{M}_{ell}^{ord}$, then $\varphi^* f(q) = f(q^p)$.
(4) There is an isomorphism of invertible sheaves $F : \boldsymbol{\omega} \xrightarrow{\sim} \varphi^*\boldsymbol{\omega}$ over $\mathcal{M}_{ell}^{ord}$, where $\boldsymbol{\omega}$ is the sheaf of invariant differentials of C.

We conclude by comparing the integral and p-adic moduli problems.

Lemma 1.4.5. *If an elliptic curve C admits a μ_N-level structure, then it is p-ordinary for all primes $p \mid N$.*

Proof. As μ_p is a subgroup scheme of μ_N when $p \mid N$, it suffices to prove the case when $N = p$. Notice μ_p is p-torsion, any embedding of μ_p into an elliptic curve C must factor through $C[p]$. When C is p-supersingular, $C[p] = \widehat{C}[p]$. Thus it reduces to showing that there is no embedding of μ_p into a height 2 formal group.

Using Dieudonné theory of finite groups schemes, we can show the only finite subgroup scheme of rank p in a height 2 formal group is étale locally isomorphic to α_p, which is not étale locally isomorphic to μ_p. □

Definition 1.4.6. Let $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$ be the stack whose R-points are

$$\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))(R) = \left\{ (C/R, \eta_p, \eta') \,\middle|\, \begin{array}{l} C \text{ is a } p\text{-ordinary elliptic curve over } R, \\ \eta_p : \mu_{p^v} \xrightarrow{\sim} \widehat{C}[p^v], \quad \eta' : \mathbb{Z}/N' \hookrightarrow C[N] \end{array} \right\}.$$

Proposition 1.4.7. *Write $N = p^v \cdot N'$, where $p \nmid N'$. Then we have*

$$(\mathcal{M}_{ell}(\mu_N))_p^\wedge \simeq \begin{cases} \mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), & \text{if } p \mid N; \\ (\mathcal{M}_{ell})_p^\wedge(\Gamma_1(N)), & \text{if } p \nmid N. \end{cases}$$

Proof. Canonical subgroups and Lemma 1.4.5. □

Proposition 1.4.8. *The forgetful map $\xi : \mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')) \to \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ is a $(\mathbb{Z}/N)^\times$-torsor of stacks.*

Proof. One can check this by unraveling the definition of G-torsors of stacks. $\square$

Proposition 1.4.9. *The stack $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$ is represented by a smooth formal affine curve over $\mathbb{Z}_p$ in the following cases:*

- $N = p^v \cdot N' \geq 4$ *for any* p.
- $N = p = 3$.
- $N = N' = 3$ *and* $p \equiv 2 \mod 3$.

Proof. By Proposition 1.4.7, $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$ is the p-completion (when $p \mid N$), or a distinguished open substack of the p-completion (when $p \nmid N$) of $\mathcal{M}_{ell}(\mu_N)$. As the latter is represented by a smooth affine curve over $\mathbb{Z}$ by Proposition 1.2.3, the first case of the claim follows.

When $N = p = 3$, $\mathcal{M}_{ell}^{ord}(3)$ is affine by Proposition 1.4.3.

When $N = 3$ and $p \neq 3$, it suffices to show the moduli problem is rigid as in the proof of Proposition 1.2.3. Let ε be a nontrivial automorphism of C that preserves a $\Gamma_1(3)$-level structure $\eta' : \underline{\mathbb{Z}/3} \hookrightarrow C[3]$. Adapting the proof of [KM85, Corollary 2.7.3] to the $N = 3$ case, we can show ε must satisfy $\varepsilon^2 + \varepsilon + 1 = 0$. This implies ε is an element of order 3 in $\mathrm{Aut}(C)$. By [Sil09, Proposition A.1.2.(c)], $\mathrm{Aut}(C)$ has an element of order 3 iff its j-invariant is 0. By [Sil09, Example V.4.4, Exercise 5.7], the $j = 0$ elliptic curve is p-supersingular when $p \equiv 2 \mod 3$. As a result, when $p \equiv 2 \mod 3$, there is no non-trivial automorphism of a p-ordinary elliptic C that preserves a $\Gamma_1(3)$-structure. This shows the moduli problem $\mathcal{M}_{ell}^{ord}(\Gamma_1(3))$ is rigid at such primes, and hence represented by a smooth formal affine curve over $\mathbb{Z}_p$. $\square$

Remark 1.4.10. The moduli problem $\mathcal{M}_{ell}^{ord}(\Gamma_1(3))$ is NOT rigid when $p \equiv 1 \mod 3$. For such primes, the $j = 0$ elliptic curve C is p-ordinary. C has an automorphism ε of order 3. As $C[3]$ is isomorphic to the constant groups scheme $\underline{\mathbb{Z}/3}^{\oplus 2}$, the automorphism ε restricts to an element of order 3 in $\mathrm{GL}_2(\mathbb{Z}/3)$. From the identity $0 = \varepsilon^3 - 1 = (\varepsilon - 1)^3$ in $\mathrm{End}(C[3]) \simeq M_2(\mathbb{Z}/3)$, ε is unipotent. Then there is a basis $\{P, Q\}$ of $C[3]$ under which ε acts by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let $\eta' : \underline{\mathbb{Z}/3} \hookrightarrow C[3]$ that sends $1 \in \underline{\mathbb{Z}/3}$ to $P \in C[3]$. The matrix representations of ε shows it is an automoprhism of the pair (C, η'). Consequently, $\mathcal{M}_{ell}^{ord}(\Gamma_1(3))$ is the not rigid and is therefore not represented by a scheme.

Proposition 1.4.11. *Let χ be a Dirichlet character of conductor N, where $N = p^v N'$ with $p \nmid N'$. Denote the Eisenstein subspace in the χ^{-1}-eigensubspace in $H^0(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])$ by $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$. Then we have a decomposition:*

$$\mathcal{E}_k(\mu_N, \chi)_p^\wedge \simeq \bigoplus_{[\sigma] \in \mathrm{Coker}\, \iota^*} \mathcal{E}_k(p^v, \Gamma_1(N'), \iota \circ \sigma \circ \chi),$$

where $\iota : \mathbb{Q}(\chi) \hookrightarrow \mathbb{C}_p$ is a field extension and $\iota^* : \mathrm{Gal}(\iota(\mathbb{Q}(\chi))/\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$ is the induced map of ι on Galois groups.

Proof. This is a result of the equivalence of p-adic $(\mathbb{Z}/N)^\times$-representations [Zha19, Corollary A.3.5]:

$$\mathbb{Z}[\chi] \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \bigoplus_{[\sigma] \in \mathrm{Coker}\, \iota^*} \mathbb{Z}_p[\iota \circ \sigma \circ \chi].$$

$\square$

Corollary 1.4.12. *Let χ_1 and χ_2 be p-adic Dirichlet characters of conductor N_1 and N_2 respectively. Then the normalized Eisenstein series E_{k,χ_1,χ_2} in Definition 1.3.3 defines a p-adic Eisenstein series in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi_2/\chi_1)$, where $N = N_1 N_2 = p^v N'$ and $p \nmid N'$.*

2. Eisenstein series and Galois representations

In this section, we adapt Katz's explanation of congruences of E_{2k} as p-adic modular forms in [Kat73b] to study the congruences of p-adic Eisenstein series with level $(\mu_{p^v}, \Gamma_1(N'))$.

Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$ be a Dirichlet character of conductor N. Write $N = p^v N'$, where $p \nmid N'$. Then χ is uniquely factorized as a product $\chi = \chi_p \cdot \chi'$, where χ_p and χ' have conductors p^v and N', respectively. Let $\mathbb{Z}_p^{\otimes k}[\chi]$ be the p-adic $(\mathbb{Z}/N)^\times$-representation, whose underlying module is $\mathbb{Z}_p[\chi]$ and where $(a,b) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$ acts on $\mathbb{Z}_p[\chi]$ by multiplication by $a^k \cdot \chi_p(a) \cdot \chi'(b)$. Throughout this section, we abbreviate the Eisenstein subspace in $H^0(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])^{\chi^{-1}}$ by $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$.

Theorem (Main Theorem 2.6.1)**.** *Let $\mathcal{I}$ be an ideal of $\mathbb{Z}_p[\chi]$. The followings are equivalent:*

(i). *There is an Eisenstein series f in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ with q-expansion $f(q) \in 1 + \mathcal{I}q[\![q]\!]$.*

(v). *The $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-representation $\mathbb{Z}_p^{\otimes k}[\chi]$ is trivial modulo $\mathcal{I}$.*

The proof of the Main Theorem relies heavily on the Dieudonné theory of formal groups and formal A-modules, which will be briefly reviewed in the next subsection. A reference for the general theory of formal groups and Dieudonné theory can be found in [Dem72].

2.1. **Review of Dieudonné modules and Galois descent of formal groups.** Let R be a smooth $\mathbb{Z}_p$-algebra such that R/p is an integrally closed domain and R admits an endomorphism $\varphi : R \to R$ that lifts the p-th power map on R.

The **Dieudonné module** $\mathbb{D}(\widehat{G})$ of a formal group $\widehat{G}_0$ over R/p is a triple

$$\mathbb{D}(\widehat{G}) = (M, F : M \longrightarrow \varphi^* M, V : \varphi^* M \longrightarrow M),$$

where $M = PH^1_{\mathrm{dR}}(\widehat{G}/R)$ is the primitives in the de-Rham cohomology for some lift $\widehat{G}$ of $\widehat{G}_0$ to R and $FV = p = VF$ on the respective domains. Formal groups of the same height $h < \infty$ over R/p are étale locally isomorphic to each other. It follows that their isomorphism classes are classified by the continuous Galois cohomology $H^1_c(\pi_1^{ét}(R/p); \mathrm{Aut}(\Gamma_h))$, where Γ_h is the height h Honda formal group. The Galois cohomology class $[\rho] \in H^1_c(\pi_1^{ét}(R/p); \mathrm{Aut}(\Gamma_h))$ that corresponds to $\widehat{G}_0$ is called the **Galois descent data** of $\widehat{G}_0$.

When $\widehat{G}$ has height (slope) 1, $PH^1_{\mathrm{dR}}(\widehat{G}/R) = \boldsymbol{\omega}(\widehat{G})$ is the sheaf of invariant differentials of $\widehat{G}$ and $F : M \longrightarrow \varphi^* M$ is an isomorphism. As a result, the Verschiebung V is determined by F. In this case, we will write $\mathbb{D}(\widehat{G}) = (\boldsymbol{\omega}(\widehat{G}), F : \boldsymbol{\omega}(\widehat{G}) \xrightarrow{\sim} \varphi^* \boldsymbol{\omega}(\widehat{G}))$.

Example 2.1.1. Let R be a $\mathbb{Z}_p$-algebra and $\varphi : R \to R$ be a lift of Frobenius map. . Denote the Dieudonné module of $\widehat{G}_m/R$, the multiplicative formal group over R by $\mathbb{D}(\widehat{G}_m) = (M, F : M \xrightarrow{\sim} \varphi^* M)$. Then M is a free R-module of rank 1 generated by an element γ such that $F(\gamma) = \gamma$.

The Galois descent data of height 1 formal groups are described by the following:

Proposition 2.1.2. $\mathrm{Hom}(\pi_1^{ét}(R), \mathbb{Z}_p^\times)$ *is an abelian group and classifies isomorphism classes of formal groups over R with height* 1 *reductions modulo p. The trivial map in* $\mathrm{Hom}(\pi_1^{ét}(R), \mathbb{Z}_p^\times)$ *corresponds to* $\widehat{G}_m$

Proof. When $h = 1$, $\Gamma_1 = \widehat{G}_m$ and $\mathrm{Aut}(\widehat{G}_m) \simeq \mathbb{Z}_p^\times$ is an abelian group. Since p is (topologically) nilpotent in R, $\pi_1^{ét}(R) \simeq \pi_1^{ét}(R/p)$. Since formal groups of height 1 over R/p are étale locally isomorphic to $\widehat{G}_m$, $H^1_c(\pi_1^{ét}(R); \mathbb{Z}_p^\times) \simeq H^1_c(\pi_1^{ét}(R/p); \mathbb{Z}_p^\times)$ classifies isomorphism classes of formal groups of height 1 over R/p. This shows the "constant 1" Galois cohomology class corresponds to $\widehat{G}_m$ over R/p. This Galois cohomology is an abelian group since $\mathbb{Z}_p^\times$ is an abelian group. As the étale fundamental group acts trivially on $\mathbb{Z}_p^\times$, we have $H^1_c(\pi_1^{ét}(R); \mathbb{Z}_p^\times) \simeq \mathrm{Hom}(\pi_1^{ét}(R), \mathbb{Z}_p^\times)$. This shows $\widehat{G}_m$ is classified by the trivial group homomorphism in $\mathrm{Hom}(\pi_1^{ét}(R), \mathbb{Z}_p^\times)$.

By Lubin-Tate deformation theory of formal groups, height 1 formal groups over R/p have unique deformations to R. This shows $\mathrm{Hom}(\pi_1^{\acute{e}t}(R), \mathbb{Z}_p^\times) \simeq H_c^1(\pi_1^{\acute{e}t}(R); \mathbb{Z}_p^\times)$ classifies isomorphism classes of formal groups over R with height 1 reductions modulo p. □

This suggests a closed symmetric monoidal structure in the category of 1-dimensional formal groups of height 1. Let $\rho_i : \pi_1^{\acute{e}t}(R) \to \mathbb{Z}_p^\times$ be the Galois descent data for the height 1 formal groups $\widehat{G}_i$, $i = 1, 2$. Then the Galois descent data for $\widehat{G}_1 \otimes \widehat{G}_2$ is $\rho_1 \cdot \rho_2$. In terms of Dieudonné modules, this monoidal structure is described by

$$\mathbb{D}(\widehat{G}_1 \otimes \widehat{G}_2) = (\boldsymbol{\omega}_1 \otimes_R \boldsymbol{\omega}_2, \quad F_1 \otimes F_2 : \boldsymbol{\omega}_1 \otimes_R \boldsymbol{\omega}_2 \xrightarrow{\sim} \varphi^*\boldsymbol{\omega}_1 \otimes_{\varphi^* R} \varphi^*\boldsymbol{\omega}_2 \simeq \varphi^*(\boldsymbol{\omega}_1 \otimes_R \boldsymbol{\omega}_2)),$$

where $\mathbb{D}(\widehat{G}_i) = (\boldsymbol{\omega}_i, F_i, V_i)$. Below are two relevant examples in this paper:

Example 2.1.3. Let C be the universal elliptic curve over $\mathcal{M}_{ell}^{ord}$ and $\widehat{C}$ be its formal group. $\widehat{C}$ is a height 1 formal group since C is a p-ordinary elliptic curve. Denote the Galois descent data for $\widehat{C}$ by $\rho^1 : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}) \to \mathbb{Z}_p^\times$. The pair $(\boldsymbol{\omega}, F : \boldsymbol{\omega} \xrightarrow{\sim} \varphi^*\boldsymbol{\omega})$ described in Proposition 1.4.4 is the Dieudonné module of $\widehat{C}$, where $F(f(q)) = f(q^p)$ on q-expansions of modular forms. Denote of the k-th monoidal power of $\widehat{C}$ by $\widehat{C}^{\otimes k}$. The Galois descent data for $\widehat{C}^{\otimes k}$ is

$$\rho^k : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}) \xrightarrow{\rho^1} \mathbb{Z}_p^\times \xrightarrow{(-)^k} \mathbb{Z}_p^\times,$$

The Dieudonné module of $\widehat{C}^{\otimes k}$ is

$$\mathbb{D}(\widehat{C}^{\otimes k}) = (\boldsymbol{\omega}^{\otimes k}, F^{\otimes k} : \boldsymbol{\omega}^{\otimes k} \xrightarrow{\sim} \varphi^*\boldsymbol{\omega}^{\otimes k}),$$

where $F^{\otimes k}(f(q)) = f(q^p)$ on q-expansions.

As the Eisenstein series we study in this paper have coefficients in $\mathbb{Z}_p[\chi]$, it is necessary to work with formal $\mathbb{Z}_p[\chi]$-modules. Let A be an algebra. A formal A-module is a formal group $\widehat{G}$ together with an embedding of algebras $i : A \hookrightarrow \mathrm{End}_{FG}(\widehat{G})$ such that the composite

$$A \hookrightarrow \mathrm{End}_{FG}(\widehat{G}) \longrightarrow \mathrm{End}(\boldsymbol{\omega}(\widehat{G}))$$

realizes $\boldsymbol{\omega}(\widehat{G})$ as an A-module. We will write the power series representation of $i(a)$ by $[a]$. Any formal group $\widehat{G}$ comes with a unique formal $\mathbb{Z}$-module structure. When $\widehat{G}$ is defined over a p-complete ring R, this formal $\mathbb{Z}$-module structure extends (uniquely) to a formal $\mathbb{Z}_p$-module structure, since $\lim_{v\to\infty}[p^v](t) = 0$ in $R[\![t]\!]$.

Construction 2.1.4. When A is $\mathbb{Z}_p$-algebra that is a finite free $\mathbb{Z}_p$-module, we define a formal A-module $\widehat{G} \otimes A$ out of a 1-dimensional formal group $\widehat{G}$. The underlying formal group of $\widehat{G} \otimes A$ is $\widehat{G}^{\oplus r}$, where r is the rank of A as a free $\mathbb{Z}_p$-module. The A-action on $\widehat{G} \otimes A = \widehat{G}^{\oplus r}$ is given by

$$A = \mathrm{End}_A(A) \hookrightarrow \mathrm{End}_{\mathbb{Z}_p}(\mathbb{Z}_p^{\oplus r}) \hookrightarrow \mathrm{End}_{FG}(\widehat{G}^{\oplus r}).$$

where the first map is induced by $A \simeq \mathbb{Z}_p^{\oplus r}$. Write $\mathbb{D}(\widehat{G}) = (\boldsymbol{\omega}(\widehat{G}), F, V)$. The Dieudonné module of $\widehat{G} \otimes A$ is

$$\mathbb{D}(\widehat{G} \otimes A) = \mathbb{D}(\widehat{G}) \otimes A = (\boldsymbol{\omega}(\widehat{G}) \otimes A, F \otimes 1, V \otimes 1).$$

If the height of $\widehat{G}$ is h, let $[\rho] \in H_c^1(\pi_1^{\acute{e}t}(R); \mathrm{Aut}(\Gamma_h))$ be the Galois descent data for $\widehat{G}$. $\widehat{G} \otimes A$ is étale locally isomorphic to $\Gamma_h \otimes A$ as a formal A-module. Then we have an embedding of algebras:

$$i : \mathrm{End}(\Gamma_h) \hookrightarrow \mathrm{End}_{\text{formal } A\text{-mod}}(\Gamma_h \otimes A) \simeq \mathrm{End}(\Gamma_h) \otimes A \quad g \longmapsto g \otimes 1.$$

i restricts to a group homomorphism on the units (automorphisms). The Galois descent data for $\widehat{G} \otimes A$ is then the image of $[\rho]$ under the induced map of i in Galois cohomology.

2.2. **Sketch of the proof.** The proof of Theorem 2.6.1 has three steps, which will be explained in details in the rest of this section. Here is a sketch:

I. By viewing the Dirichlet character χ as a Galois cohomology class, we construct a formal $\mathbb{Z}_p[\chi]$-module $\widehat{C}^{k,\chi}$ of height 1 over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ such that
$$H^0(\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N')),\boldsymbol{\omega}^{\otimes k}\otimes\mathbb{Z}_p[\chi])^{\chi^{-1}}\simeq H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')),\boldsymbol{\omega}(\widehat{C}^{k,\chi})).$$
In this way, we translate congruences of Eisenstein series in $\mathcal{E}_k(p^v,\Gamma_1(N'),\chi)$ to those of elements in the Dieudonné module of $\widehat{C}^{k,\chi}$.

II. By reformulating a Riemann-Hilbert type correspondence in [Kat73b] using the Dieudonné theory of height 1 formal A-modules and their finite subgroups, we relate the congruence of the Dieudonné module $\mathbb{D}(\widehat{C}^{k,\chi})$ with that of the Galois descent data $[\rho^{k,\chi}]$ for $\widehat{C}^{k,\chi}$.

III. The Galois cohomology class $[\rho^{k,\chi}]\in H_c^1(\pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')));(\mathbb{Z}_p[\chi])^\times)$ is represented by a group homomorphism that factorizes as
$$\rho^{k,\chi}:\pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))\xrightarrow{\rho^1\times\lambda_{N'}}\mathbb{Z}_p^\times\times(\mathbb{Z}/N')^\times\xrightarrow{(a,b)\mapsto\chi_p(a)\chi'(b)a^k}(\mathbb{Z}_p[\chi])^\times.$$
Here $\rho^1:\pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))\to\mathbb{Z}_p^\times$ is the Galois descent data for $\widehat{C}$ described in Example 2.1.3 and $\lambda_{N'}:\pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))\to(\mathbb{Z}/N')^\times$ classifies the $(\mathbb{Z}/N')^\times$-torsor $\mathcal{M}_{ell}^{ord}(\Gamma_1(N'))\to\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. The theorem then follows from the surjectivity of $\rho^1\times\lambda_{N'}$.

2.3. **Step I: Dirichlet characters and Galois descent.** The first step in the proof of the Main Theorem is to view the Dirichlet character $\chi:(\mathbb{Z}/N)^\times\to\mathbb{C}_p^\times$ as the Galois descent data for a formal A-module $\widehat{C}^{k,\chi}$ of height 1 over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ along the $(\mathbb{Z}/N)^\times$-torsor $\xi:\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N'))\longrightarrow\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. (See Proposition 1.4.8 for a proof that ξ is a $(\mathbb{Z}/N)^\times$-torsor.)

Construction 2.3.1. Let (C,η_p,η') be the universal elliptic curve with the given level structures over $\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N'))$ and $\widehat{C}$ be its formal group. Then $\widehat{C}^{\otimes k}\otimes\mathbb{Z}_p[\chi]$ is a formal $\mathbb{Z}_p[\chi]$-module of height 1. Notice that:

- The automorphism group of $\widehat{C}^{\otimes k}\otimes\mathbb{Z}_p[\chi]$ as a formal $\mathbb{Z}_p[\chi]$-module is $(\mathbb{Z}_p[\chi])^\times$.
- The forgetful map $\xi:\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N'))\to\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ is a $(\mathbb{Z}/p^v)^\times\times(\mathbb{Z}/N')^\times\simeq(\mathbb{Z}/N)^\times$-torsor.

The Dirichlet character $\chi:(\mathbb{Z}/N)^\times\to\mathbb{C}_p^\times$ then represents a cohomology class
$$\begin{aligned}[\chi]\in&H^1((\mathbb{Z}/N)^\times;(\mathbb{Z}_p[\chi])^\times)\\ \simeq&H^1(\mathrm{Aut}_{\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))}(\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N')));\mathrm{Aut}_{\text{formal }\mathbb{Z}_p[\chi]\text{-mod}}(\widehat{C}^{\otimes k}\otimes\mathbb{Z}_p[\chi])),\end{aligned}$$
where $(\mathbb{Z}/N)^\times$ acts on $(\mathbb{Z}_p[\chi])^\times$ trivially. This cohomology group classifies isomorphism classes of formal $\mathbb{Z}_p[\chi]$-modules $\widehat{G}$ over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ such that $\xi^*\widehat{G}\simeq\widehat{C}^{\otimes k}\otimes\mathbb{Z}_p[\chi]$ over $\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N'))$. In this way, the cohomology class $[\chi]$ corresponds to a formal $\mathbb{Z}_p[\chi]$-module $\widehat{C}^{k,\chi}$ over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. More precisely, fix an isomorpshim $\eta:\xi^*\widehat{C}^{k,\chi}\xrightarrow{\sim}\widehat{C}^{\otimes k}\otimes\mathbb{Z}_p[\chi]$, then for any $\sigma\in(\mathbb{Z}/N)^\times\simeq\mathrm{Aut}_{\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))}(\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N')))$, we have a commutative diagram of isomorphisms:
$$\begin{tikzcd}
\xi^*\widehat{C}^{k,\chi} \arrow[r, "\sigma\otimes 1"] \arrow[d, "\eta"] & \sigma^*\xi^*\widehat{C}^{k,\chi} \arrow[r, equal] \arrow[d, "\sigma^*\eta"] & \xi^*\widehat{C}^{k,\chi} \arrow[d, "\sigma^*\eta"] \\
\widehat{C}^{\otimes k}\otimes\mathbb{Z}_p[\chi] \arrow[r] \arrow[rr, bend right, "{[\chi(\sigma)]}"'] & \sigma^*(\widehat{C}^{\otimes k}\otimes\mathbb{Z}_p[\chi]) \arrow[r, equal] & \widehat{C}^{\otimes k}\otimes\mathbb{Z}_p[\chi]
\end{tikzcd}$$

In this diagram,

- $[\chi(\sigma)]$ is defined in Construction 2.1.4.
- $\sigma^*\eta = \eta$ since $(\mathbb{Z}/N)^\times$ acts on $(\mathbb{Z}_p[\chi])^*$ trivially.
- The correspondence between $\widehat{C}^{k,\chi}$ and χ is independent of the choice of the isomorpshism η, since $\mathrm{Aut}_{\mathbb{Z}_p[\chi]}(\widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]) = (\mathbb{Z}_p[\chi])^\times$ is abelian.

Let $\boldsymbol{\omega}^{k,\chi} := \boldsymbol{\omega}(\widehat{C}^{k,\chi})$ be the sheaf of invariant differentials of $\widehat{C}^{k,\chi}$. $\boldsymbol{\omega}^{k,\chi}$ is a locally free finitely generated sheaf over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$, since it is the cotangent sheaf of a formal scheme that is étale locally isomorphic to $\widehat{\mathbb{A}}^r$, where r is the rank of $\mathbb{Z}_p[\chi]$ as a $\mathbb{Z}_p$-module.

Proposition 2.3.2. *$\xi^*\boldsymbol{\omega}^{k,\chi} \simeq \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi]$ over $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$. The sheaf cohomology of $\boldsymbol{\omega}^{k,\chi}$ is computed as follows:*

(1). $H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi}) \simeq H^0(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])^{\chi^{-1}}$ *for all* $N > 1$.

(2). *When $N > 3$ or $N = 3$ and $p \not\equiv 1 \mod 3$, we have for all $s \geq 0$:*

$$H^s(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi}) \simeq H^s((\mathbb{Z}/N)^\times; H^0(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])).$$

(3). *When $p \nmid \phi(N) = |(\mathbb{Z}/N)^\times|$, we have for all $t \geq 0$:*

$$H^t(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi}) \simeq H^t(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])^{\chi^{-1}}.$$

(4). *In particular, when N and p satisfy both conditions above, we further have:*

$$H^s(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi}) = \begin{cases} H^0(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])^{\chi^{-1}}, & s = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The functor $\boldsymbol{\omega}$ is compatible with pullbacks, yielding

$$\xi^*\boldsymbol{\omega}^{k,\chi} = \xi^*\boldsymbol{\omega}(\widehat{C}^{k,\chi}) \simeq \boldsymbol{\omega}(\xi^*\widehat{C}^{k,\chi}) \simeq \boldsymbol{\omega}(\widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]) = \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi].$$

To compute $H^s(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi})$, we use the Hochschild-Serre spectral sequence [Mil80, Theorem 2.20]:

$$E_2^{s,t} = H^s((\mathbb{Z}/N)^\times; H^t(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \xi^*\boldsymbol{\omega}^{k,\chi})) \Longrightarrow H^{s+t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi}), \tag{2.3.3}$$

where $\sigma \in (\mathbb{Z}/N)^\times$ acts on $\xi^*\boldsymbol{\omega}^{k,\chi} \simeq \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi]$ by the Galois descent data $1 \otimes \chi(\sigma)$. As the spectral sequence is concentrated in the first quadrant, its $E_2^{0,0}$-term receives and supports no differentials. This implies (1).

By Proposition 1.4.9, the stack $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$ is a formal affine scheme when $N \geq 4$ or $N = 3$ and $p \not\equiv 1 \mod 3$. It follows that (2.3.3) is concentrated in the $t = 0$ line in those cases. As a result, the spectral sequence collapses on the E_2-page and we have proved (2).

When $p \nmid \phi(N) = |(\mathbb{Z}/N)^\times|$, the group cohomology of $(\mathbb{Z}/N)^\times$ with coefficients in $\mathbb{Z}_p$-modules vanishes in positive degrees. It follows that (2.3.3) is concentrated in the $s = 0$ line in this case and thus collapses on the E_2-page. This implies (3).

(4) is the intersection of (2) and (3). $\square$

Remark 2.3.4. Note that 2 is the only prime p dividing $\phi(3) = 2$. The spectral sequence (2.3.3) collapses on the E_2-page for all $N \geq 3$ and p.

We have proved in Proposition 2.3.2:

$$H^0(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])^{\chi^{-1}} \simeq H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi}). \tag{2.3.5}$$

Write $\mathbb{D}(\widehat{C}^{k,\chi}) = (\boldsymbol{\omega}^{k,\chi}, F^{k,\chi} : \boldsymbol{\omega}^{k,\chi} \xrightarrow{\sim} \varphi^*\boldsymbol{\omega}^{k,\chi})$. The Frobenius homomorphism $F^{k,\chi}$ of $\widehat{C}^{k,\chi}$ descends from that of $\xi^*\widehat{C}^{k,\chi} \simeq \widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]$. By Example 2.1.3 and Construction 2.1.4, we have

$$\xi^*F^{k,\chi} = F^{\otimes k} \otimes 1 : \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi] \xrightarrow{\sim} \varphi^*\boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi].$$

Notice $F^{\otimes k}\otimes 1$ commutes with the Galois descent data $1\otimes\chi(\sigma)$ for $\sigma\in(\mathbb{Z}/N)^\times$, we have shown

Proposition 2.3.6 (Step I)**.** *Let $f\in H^0(\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N')),\boldsymbol{\omega}^{\otimes k}\otimes\mathbb{Z}_p[\chi])^{\chi^{-1}}\simeq H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')),\boldsymbol{\omega}^{k,\chi})$ be an Eisenstein series, then $F^{k,\chi}(f(q))=(F^{\otimes k}\otimes 1)(f(q))=f(q^p)$. Let $\mathcal{I}\trianglelefteq\mathbb{Z}_p[\chi]$ be an ideal. Then the followings are equivalent:*

(i). *There is an Eisenstein series $f\in\mathcal{E}_k(p^v,\Gamma_1(N'),\chi)$ such that $f(q)\in 1+\mathcal{I}q[\![q]\!]$.*
(ii). *There is a generator $\gamma\in H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')),\boldsymbol{\omega}^{k,\chi})$ as an $H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')),\mathcal{O})\otimes\mathbb{Z}_p[\chi]$-module such that $F^{k,\chi}(\gamma)\equiv\gamma\mod\mathcal{I}$.*

This concludes step I in Section 2.2.

2.4. **Step II: From Dieudonné modules to Galois representations.** One major tool Katz used in [Kat73b, Chapter 4] to explain the congruences of the normalized Eisenstein series E_{2k} of level 1 is a Riemann-Hilbert type correspondence. In this subsection, we reformulate the correspondence in terms of formal A-modules and their finite subgroup schemes, and then apply it to the formal $\mathbb{Z}_p[\chi]$-module $\widehat{C}^{k,\chi}$ over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ we constructed in Construction 2.3.1.

Let κ be a perfect field of characteristic p containing $\mathbb{F}_q$ and $\mathbb{W}_m(\mathbb{F}_q)$ be the ring of Witt vectors of length m on $\mathbb{F}_q$. Let S_m be a flat affine $\mathbb{W}_m(\kappa)$-scheme whose special fiber is normal, reduced, and irreducible. Assume S_m is formally smooth, so that it admits an endomorphism $\varphi:S_m\to S_m$ that lifts the q-th power map on S_m/p. Then Katz proved

Theorem 2.4.1. [Kat73b, Proposition 4.1.1, Remark 4.1.2.1]
There is an equivalence of closed symmetric monoidal categories:

$$\left\{\begin{array}{c}\textit{Finite locally free sheaves } \mathscr{F} \textit{ on } S_m\\ \textit{with an isomorphism } F:\varphi^*\mathscr{F}\xrightarrow{\sim}\mathscr{F}\end{array}\right\}\cong\left\{\begin{array}{c}\textit{Finite free } \mathbb{W}_m(\mathbb{F}_q)\textit{-modules}\\ \textit{with continuous } \pi_1^{\acute{e}t}(S_m)\textit{-actions}\end{array}\right\}.$$

Proposition 2.4.2. [Kat73a, Remark 5.5] *Theorem 2.4.1 holds for affine formal schemes S over $\mathbb{W}(\kappa)$ under the same assumption. That is, there is an equivalence of closed symmetric monoidal category:*

$$\left\{\begin{array}{c}\textit{Finite locally free sheaves } \mathscr{F} \textit{ on } S\\ \textit{with an isomorphism } F:\varphi^*\mathscr{F}\xrightarrow{\sim}\mathscr{F}\end{array}\right\}\cong\left\{\begin{array}{c}\textit{Finite free } \mathbb{W}(\mathbb{F}_q)\textit{-modules}\\ \textit{with continuous } \pi_1^{\acute{e}t}(S)\textit{-actions}\end{array}\right\}.$$

This equivalence of Katz is essentially an equivalence of Dieudonné module and Galois descent data of a formal group and its finite subgroups. Let A be a $\mathbb{Z}_p$-algebra that is finite free as a $\mathbb{Z}_p$-module and $\widehat{G}$ be formal A-module of height 1. Let $\mathcal{I}\trianglelefteq A$ be an ideal.

Definition 2.4.3. Define $\widehat{G}[\mathcal{I}]$ to be the kernel of all the endomorphisms in $\mathcal{I}\trianglelefteq A\hookrightarrow\operatorname{End}(\widehat{G})$. If $\widehat{G}=\operatorname{Spf}R[\![t]\!]$ has a coordinate, then $\widehat{G}[\mathcal{I}]=\operatorname{Spf}R[\![t]\!]/([a](t)\mid a\in\mathcal{I})$ as a finite flat scheme. When $\mathcal{I}=(a)$ is a principal ideal, $\widehat{G}[\mathcal{I}]=\widehat{G}[a]=\operatorname{Spf}R[\![t]\!]/([a](t))$.

Proposition 2.4.4. *Let $\widehat{G}$ be a formal A-module. Write the Dieudonné module of $\widehat{G}$ as $\mathbb{D}(\widehat{G})=(M,F,V)$. Then M has an A-module structure and the homomorphisms F and V are A-linear. The Dieudonné module of $\widehat{G}[\mathcal{I}]$ is $\mathbb{D}(\widehat{G})/\mathcal{I}\coloneqq(M/\mathcal{I}M,F:M/\mathcal{I}M\to\varphi^*(M/\mathcal{I}M),V:\varphi^*(M/\mathcal{I}M)\to M/\mathcal{I}M)$.*

Proposition 2.4.5. *Let $\widehat{G}$ be a formal A-module over R that is isomorphic to $\widehat{G}'$ over the separable closure R^{sep} of R. Let the cohomology class $[\rho]\in H_c^1(\pi_1^{\acute{e}t}(R);\operatorname{Aut}_A(\widehat{G}'))$ be the Galois descent data for $\widehat{G}$. $[\rho]$ is represented by some crossed homomorphism $\rho:\pi_1^{\acute{e}t}(R)\to\operatorname{Aut}_A(\widehat{G}')$. Then the Galois descent data for the finite flat group scheme $\widehat{G}[\mathcal{I}]$ is represented by the crossed homomorphism:*

$$\rho_{\mathcal{I}}:\pi_1^{\acute{e}t}(R)\xrightarrow{\rho}\operatorname{Aut}_A(\widehat{G}')\longrightarrow\operatorname{Aut}(\widehat{G}'[\mathcal{I}]),$$

where the last map $\mathrm{Aut}_A(\widehat{G}') \longrightarrow \mathrm{Aut}(\widehat{G}'[\mathcal{I}])$ *is the restriction of the quotient map*

$$\mathrm{End}_A(\widehat{G}') \longrightarrow\!\!\!\!\rightarrow \mathrm{End}_A(\widehat{G}')/(\mathcal{I}\otimes_A \mathrm{End}_A(\widehat{G}')) \simeq \mathrm{End}_A(\widehat{G}'[\mathcal{I}])$$

to the units.

In the view of Proposition 2.4.4 and Proposition 2.4.5, Katz's Riemann-Hilbert correspondence (Theorem 2.4.1) can be generalized as:

Theorem 2.4.6. *Let $\widehat{G}$ be a formal A-module of height 1 over R, where $\mathrm{Spf}\, R$ satisfies the same assumptions as in Theorem 2.4.1. Let $\mathbb{D}(\widehat{G}) = (M, F : M \xrightarrow{\sim} \varphi^* M)$ and $\rho : \pi_1^{\acute{e}t}(R) \to A^\times$ be the Dieudonné module and Galois descent data for $\widehat{G}$, respectively. Then the followings are equivalent:*

(1) There is a generator γ of M as an $R\otimes A$-module such that $F\gamma \equiv \gamma \mod \mathcal{I}$.
(2) $\widehat{G}[\mathcal{I}] \simeq (\widehat{G}_m \otimes A)[\mathcal{I}]$.
(3) The composition homomorphism $\rho_{\mathcal{I}} : \pi_1^{\acute{e}t}(R) \xrightarrow{\rho} A^\times \twoheadrightarrow (A/\mathcal{I})^\times$ is trivial.

Proof. Let's prove the case when $R = R/p$. By [Jon95, Main Theorem 1], the functor $\mathbb{D}$ is an equivalence over R. The claim then follows from the computation of the Dieudonné module and the Galois descent data of $\widehat{G}_m$ in Example 2.1.1, as well as Proposition 2.4.4 and Proposition 2.4.5.

Now let R be a $\mathbb{W}\kappa$-algebra. Using the Lubin-Tate deformation theory, we can show there is an equivalence between height 1 formal groups over R/p and their deformations to R/p. The claim now follows from the $R = R/p$-case. ☐

Remark 2.4.7. Katz's Theorem 2.4.1 is the $\mathcal{I} = (p^m) \trianglelefteq A = \mathbb{W}\mathbb{F}_q$ case of Theorem 2.4.6.

Remark 2.4.8. We can generalize Theorem 2.4.1 and Proposition 2.4.2 in terms of formal groups and formal A-modules of height $h > 1$. In that case, we need to study the Dieudonné module of the height h Honda formal group Γ_h and its finite subgroup schemes.

Now apply Theorem 2.4.6 to the formal $\mathbb{Z}_p[\chi]$-module $\widehat{C}^{k,\chi}$ over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ constructed in Construction 2.3.1, we have established Step II in Section 2.2:

Corollary 2.4.9 (Step II)**.** *Let $\mathcal{I} \trianglelefteq \mathbb{Z}_p[\chi]$ be an ideal. The followings are equivalent:*

(ii). *There is a generator $\gamma \in H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi})$ such that $F^{k,\chi}(\gamma) \equiv \gamma \mod \mathcal{I}$.*
(iii). $\widehat{C}^{k,\chi}[\mathcal{I}] \simeq (\widehat{G}_m \otimes \mathbb{Z}_p[\chi])[\mathcal{I}]$.
(iv). *The Galois descent data $\rho^{k,\chi} : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')) \to (\mathbb{Z}_p[\chi])^\times$ of $\widehat{C}^{k,\chi}$ is trivial modulo $\mathcal{I}$.*

2.5. Step III: Factorizations of the Galois descent data.

The final step is to study the Galois descent data $\rho^{k,\chi}$ for $\widehat{C}^{k,\chi}$. Recall from Construction 2.3.1, $\widehat{C}^{k,\chi}$ is constructed using the following data:

- $\xi^*\widehat{C}^{k,\chi} \simeq \widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]$ over $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$, where $\xi : \mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')) \to \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ is the forgetful map.
- $\widehat{C}^{k,\chi}$ corresponds to the character $[\chi] \in H^1((\mathbb{Z}/N)^\times; (\mathbb{Z}_p[\chi])^\times)$.

Proposition 2.5.1. $\rho^{k,\chi} : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')) \to (\mathbb{Z}_p[\chi])^\times$ *factorizes as*

$$\rho^{k,\chi} : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \xrightarrow{\rho^1 \times \lambda_\xi} \mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times \xrightarrow{(-)^k \cdot \chi(-)} (\mathbb{Z}_p[\chi])^\times,$$

where $\lambda_\xi : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \to (\mathbb{Z}/N)^\times$ is the character that classifies the $(\mathbb{Z}/N)^\times$-torsor ξ.

Proof. Recall in Construction 2.3.1, we used the following correspondence to construct $\widehat{C}^{k,\chi}$ from the character χ:

$$H^1((\mathbb{Z}/N)^\times;(\mathbb{Z}_p[\chi])^\times)\simeq\left\{\begin{matrix}\text{Formal } \mathbb{Z}_p[\chi]\text{-modules } \widehat{G} \text{ over } \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))\\ \text{such that } \xi^*\widehat{G}\simeq\widehat{C}^{\otimes k}\otimes\mathbb{Z}_p[\chi] \text{ over } \mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N'))\end{matrix}\right\}\Big/\sim \tag{2.5.2}$$

Here, the constant group homomorphism on the left hand side corresponds to the formal $\mathbb{Z}_p[\chi]$-module $\widehat{C}^{\otimes k}\otimes\mathbb{Z}_p[\chi]$ over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. Now we need to describe this correspondence in terms of the Galois descent data $\rho_{\widehat{G}}$ of $\widehat{G}$. On the one hand, since $\xi^*\widehat{G}\simeq\widehat{C}^{\otimes k}\otimes\mathbb{Z}_p[\chi]$, the composition

$$\pi_1^{ét}(\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N')))\xrightarrow{\pi_1^{ét}(\xi)}\pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))\xrightarrow{\rho_{\widehat{G}}}(\mathbb{Z}_p[\chi])^\times \tag{2.5.3}$$

is the same as the Galois descent data for the formal $\mathbb{Z}_p[\chi]$-module $\widehat{C}^{\otimes k}\otimes\mathbb{Z}_p[\chi]$ over $\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N'))$. On the other hand, by Example 2.1.3 and Construction 2.1.4, this Galois descent data also factorizes as

$$\pi_1^{ét}(\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N')))\xrightarrow{\pi_1^{ét}(\xi)}\pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))\xrightarrow{\rho^1}\mathbb{Z}_p^\times\xrightarrow{(-)^k}\mathbb{Z}_p^\times\xrightarrow{i}(\mathbb{Z}_p[\chi])^\times. \tag{2.5.4}$$

Denote the composition $i\circ(-)^k\circ\rho^1$ in (2.5.4) by ρ^k. Since the first maps in (2.5.3) and (2.5.4) are both $\pi_1^{ét}(\xi)$ and the compositions are the same, the difference of $\rho_{\widehat{G}}$ and ρ^k must factor through the cokernel of $\pi_1^{ét}(\xi)$. We have the following diagram:

$$\begin{tikzcd}
\pi_1^{ét}(\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N'))) \arrow[r, "\pi_1^{ét}(\xi)"] & \pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \arrow[r, "\lambda_\xi"] \arrow[d, shift right, "\rho_{\widehat{G}}"'] \arrow[d, shift left, "\rho^k"] & (\mathbb{Z}/N)^\times \arrow[r] \arrow[dl, dashed, "\exists!\ \chi_{\widehat{G}}"] & 1 \\
 & (\mathbb{Z}_p[\chi])^\times & &
\end{tikzcd}$$

As the cokernel of $\pi_1^{ét}(\xi)$, λ_ξ classifies the $(\mathbb{Z}/N)^\times$-torsor $\xi:\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N'))\to\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. It follows the that there exists a unique character $\chi_{\widehat{G}}:(\mathbb{Z}/N)^\times\to\mathbb{Z}_p[\chi]$ such that for any $\sigma\in\pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))$, $\rho_{\widehat{G}}(\sigma)=(\rho^1(\sigma))^k\cdot(\chi_{\widehat{G}}\circ\lambda_\xi)(\sigma)$.

This $\chi_{\widehat{G}}$ is the character corresponding to $\widehat{G}$ in (2.5.2). Since $\widehat{C}^{k,\chi}$ is constructed using χ, we have

$$\rho^{k,\chi}(\sigma)=(\rho^1(\sigma))^k\cdot(\chi\circ\lambda_\xi)(\sigma)=((-)^k\cdot\chi(-))\circ(\rho^1\times\lambda_\xi)(\sigma)$$

for all $\sigma\in\pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))$. □

Now we need to find the image of $\rho^1\times\lambda_\xi$.

Proposition 2.5.5. $\rho^1\times\lambda_\xi:\pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))\longrightarrow\mathbb{Z}_p^\times\times(\mathbb{Z}/N)^\times$ *factorizes as:*

$$\rho^1\times\lambda_\xi:\pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))\xrightarrow{\rho^1\times\lambda_{N'}}\mathbb{Z}_p^\times\times(\mathbb{Z}/N')^\times\xrightarrow{(a,b)\mapsto(a,[a],b)}\mathbb{Z}_p^\times\times(\mathbb{Z}/p^v)^\times\times(\mathbb{Z}/N')^\times\simeq\mathbb{Z}_p^\times\times(\mathbb{Z}/N)^\times,$$

where $\lambda_{N'}:\pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))\to(\mathbb{Z}/N')^\times$ *classifies the* $(\mathbb{Z}/N')^\times$*-torsor* $\mathcal{M}_{ell}^{ord}(\Gamma_1(N'))\to\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$.

Proof. We prove the factorization by translating Galois representations into torsors over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$.

Lemma 2.5.6. *The character* $\rho^1:\pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))\to\mathbb{Z}_p^\times$ *classifies the* $\mathbb{Z}_p^\times$*-torsor* $\mathcal{M}_{ell}^{triv}(\Gamma_0(N'))\to\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$, *where* $\mathcal{M}_{ell}^{triv}(\Gamma_0(N'))$ *is a stack whose* R*-points are*

$$\mathcal{M}_{ell}^{triv}(\Gamma_0(N'))(R)\coloneqq\{(C/R,\eta:\widehat{G}_m\xrightarrow{\sim}\widehat{C},H\subseteq C[N'])\mid H\simeq\underline{\mathbb{Z}/N'}\}.$$

Proof of the Lemma. Recall that $[\rho^1] \in H^1_c(\pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))); \mathbb{Z}_p^\times)$ is the Galois descent data for $\widehat{C}$, the formal group of the universal elliptic curve over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. The character ρ^1 then corresponds to a $\mathbb{Z}_p^\times$-torsor over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ such that its fiber over the an R-point $(C/R, H \subseteq C[N'])$ is the set of triples $(C/R, \eta : \widehat{G}_m \xrightarrow{\sim} \widehat{C}, H \subseteq C[N'])$. □

Lemma 2.5.6 implies that the character $\rho^1 \times \lambda_\xi$ classifies the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times$-torsor $\mathcal{M}_{ell}^{triv}(p^v, \Gamma_1(N')) \to \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$, where $\mathcal{M}_{ell}^{triv}(p^v, \Gamma_1(N'))$ is a stack whose R-points are

$$\mathcal{M}_{ell}^{triv}(p^v, \Gamma_1(N'))(R) \coloneqq \{(C/R, \eta, \eta_p, \eta') \mid \eta : \widehat{G}_m \xrightarrow{\sim} \widehat{C}, \eta_p : \mu_{p^v} \xrightarrow{\sim} \widehat{C}[p^v], \eta' : \underline{\mathbb{Z}/N'} \hookrightarrow C[N']\}.$$

Sitting in between $\mathcal{M}_{ell}^{triv}(p^v, \Gamma_1(N'))$ and $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ is the stack $\mathcal{M}_{ell}^{triv}(\Gamma_1(N'))$, whose R-points are

$$\mathcal{M}_{ell}^{triv}(\Gamma_1(N'))(R) \coloneqq \{(C/R, \eta, \eta') \mid \eta : \widehat{G}_m \xrightarrow{\sim} \widehat{C}, \eta' : \underline{\mathbb{Z}/N'} \hookrightarrow C[N']\}.$$

In the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times$-torsor

$$\mathcal{M}_{ell}^{triv}(p^v, \Gamma_1(N')) \longrightarrow \mathcal{M}_{ell}^{triv}(\Gamma_1(N')) \longrightarrow \mathcal{M}_{ell}^{ord}(\Gamma_0(N')),$$

the first map $\mathcal{M}_{ell}^{triv}(p^v, \Gamma_1(N')) \longrightarrow \mathcal{M}_{ell}^{triv}(\Gamma_1(N'))$ is a $(\mathbb{Z}/p^v)^\times$-torsor that admits a section:

$$s : \mathcal{M}_{ell}^{triv}(\Gamma_1(N')) \longrightarrow \mathcal{M}_{ell}^{triv}(p^v, \Gamma_1(N')), \quad (C/R, \eta, \eta') \longmapsto (C/R, \eta, \eta|_{\widehat{C}[p^v]}, \eta').$$

The existence of this section implies that $\rho^1 \times \lambda_\xi$ must factor through $\rho^1 \times \lambda_{N'} : \pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \to \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$, the character corresponding to the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-torsor $\mathcal{M}_{ell}^{triv}(\Gamma_1(N')) \to \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. The formula of s then yields a commutative diagram:

$$\begin{tikzcd}
\pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \arrow[r, "\rho^1\times\lambda_\xi"] \arrow[d, "\rho^1\times\lambda_{N'}"'] & \mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times \arrow[d, equal] \\
\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times \arrow[r, "{(a,b)\mapsto(a,[a],b)}"] & \mathbb{Z}_p^\times \times (\mathbb{Z}/p^v)^\times \times (\mathbb{Z}/N')^\times
\end{tikzcd}$$

□

Combining Proposition 2.5.1 and Proposition 2.5.5, we have shown

Corollary 2.5.7. *$\rho^{k,\chi} : \pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')) \to (\mathbb{Z}_p[\chi])^\times$ factorizes as*

$$\rho^{k,\chi} : \pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \xrightarrow{\rho^1\times\lambda_{N'}} \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times \xrightarrow{(a,b)\mapsto \chi_p(a)\chi'(b)a^k} (\mathbb{Z}_p[\chi])^\times. \tag{2.5.8}$$

To relate the congruence of $\rho^{k,\chi}$ with that of the second map in (2.5.8), it remains to show:

Proposition 2.5.9. *$\rho^1 \times \lambda_{N'} : \pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \longrightarrow \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$ is surjective.*

Proof. By [Sza09, Theorem 5.4.2], the surjectivity of $\rho^1 \times \lambda_{N'}$ is equivalent to the connectivity of the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-torsor it classifies. As $\rho^1 \times \lambda_{N'}$ classifies the torsor $\mathcal{M}_{ell}^{triv}(\Gamma_1(N')) \to \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$, we need to show $\mathcal{M}_{ell}^{triv}(\Gamma_1(N'))$ is connected.

By a relative version of Igusa's theorem in [KM85, Corrollary 12.6.2.(2)], $\mathcal{M}_{ell}^{triv}(\Gamma_1(N'))$ is connected whenever $\mathcal{M}_{ell}^{ord}(\Gamma_1(N'))$ is. The integral stack $\mathcal{M}_{ell}(\Gamma_1(N'))$ has geometrically connected fiber by [Con07, Theorem 1.2.1]. It is also smooth by [KM85, Corollary 4.7.1]. It follows that $\mathcal{M}_{ell}(\Gamma_1(N'))$ is irreducible and so is its p-completion $\mathcal{M}_{ell}(\Gamma_1(N'))_p^\wedge$. From this we conclude $\mathcal{M}_{ell}^{ord}(\Gamma_1(N'))$ is irreducible (hence connected), since it is an open substack of an irreducible stack. □

Now by Corollary 2.5.7 and Proposition 2.5.9, we have proved:

Corollary 2.5.10 (Step III)**.** *Let $\mathcal{I} \trianglelefteq \mathbb{Z}_p[\chi]$ be an ideal. The followings are equivalent:*

(iv). *The composition $\rho^{k,\chi} : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \longrightarrow (\mathbb{Z}_p[\chi])^\times \twoheadrightarrow (\mathbb{Z}_p[\chi]/\mathcal{I})^\times$ is trivial.*

(v). *The composition $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times \xrightarrow{(a,b)\mapsto \chi_p(a)\chi'(b)a^k} (\mathbb{Z}_p[\chi])^\times \twoheadrightarrow (\mathbb{Z}_p[\chi]/\mathcal{I})^\times$ is trivial.*

2.6. **Restatement of the Main Theorem.** Combining Proposition 2.3.6, Corollary 2.4.9, and Corollary 2.5.10, we now restate the Main Theorem:

Theorem 2.6.1 (Main Theorem, restated)**.** *Let $\mathcal{I} \trianglelefteq \mathbb{Z}_p[\chi]$ be an ideal. Then the followings are equivalent:*

(i). *There is an Eisenstein series $f \in \mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ such that $f(q) \in 1 + \mathcal{I}q[\![q]\!]$.*
(ii). *There is a generator $\gamma \in H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi})$ such that $F^{k,\chi}(\gamma) \equiv \gamma \mod \mathcal{I}$.*
(iii). *$\widehat{C}^{k,\chi}[\mathcal{I}] \simeq (\widehat{G}_m \otimes \mathbb{Z}_p[\chi])[\mathcal{I}]$.*
(iv). *The Galois descent data $\rho^{k,\chi} : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \to (\mathbb{Z}_p[\chi])^\times$ of $\widehat{C}^{k,\chi}$ is trivial modulo $\mathcal{I}$.*
(v). *The character $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times \xrightarrow{(a,b)\mapsto \chi_p(a)\chi'(b)a^k} (\mathbb{Z}_p[\chi])^\times$ is trivial modulo $\mathcal{I}$.*

Remark 2.6.2. When the character χ is trivial, we recover Katz's algebro-geometric explanation of congruences of p-adic Eisenstein series of level 1 in [Kat73b, Corollary 4.4.1]. In that case, Step I in the proof above is not needed.

3. The maximal congruence of Eisenstein series

Theorem 2.6.1 identifies the maximal congruence of Eisenstein series in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ with that of $\mathbb{Z}_p^{\otimes k}[\chi]$ as a $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-representation in $\mathbb{Z}_p[\chi]$-modules. In this section, we first compute the maximal congruence of $\mathbb{Z}_p^{\otimes k}[\chi]$ and then find explicit examples of Eisenstein series that realize this congruence in certain cases.

3.1. Congruences of p-adic representations.

Definition 3.1.1. Let R be a p-complete local ring and M be a torsion-free R-module with a continuous R-module action by a profinite group G. M is said to be a trivial G-representation modulo an ideal $\mathcal{I} \trianglelefteq R$ if G acts on $M/\mathcal{I}M$ trivially, or equivalently $(M/\mathcal{I}M)^G = M/\mathcal{I}M$. The maximal congruence of M as a G-representation is the smallest ideal $\mathcal{I}$ such that $M/\mathcal{I}M$ is a trivial G-representation.

Remark 3.1.2. The G-action on the quotient $M/\mathcal{I}M$ is well defined since G acts by R-linear maps. Otherwise, we need to assume $\mathcal{I} \trianglelefteq R$ is a G-invariant ideal, i.e. $g\mathcal{I} = \mathcal{I}$ for all $g \in G$.

Lemma 3.1.3. *When the underlying R-module of the G-representation M is R, the G-action of M is then associated to a character $\chi : G \to R^\times$. Let $\{g_i \mid i \in I\}$ be a set of generators of G. The maximal congruence of M is the ideal $(1 - \chi(g_i) \mid i \in I)$.*

Proof. The maximal congruence of M is by definition the ideal $(1-\chi(g) \mid g \in G)$. Notice that

$$(1-\chi(gg')) = (1-\chi(g)+\chi(g)-\chi(gg')) \subseteq (1-\chi(g)) + (\chi(g)-\chi(gg')) = (1-\chi(g)) + (1-\chi(g')).$$

and that $(1-\chi(g^{-1})) = (\chi(g)-1)$, we have $(1-\chi(g) \mid g \in G) = (1-\chi(g_i) \mid i \in I)$. ☐

When $p > 2$, $\mathbb{Z}_p^\times$ is topologically cyclic. When $p = 2$, $\mathbb{Z}_2^\times = \{\pm 1\} \times (1+4\mathbb{Z}_2)$ and $1+4\mathbb{Z}_2$ is topologically cyclic. Let g be a topological generator of $\mathbb{Z}_p^\times$ when $p > 2$ and a topological generator of $1+4\mathbb{Z}_2$ when $p = 2$.

Theorem 3.1.4. *The congruences of $\mathbb{Z}_p^{\otimes k}[\chi]$ have seven cases:*

I. *$p > 2$ and the conductor of χ is p or 1. In this case, $\chi = \omega^a$ for some integer $0 \le a \le p-2$, where $\omega : (\mathbb{Z}/p)^\times \to \mathbb{Z}_p^\times$ is the p-**adic Teichmüller character**. The image of χ is contained in $\mathbb{Z}_p^\times$. Then the maximal congruence of $\mathbb{Z}_p^{\otimes k}[\omega^a]$ is the following ideal in $\mathbb{Z}_p = \mathbb{Z}_p[\omega^a]$:*

$$(1 - g^k\chi(g)) = (1 - g^k\omega^a(g)) = \begin{cases} (p^{v_p(k)+1}), & (p-1) \mid (k+a); \\ (1) & \text{otherwise.} \end{cases}$$

II. *$p = 2$ and the conductor of χ is 4 or 1. In this case $\chi = \omega^a$ for $a = 0$ or 1, where $\omega : (\mathbb{Z}/4)^\times \to \mathbb{Z}_2^\times$ is the 2-adic Teichmüller character. As $g \in 1 + 4\mathbb{Z}_2$, $\omega(g) = 1$. Again the image of χ is contained in $\mathbb{Z}_2^\times$. Then the maximal congruence of $\mathbb{Z}_2^{\otimes k}[\omega^a]$ is the following ideal in $\mathbb{Z}_2 = \mathbb{Z}_2[\omega^a]$:*

$$(1 - g^k\omega^a(g), 1 - (-1)^k\omega^a(-1)) = \begin{cases} (2^{v_p(k)+2}), & 2 \mid (k+a); \\ (2), & \text{otherwise.} \end{cases}$$

III. *$p > 2$ and the conductor of χ is $p^v > p$. In this case, $(\mathbb{Z}/p^v)^\times \simeq (\mathbb{Z}/p)^\times \times C_{p^{v-1}}$ and As χ is primitive of conductor p^v, $\chi|_{C_{p^{v-1}}}$ is injective. As a result, $\mathbb{Z}_p[\chi] = \mathbb{Z}_p[\zeta_{p^{v-1}}]$. $\mathbb{Z}_p[\zeta_{p^{v-1}}]$ is a p-complete local ring with uniformizer $1 - \zeta_{p^{v-1}}$. Write $\chi|_{(\mathbb{Z}/p)^\times} = \omega^a$ for some $0 \le a \le p-2$. Then the maximal congruence of $\mathbb{Z}_p^{\otimes k}[\omega^a]$ is the following ideal in $\mathbb{Z}_p[\zeta_{p^{v-1}}] = \mathbb{Z}_p[\chi]$:*

$$(1 - g^k\chi(g)) = (1 - \zeta_{p^{v-1}} g^k\omega^a(g)) = \begin{cases} (1 - \zeta_{p^{v-1}}), & (p-1) \mid (k+a); \\ (1), & \text{otherwise.} \end{cases}$$

IV. *$p = 2$ and the conductor of χ is $2^v > 4$. In this case, $(\mathbb{Z}/2^v)^\times \simeq (\mathbb{Z}/4)^\times \times C_{2^{v-2}}$. As χ is primitive of conductor 2^v, $\chi|_{C_{2^{v-2}}}$ is injective. As a result, $\mathbb{Z}_2[\chi] = \mathbb{Z}_2[\zeta_{2^{v-2}}]$. $\mathbb{Z}_2[\zeta_{2^{v-2}}]$ is a 2-complete local ring with uniformizer $1 - \zeta_{2^{v-2}}$. Write $\chi|_{(\mathbb{Z}/4)^\times} = \omega^a$ for $a = 0$ or 1. Then the maximal congruence of $\mathbb{Z}_2^{\otimes k}[\chi]$ is the following ideal in $\mathbb{Z}_2[\zeta_{2^{v-2}}] = \mathbb{Z}_2[\chi]$:*

$$(1 - \zeta_{2^{v-2}} g^k\omega^a(g), 1 - (-1)^k\omega^a(-1)) = (1 - \zeta_{2^{v-2}}) \quad \textit{for all } k \textit{ and } a.$$

V. *$N' \ne 1$ and $|\operatorname{Im}\chi'|$ is not a power of p. In this case, $\operatorname{Im}\chi'$ contains of a root of unity $\zeta_{n'}$ whose order n' is coprime to p. As $1 - \zeta_{n'}$ is invertible in $\mathbb{Z}_p[\zeta_{n'}] \subseteq \mathbb{Z}_p[\chi]$, we have the maximal congruence of $\mathbb{Z}_p^{\otimes k}[\chi]$ is the ideal (1) in $\mathbb{Z}_p[\chi]$.*

VI. *$p > 2$, $N' \ne 1$ and $|\operatorname{Im}\chi'|$ is a power of p greater than 1. In the case, $\operatorname{Im}\chi'$ is generated by $\zeta_{p^{v'}}$ for some $v' \ge 1$. We have $\mathbb{Z}_p^{\otimes k}[\chi] = \mathbb{Z}_p[\zeta_{p^{\max(v-1,v')}}]$. Write $\chi_p|_{(\mathbb{Z}/p)^\times} = \omega^a$ for some $0 \le a \le p-2$. Then the maximal congruence of $\mathbb{Z}_p^{\otimes k}[\chi]$ is the following ideal in $\mathbb{Z}_p^{\otimes k}[\chi] = \mathbb{Z}_p[\zeta_{p^{\max(v-1,v')}}]$:*

$$(1 - g^k\chi(g), 1 - \zeta_{p^{v'}}) = (1 - \zeta_{p^{v-1}} g^k\omega^a(g), 1 - \zeta_{p^{v'}}) = \begin{cases} (1 - \zeta_{p^{\max(v-1,v')}}), & (p-1) \mid (k+a); \\ (1), & \text{otherwise.} \end{cases}$$

VII. *$p = 2$, $N' \ne 1$ and $\mathbb{Q}_2(\chi')$ is a totally ramified extension of $\mathbb{Q}_2$. In the case, the image of χ' is generated by $\zeta_{2^{v'}}$ for some $v' \ge 1$. We have $\mathbb{Z}_2^{\otimes k}[\chi] = \mathbb{Z}_2[\zeta_{2^{\max(v',v-2)}}]$. Write $\chi_2|_{(\mathbb{Z}/4)^\times} = \omega^a$ for $a = 0, 1$. Then the maximal congruence of $\mathbb{Z}_2^{\otimes k}[\chi]$ is the following ideal in $\mathbb{Z}_2[\zeta_{2^{\max(v',v-2)}}] = \mathbb{Z}_2[\chi]$:*

$$(1 - \zeta_{2^{v'}}, 1 - \zeta_{2^{v-2}} g^k\omega^a(g), 1 - (-1)^k\omega^a(-1)) = (1 - \zeta_{2^{\max(v',v-2)}}) \quad \textit{for all } k \textit{ and } a.$$

3.2. **Realizations of the maximal congruence.** Having computed the maximal congruence of the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-representation $\mathbb{Z}_p^{\otimes}[\chi]$, now we give explicit examples of Eisenstein series in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ whose q-expansions realize the maximal congruence.

Let k be an integer such that $(-1)^k = \chi(-1)$. Recall from Theorem 1.1.4 and Corollary 1.4.12 that Eisenstein subspace $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi) \otimes \mathbb{Q}_p$ is spanned by Eisenstein series of the forms:

$$E_{k,\chi^0,\chi}(q^t) = c \cdot E_{k,\chi}(q^t) = c \cdot \left(1 - \frac{2k}{B_{k,\chi}} \sum_{n\geq 1} \left(\sum_{0<d|n} \chi(d) d^{k-1}\right) q^{nt}\right),$$

$$E_{k,\chi_1,\chi_2}(q^t) = \sum_{n\geq 1} \left(\sum_{0<d|n} \chi_1^{-1}(n/d)\chi_2(d) d^{k-1}\right) q^{nt},$$

where

- c is $\mathbb{Z}_p[\chi]$ with the smallest valuation so that $E_{k,\chi^0,\chi}(q^t) \in \mathbb{Z}_p[\chi][\![q]\!]$.
- χ_1 and χ_2 are characters of conductors N_1 and N_2 satisfying $\chi_1/\chi_2 = \chi^{-1}$ and $(N_1 N_2 t) \mid N$.

By the q-expansion principle Proposition 1.3.2, an element of $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ is a $\mathbb{Q}_p$-linear combination $f(q)$ of these $E_{k,\chi_1,\chi_2}(q)$ such that $f(q) \in \mathbb{Z}_p[\chi][\![q]\!]$. Write $E_{k,\chi^0,\chi}$ and E_{k,χ_1,χ_2} for $E_{k,\chi^0,\chi}(q)$ and $E_{k,\chi_1,\chi_2}(q)$, respectively. Using the arithmetic properties of generalized Bernoulli numbers in [Car59, Theorem 1, 3], we can check

Proposition 3.2.1. *In Cases I-V in Theorem 3.1.4, the Eisenstein series $E_{k,\chi}$ realizes the maximal congruence predicted in Theorem 2.6.1.*

By [Car59, Theorem 1], $\frac{B_{k,\chi}}{k}$ is an algebraic p-adic integer when N is not a power of p. As a result $E_{k,\chi}(q)$ does not realize the maximal congruence in Cases VI and VII in Theorem 3.1.4. Instead, we need to consider linear combinations of basis in the Eisenstein subspace. In general, it is hard to write down the explicit formulas of Eisenstein series that satisfies the maximal congruence predicted in Theorem 2.6.1 and Theorem 3.1.4 in cases VI and VII. Here, we work out one of the simplest cases in the rest of this subsection.

Example 3.2.2. Consider the character $\chi : (\mathbb{Z}/\ell)^\times \to \mathbb{C}_p^\times$, where ℓ is a prime different from $p > 2$ and $\mathbb{Q}_p(\chi)$ is a totally ramified extension of $\mathbb{Q}_p$. In this case, $\mathbb{Z}_p[\chi] = \mathbb{Z}_p[\zeta_{p^m}]$ for some $m \geq 1$ is a p-complete local ring with uniformizer $\varpi = 1 - \zeta_{p^m}$. Write the maximal ideal of $\mathbb{Z}_p[\chi]$ by $\mathfrak{m}$. By [Car59, Theorem 1], $\frac{B_{k,\chi}}{2k}$ is an algebraic p-adic integer. As a result, we can take $E_{k,\chi^0,\chi}$ to be:

$$E_{k,\chi^0,\chi} = \frac{B_{k,\chi}}{2k} - \sum_{n\geq 1} \left(\sum_{0<d|n} \chi(d) d^{k-1}\right) q^n.$$

Comparing Theorem 2.6.1 and Case VI in Theorem 3.1.4, we should expect to find Eisenstein series of weight k, level $\Gamma_1(\ell)$, and character χ that is congruent to 1 modulo $\mathfrak{m} = (\varpi)$ only when $(p-1) \mid k$, and there is no Eisenstein series of level $\Gamma_1(\ell)$ and character χ that is congruent to 1 modulo $\mathfrak{m}^2$. The Eisenstein subspace in this case $\mathcal{E}_k(\Gamma_1(\ell), \chi)$ is spanned by $E_{k,\chi^0,\chi}(q)$ and $E_{k,\chi^{-1},\chi^0}(q)$.

When $(p-1) \mid k$, the maximal congruence is the realized by a linear combination of the two basis Eisenstein series, since neither of them satisfies the maximal congruence relation. As $\frac{B_{k,\chi}}{2k} \in \mathbb{Z}_p[\chi]$, we have $\frac{B_{k,\chi}}{2k} \cdot E_{k,\chi} \in \mathbb{Z}_p[\chi][\![q]\!]$. Notice that the coefficients of q in $E_{k,\chi^0,\chi}$ and E_{k,χ^{-1},χ^0} are -1 and 1, respectively. Consider the q-expansion of their sum:

$$E_{k,\chi^0,\chi} + E_{k,\chi^{-1},\chi^0} = \frac{B_{k,\chi}}{2k} + \sum_{n\geq 1} a_n q^n = \frac{B_{k,\chi}}{2k} + \sum_{n\geq 1} \left(\sum_{0<d|n} (\chi^{-1}(n/d) - \chi(d)) d^{k-1}\right) q^n. \tag{3.2.3}$$

Lemma 3.2.4. $E_{k,\chi^0,\chi} + E_{k,\chi^{-1},\chi^0} \equiv \frac{B_{k,\chi}}{2k} \mod \mathfrak{m}q[\![q]\!]$ *for all k with $(-1)^k = \chi(-1)$.*

Proof. We need to show the coefficient a_n of q^n in (3.2.3) is in $\mathfrak{m}$ for all $n \geq 1$. Write $n = \ell^v n'$ where $\ell \nmid n'$. Since the conductor of χ is the prime number ℓ, $\chi(a) = 0$ iff $\ell \mid a$. As a result, we have

$$\begin{aligned} a_n = \sum_{0<d|n} (\chi^{-1}(n/d) - \chi(d)) d^{k-1} &= \sum_{0<d|n} \chi^{-1}(n/d) d^{k-1} - \sum_{0<d|n} \chi(d) d^{k-1} \\ &= \sum_{\ell^v|d|n} \chi^{-1}(n/d) d^{k-1} - \sum_{0<d|n'} \chi(d) d^{k-1} \\ \text{(set } d = \ell^v d' \text{ in the first summation)} \quad &= \sum_{0<d'|n'} \chi^{-1}(n'/d')(\ell^v d')^{k-1} - \sum_{0<d|n'} \chi(d) d^{k-1} \\ &= \sum_{0<d'|n'} \chi^{-1}(n')\chi(d')\ell^{v(k-1)} d'^{k-1} - \sum_{0<d|n'} \chi(d) d^{k-1} \\ &= (\chi^{-1}(n')\ell^{v(k-1)} - 1) \sum_{0<d|n'} \chi(d) d^{k-1}. \end{aligned} \tag{3.2.5}$$

Since $(\mathbb{Z}/\ell)^\times$ surjects onto C_{p^m} by assumption, $\ell \equiv 1 \mod p$. This implies $\ell^{v(k-1)} \equiv 1 \mod p$. Also, as $\chi^{-1}(n') \neq 0$ is a p-power root of unity, $1 - \chi^{-1}(n') \in \mathfrak{m}$. Combining these two facts, we conclude

$$\chi^{-1}(n')\ell^{v(k-1)} - 1 = \chi^{-1}(n') - 1 + \chi^{-1}(n')(\ell^{v(k-1)} - 1) \in \mathfrak{m}$$

for all n' not divided by ℓ. This shows $a_n \in \mathfrak{m}$ for all n. From this, we conclude $E_{k,\chi^0,\chi} + E_{k,\chi^{-1},\chi^0} \equiv \frac{B_{k,\chi}}{2k}$ mod $\mathfrak{m}q[\![q]\!]$. □

Proposition 3.2.6. *The algebraic p-adic integer* $\frac{B_{k,\chi}}{2k}$ *is in* $\mathfrak{m}$ *iff* $(p-1) \nmid k$.

Proof. When $(p-1) \nmid k$, there is no Eisenstein series in $\mathcal{E}_k(\Gamma_1(\ell), \chi)$ whose q-expansion is in $1 + \mathfrak{m}q[\![q]\!]$ by Theorem 2.6.1 and Case VI in Theorem 3.1.4. In Lemma 3.2.4, we showed all the a_n's in (3.2.3) are in $\mathfrak{m}$. This implies its constant term $\frac{B_{k,\chi}}{2k}$ must also be in $\mathfrak{m}$ so that there is a common factor.

When $(p-1) \mid k$, Theorem 2.6.1 and Case VI in Theorem 3.1.4 predicts an Eisenstein series in $\mathcal{E}_k(\Gamma_1(\ell), \chi)$ whose q-expansion is in $1 + \mathfrak{m}q[\![q]\!]$, or equivalently $(\mathbb{Z}_p[\chi])^\times + \mathfrak{m}q[\![q]\!]$. We can write this Eisenstein series as $\varpi^{-v}(aE_{k,\chi^0,\chi} + bE_{k,\chi^{-1},\chi^0})$ for some $v \geq 0$ and $a, b \in \mathbb{Z}_p[\chi]$ such that one of them is in $(\mathbb{Z}_p[\chi])^\times$. Notice the coefficients of q in the q-expansions of this Eisenstein series is $\varpi^{-v}(-a+b)$. We have $b-a$ is in $\mathfrak{m}^{v+1} \subseteq \mathfrak{m}$. This implies $v_p(a) = v_p(b) = 0$. Without loss of generality, we can then assume $a = 1$ and $b \equiv 1 \mod \mathfrak{m}^{v+1} \subseteq \mathfrak{m}$. It now suffices to prove $v = 0$, for that implies $\frac{B_{k,\chi}}{2k}$, the constant term of $E_{k,\chi^0,\chi} + bE_{k,\chi^{-1},\chi^0}$ is in $(\mathbb{Z}_p[\chi])^\times$.

Suppose $v > 0$. Following (3.2.5), we have

$$E_{k,\chi^0,\chi} + bE_{k,\chi^{-1},\chi^0} = E_{k,\chi^0,\chi} + E_{k,\chi^{-1},\chi^0} + (b-1)E_{k,\chi^{-1},\chi^0} \tag{3.2.7}$$

$$= \frac{B_{k,\chi}}{2k} + \sum_{n\geq 1} b_n q^n \tag{3.2.8}$$

$$\text{(set } n = \ell^v n'\text{)} \quad = \frac{B_{k,\chi}}{2k} + \sum_{n\geq 1} \left((b\chi^{-1}(n')\ell^{v(k-1)} - 1) \sum_{0<d|n'} \chi(d) d^{k-1} \right) q^n$$

Lemma 3.2.4 and (3.2.7) imply $E_{k,\chi^0,\chi} + bE_{k,\chi^{-1},\chi^0} \equiv \frac{B_{k,\chi}}{2k} \mod \mathfrak{m}q[\![q]\!]$. Now we want to find a b_n in (3.2.8) that is not in $\mathfrak{m}^2$. Let p' be a prime number such that $\chi(p')$ is a primitive p^m-th root of unity and that $p' \not\equiv (-1) \mod p$. This assumption implies $p' \neq \ell$. If $p' = p$ does not satisfy the assumption (i.e. $\chi(p)$ is not a primitive p^m root of unity), then there are infinitely many choices of the prime p'. This is because the conditions on p' depend only on its residual class modulo $p \cdot \ell$. In this case, we have by (3.2.8),

$$b_{p'} = (b\chi^{-1}(p') - 1)(1 + \chi(p')p'^{k-1}) = ((b-1) \cdot \chi^{-1}(p') + \chi^{-1}(p') - 1)(1 + \chi(p')p'^{k-1}).$$

Notice:

- $1-\chi^{-1}(p')$ is a uniformizer in $\mathbb{Z}_p[\chi]$, since $\mathbb{Z}_p[\chi]=\mathbb{Z}_p[\zeta_{p^m}]=\mathbb{Z}_p[\chi(p')]$.
- $b-1\in\mathfrak{m}^{v+1}\subseteq\mathfrak{m}^2$, since $v>0$ by assumption. We have $(b\chi^{-1}(p')-1)$ is also a uniformizer.
- $p'^{k-1}\equiv p'^{-1}\not\equiv(-1) \mod p$, since $(p-1)\mid k$.

We have $1+\chi(p')p'^{k-1}\notin\mathfrak{m}$. As a result, $b_{p'}\in\mathfrak{m}-\mathfrak{m}^2$ and $\varpi^{-v}b_{p'}\notin\mathfrak{m}$. This contradicts the assumption that $\varpi^{-v}(E_{k,\chi^0,\chi}+bE_{k,\chi^{-1},\chi^0})\in(\mathbb{Z}_p[\chi])^\times+\mathfrak{m}q[\![q]\!]$ is an Eisenstein series that realizes the maximal congruence. Consequently, we have $v=0$, $\frac{B_{k,\chi}}{2k}\in(\mathbb{Z}_p[\chi])^\times$ when $(p-1)\mid k$. ☐

Remark 3.2.9. When $(p-1)\nmid k$, it is possible that $\frac{B_{k,\chi}}{2k}\in\mathfrak{m}^s$ for some $s>1$.

It follows that when $(p-1)\nmid k$, the maximal congruence in $\mathcal{E}_k(\Gamma_1(\ell),\chi)$ is realized by:

$$\frac{2k\cdot\left(E_{k,\chi^0,\chi}+bE_{k,\chi^{-1},\chi^0}\right)}{B_{k,\chi}}=1+\frac{2k}{B_{k,\chi}}\sum_{n\geq1}\left(\sum_{0<d\mid n}(b\chi^{-1}(n/d)-\chi(d))d^{k-1}\right)q^n\in1+\mathfrak{m}q[\![q]\!],$$

where $b\in1+\mathfrak{m}\subseteq\mathbb{Z}_p[\chi]$.

4. Relations with Dirichlet $K(1)$-local spheres and J-spectra

In this section, we first relate the maximal congruence of $\mathbb{Z}_p^{\otimes k}[\chi]$ to the group cohomology of $\mathbb{Z}_p^\times\times(\mathbb{Z}/N')^\times$. This group cohomology is on the E_2-page of a spectra sequence to compute homotopy groups of the Dirichlet $K(1)$-local sphere attached to χ. This spectrum was introduced by the author in [Zha19].

Combined with Theorem 2.6.1, this connects congruences of Eisenstein series of level $\Gamma_1(N)$ to chromatic homotopy theory. In particular, when the character is trivial, we have given a new explanation of the relation between congruences of Eisenstein series of level 1 and the image of the J-homomorphism in the stable homotopy groups of spheres.

4.1. Congruence and group cohomology.

Let $\mathbb{Z}_p^{\otimes k}[\chi]$ be the $\mathbb{Z}_p^\times\times(\mathbb{Z}/N')^\times$-representation associated to the character $\mathbb{Z}_p^\times\times(\mathbb{Z}/N')^\times\xrightarrow{(a,b)\mapsto\chi_p(a)\chi'(b)a^k}(\mathbb{Z}_p[\chi])^\times$. The maximal congruence of $\mathbb{Z}_p^{\otimes k}[\chi]$ as a $\mathbb{Z}_p^\times\times(\mathbb{Z}/N')^\times$-representation is closely related to its group cohomology.

Lemma 4.1.1. *Suppose $(R,\mathfrak{m})$ is a p-complete discrete valuation ring and let $\varpi\in\mathfrak{m}$ be a uniformizer. The maximal congruence of M is the ideal $\mathfrak{m}^k$ such that*

$$\operatorname*{colim}_m((M/\varpi^m)^G)=\operatorname*{colim}_m((M/\mathfrak{m}^m)^G)=M/\mathfrak{m}^k.$$

Proposition 4.1.2. *When G is topologically finitely generated, the natural map*

$$\operatorname*{colim}_m((M/\varpi^m)^G)\longrightarrow\left(\operatorname*{colim}_m(M/\varpi^m)\right)^G=:(M/\varpi^\infty)^G$$

is an isomorphism.

Proof. When G is topologically finitely generated, taking G-fixed points is a equivalent to a finite limit (in the 1-categorical sense). As a result, $(-)^G$ commutes with the sequential colimit $(-/\varpi^\infty)$ by [Mac71, Theorem 1 in Section IX.2]. ☐

As M is a torsion-free R-module, the total quotient module M/ϖ^∞ can also be obtained from a short exact sequence of G-representations in R-modules:

$$0\longrightarrow M\longrightarrow\varpi^{-1}M\longrightarrow M/\varpi^\infty\longrightarrow0. \tag{4.1.3}$$

Remark 4.1.4. (4.1.3) is the colimit of the following tower of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\varpi"] \arrow[d, equal] & M \arrow[r] \arrow[d, "\varpi"] & M/\varpi \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "\varpi^2"] \arrow[d, equal] & M \arrow[r] \arrow[d, "\varpi"] & M/\varpi^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "\varpi^3"] \arrow[d, equal] & M \arrow[r] \arrow[d, "\varpi"] & M/\varpi^3 \arrow[r] \arrow[d] & 0 \\
& \cdots & \cdots & \cdots &
\end{tikzcd}$$

Proposition 4.1.5. *Assumptions and notations as above. When $M^G = 0$, there is a natural injection $\delta : (M/\varpi^\infty)^G \to H^1_c(G; M)$.*

Proof. Apply $H^*_c(G; -)$ on (4.1.3), we get a long exact sequence of G cohomology that start with:

$$0 \longrightarrow M^G \longrightarrow (\varpi^{-1}M)^G \longrightarrow (M/\varpi^\infty)^G \xrightarrow{\delta} H^1_c(G; M) \longrightarrow H^1_c(G; \varpi^{-1}M) \longrightarrow \cdots$$

The fixed points $(\varpi^{-1}M)^G = 0$ since

$$(\varpi^{-1}M)^G = \left(\operatorname{colim}(M \xrightarrow{\varpi} M \xrightarrow{\varpi} \cdots)\right)^G \simeq \operatorname{colim}(M^G \xrightarrow{\varpi} M^G \xrightarrow{\varpi} \cdots) = \varpi^{-1}(M^G) = 0.$$

This shows δ is injective. □

Theorem 4.1.6. *The connecting homomorphism $\delta : (\mathbb{Z}_p^{\otimes k}[\chi]/\varpi^\infty)^{\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times} \to H^1_c(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes k}[\chi])$ is an isomorphism.*

Proof. In this case, $G = \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$ is topologically finitely generated, $R = \mathbb{Z}_p[\chi]$ and $M = \mathbb{Z}_p^{\otimes k}[\chi]$. $R = \mathbb{Z}_p[\chi]$ is a p-complete discrete valuation ring since it is isomorphic to the form $\mathbb{Z}_p[\zeta_n]$ for some n. As $\mathbb{Z}_p[\zeta_n]$ is an integral domain, the action of an non-identity element $(a, b) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$ on $\mathbb{Z}_p[\zeta_n]$ by multiplication by $\chi_p(a)\chi'(b)a^k$ has no fixed points. By Proposition 4.1.5, the connecting homomorphism δ is injective.

As p is a power of the uniformizer ϖ, $\varpi^{-1}M = p^{-1}M = \mathbb{Q}_p^{\otimes k}(\chi)$. We now show $H^1_c(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Q}_p^{\otimes k}(\chi)) = 0$, which would imply δ is surjective. Write $G = \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times = G_{fin} \times G_{pro}$, where

- G_{fin} is the maximal finite subgroup of G.
- $G_{pro} = 1 + p\mathbb{Z}_p$ when $p > 2$ and $G_{pro} = 1 + 4\mathbb{Z}_p$ when $p = 2$.

Since $\mathbb{Q}_p^{\otimes k}(\chi)$ is a $\mathbb{Q}_p$-module, we have $H^1_c(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Q}_p^{\otimes k}(\chi)) \simeq H^0(G_{fin}; H^1_c(G_{pro}; \mathbb{Q}_p^{\otimes k}(\chi)))$. This is because the associated Hochschild-Serre spectral sequences is concentrated in the 0-line. Let g be a pro-generator of G_{pro}. Then we have

$$H^1_c\left(G_{pro}; \mathbb{Q}_p^{\otimes k}(\chi)\right) \simeq \mathbb{Q}_p(\chi) \big/ \left(1 - \chi_p(g)g^k\right),$$

where g is viewed as an element of $\mathbb{Q}_p(\chi)$ via $g \in G_{pro} = 1 + 2p\mathbb{Z}_p \subseteq \mathbb{Q}_p(\chi)$. The quotient is zero since $1 - \chi_p(g)g^k \neq 0$ and $\mathbb{Q}_p(\chi) = \mathbb{Q}_p(\zeta_n)$ is a field. This implies $H^1_c(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Q}_p^{\otimes k}(\chi)) = 0$. Consequently, the connecting homomorphism δ is surjective. □

Remark 4.1.7. Unlike the finite group case, it is in general NOT true that $H^s_c(G; M) = 0$ for $s > 0$ when G is profinite and M is a $\mathbb{Q}_p$-module. Using Kummer theory, one can construct explicit examples G and M when the group cohomology $H^1_c(G; M)$ is non-zero.

However, it is true that $H^s_c(G;M) = 0$ when $s > 0$, if $M = \bigcup_{H\leq G} M^H$ where H ranges over all open subgroups of G. Such an M is called a discrete G-module. In this case, we have by [Ser97, Corollary 1 in §2.2]:

$$H^s_c(G;M) \simeq \operatorname*{colim}_{H\leq G \text{ open}} H^s(G/H;M^H).$$

This is group cohomology is zero when $s > 0$, since G/H is finite for any open subgroup H of G and $M^H \subseteq M$ is a $\mathbb{Q}_p$-module. It is straight forward to check that $\mathbb{Z}_p[\chi]^{\otimes k}$ is not a discrete $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$. That is why we have to explicitly compute $H^1_c(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Q}_p^{\otimes k}(\chi)) = 0$ in Theorem 4.1.6.

Now combining Theorem 2.6.1 and Theorem 4.1.6 yields:

Corollary 4.1.8. *The followings are equivalent:*
(1) $\mathcal{I} \trianglelefteq \mathbb{Z}_p[\chi]$ *is the maximal congruence of Eisenstein series in* $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$.
(2) $H^1_c(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes k}[\chi]) \simeq \mathbb{Z}_p[\chi]/\mathcal{I}$.

Remark 4.1.9. Comparing Corollary 4.1.8 with Proposition 3.2.1 and Proposition 3.2.6, the group cohomology $H^1_c(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes k}[\chi])$ computes the *denominator* of $\frac{B_{k,\chi}}{2k} \in \mathbb{Q}_p(\chi)$ under the assumptions in Cases I-V in Theorem 3.1.4. In Cases VI and VII, this cohomological computation sheds light on the *numerator* of $\frac{B_{k,\chi}}{2k}$ (still does not determine the valuation in general).

4.2. **The Dirichlet $K(1)$-local spheres and J-spectra.** The group cohomology $H^1_c(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes t})$ is on the E_2-page of a **homotopy fixed point spectral sequence** (HFPSS):

$$E_2^{s,t} = H^s_c(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes t}) \Longrightarrow \pi_{2t-s}\left((KU_p^\wedge)^{h\mathbb{Z}_p^\times}\right),$$

where $a \in \mathbb{Z}_p^\times$ acts on the p-adic K-theory spectrum $KU_p^\wedge$ by the Adams operation ψ^a. The homotopy fixed point spectrum is equivalent to $S^0_{K(1)}$, the Bousfield localization of the sphere spectrum at $K(1)$, the Morava K-theory of height 1 at prime p. The HFPSS collapses on the E_2-page except when $p = 2$. This yields isomorphisms

$$H^1_c(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes 2k}) \simeq \pi_{4k-1}(S^0_{K(1)})$$

for all primes p. The $K(1)$-local sphere spectrum is the p-completion of the K-local sphere spectrum S^0_{KU}. S^0_{KU} by construction is a KU-local $\mathbb{E}_\infty$-ring spectum. The Hurewicz image in $\pi_*(S^0_{KU})$ detects the image of the stable J-homomorphisms in $\pi_*(S^0)$. In way, when the character χ is trivial, we have given a new explanation of the connection between the congruences E_{2k} and the image of J-homomorphism in $\pi_{4k-1}(S^0)$ in Corollary 4.1.8.

In [Zha19], the author constructed the **Dirichlet $K(1)$-local sphere** for each p-adic Dirichlet character $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$. These Dirichlet $K(1)$-local spheres are defined by

$$S^0_{K(1)}(p^v)^{h\chi} := \operatorname{Map}\left(M(\mathbb{Z}_p[\chi]), S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/N)^\times}.$$

In this construction

- $M(\mathbb{Z}_p[\chi])$ is a Moore spectrum of $\mathbb{Z}_p[\chi]$ together with a $(\mathbb{Z}/N)^\times$-action such that the induced action on $\pi_0(M(\mathbb{Z}_p[\chi]))$ is equivalent to that on $\mathbb{Z}_p[\chi]$.
- $S^0_{K(1)}(p^v) := \left(KU_p^\wedge\right)^{h(1+p^v\mathbb{Z}_p)}$ is a $(\mathbb{Z}/p^v)^\times$-Galois extension of the $K(1)$-local sphere.

$S^0_{K(1)}(p^v)^{h\chi}$ can be identified with

$$S^0_{K(1)}(p^v)^{h\chi} \simeq \operatorname{Map}\left(M(\mathbb{Z}_p[\chi]), KU_p^\wedge\right)^{\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times},$$

where $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$ acts on $M(\mathbb{Z}_p[\chi])$ through the $(\mathbb{Z}/p^v)^\times \times (\mathbb{Z}/N')^\times \simeq (\mathbb{Z}/N)^\times$-action on $M(\mathbb{Z}_p[\chi])$ and on $KU_p^\wedge$ through the Adams operations on $KU_p^\wedge$ by $\mathbb{Z}_p^\times$. Then the group cohomology $H_c^1(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes k}[\chi^{-1}])$ is on the E_2-page of another HFPSS:

$$H_c^s(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes t}[\chi^{-1}]) \Longrightarrow \pi_{2t-s}\left(S^0_{K(1)}(p^v)^{h\chi}\right).$$

This spectral collapses at the E_2-page under the assumptions of Cases I, III, and V in Theorem 3.1.4. In those cases, we get

$$H_c^1(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes t}[\chi^{-1}]) \simeq \pi_{2t-1}\left(S^0_{K(1)}(p^v)^{h\chi}\right).$$

Like the classical $K(1)$-local sphere, the Dirichlet $K(1)$-local sphere is a summand of the p-completions of the **Dirichlet J-spectra** $J(N)^{h\chi}$, where $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ is $\mathbb{C}$-valued Dirichlet character. It is defined by

$$J(N)^{h\chi} := \operatorname{Map}(M(\mathbb{Z}[\chi]), J(N))^{h(\mathbb{Z}/N)^\times}$$

In this construction:

- $M(\mathbb{Z}[\chi])$ is the Moore spetrum of $\mathbb{Z}[\chi]$ with an $(\mathbb{Z}/N)^\times$-action such that the induced action on its zeroth homomotopy group is equivalent to that on $\mathbb{Z}[\chi]$.
- $J(N)$ is a "J-spectrum of level μ_N". It is constructed with the arithmetic pullback square:

$$\begin{tikzcd}
J(N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_p S^0_{KU/p}\left(p^{v_p(N)}\right) \arrow[d, "\text{Rationalization}"] \\
S^0_{\mathbb{Q}} \arrow[r, "\text{Hurewicz}"] & \left(\prod_p S^0_{KU/p}\left(p^{v_p(N)}\right)\right)_{\mathbb{Q}}
\end{tikzcd}$$

  $J(N)$ is a K-local $\mathbb{E}_\infty$-ring spectrum with an $(\mathbb{Z}/N)^\times$-action by $\mathbb{E}_\infty$-ring automorphisms. This $(\mathbb{Z}/N)^\times$-action is inherited from the $(\mathbb{Z}/p^v)^\times$-actions on $S^0_{K/p}\left(p^{v_p(N)}\right)$ for all primes $p \mid N$.

The splitting of $\left(J(N)^{h\chi}\right)^\wedge_p$ is parallel to that of the Eisenstein subspace in Proposition 1.4.11. In this way, we have connected congruences of Eisenstein series in $\mathcal{E}_k(\mu_N, \chi)$ to homotopy groups of the Dirichlet J-spectra $\pi_{2k-1}(J(N)^{h\chi^{-1}})$ in Corollary 4.1.8. In addition, this explains how these homotopy groups are related to special values of the corresponding Dirichlet L-function $L(s;\chi)$.

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