$RO(G)$-GRADED HOMOTOPY MACKEY FUNCTOR OF $H\underline{\mathbb{Z}}$ FOR C_{p^2} AND HOMOLOGICAL ALGEBRA OF $\underline{\mathbb{Z}}$-MODULES

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Abstract. In this paper we compute the $RO(G)$-graded homotopy Mackey functor for $H\underline{\mathbb{Z}}$, the Eilenberg-Mac Lane spectrum of constant Mackey functor for $G = C_{p^2}$, and give some computation for larger G. As an application, we use it to give some computation of homological algebra of $\underline{\mathbb{Z}}$-modules.

1. Introduction

In the Kervaire invariant one paper [HHR16], Hill, Hopkins and Ravenel developed a computational tool for equivariant stable homotopy theory, namely the slice spectral sequence. It is later refined by Ullman in [Ull13] to obtain a better multiplicative property. The slice spectral sequence computes $\underline{\pi}_\bigstar X$, the $RO(G)$-graded homotopy Mackey functors of a G-spectrum X, from $RO(G)$-graded homotopy Mackey functors of "slices" of X. In many cases, slices of an interesting spectrum (e.g. MU and its norm in [HHR16], $Tmf_1(3)$ in [HM17] and Morava E-theories in [HS]) are suspensions of $H\underline{\mathbb{Z}}$ or its variants by virtual representations. Therefore, if we want to compute the $RO(G)$-graded homotopy Mackey functor of these spectra, we need to understand the $RO(G)$-graded homotopy Mackey functor of $H\underline{\mathbb{Z}}$. This is the main topic of this paper.

In this paper, we will focus on the computation of $RO(G)$-graded homotopy Mackey functor of $H\underline{\mathbb{Z}}$ for $G = C_{p^2}$. The main result, including the $RO(G)$-graded multiplicative structure, is Theorem 4.8. $RO(G)$-graded homotopy Mackey functor of $H\underline{\mathbb{Z}}$ for $G = C_p$ is known for decades, and its computation can be found in [Dug05] and [Gre]. Computing everything in terms of Mackey functor is essential, since we need to make use of homological algebra of Mackey functors. We will use four different ways to compute and explain $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}$, for $G = C_{p^n}$:

(1) **Cellular Method.** Notice that $\underline{\pi}_V H\underline{\mathbb{Z}} \cong \underline{H}_0(S^{-V}; \underline{\mathbb{Z}})$, where $\underline{H}_\bigstar(-;\underline{\mathbb{Z}})$ is the Mackey functor valued ordinary homology with coefficient $\underline{\mathbb{Z}}$. Then we can use G-cellular structure of S^{-V} to compute its homology. When V is an actual representation or the opposite of one, S^V has at most one G-cell in each dimension, therefore the chain complex can be easily computed. In fact, this chain complex of Mackey functors is determined by its underlying chain complex of abelian groups. If $V = V_1 - V_2$, where V_1 and V_2 are two actual representations, we can use the product cell structure on S^V. In this case, we get a double complex. When we filter the double complex by cellular structure of S^{V_1} and S^{-V_2}, we get two spectral sequences. Comparison between these two spectral sequences can deduce a lot of differentials, and gives a complete answer when $G = C_{p^2}$.

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(2) **Cofibre of a Method.** We can build G-spectra L_n with the following properties:
   - $\underline{\pi}_*(L_n)$ can be computed.
   - For oriented representation V, $\Sigma^V L_n \simeq \Sigma^{|V|} L_n$, therefore its $RO(G)$-graded homotopy Mackey functor is just shift of integer graded one.
   - There are cofibre sequences $S^V \wedge H\underline{\mathbb{Z}} \to S^{V+n\lambda_0} \wedge H\underline{\mathbb{Z}} \to S^V \wedge L_n$, where λ_0 is the representation given by multiplication by a primitive p^n-th root of unity on the complex plane.

   Assume we know $\underline{\pi}_*(S^V \wedge H\underline{\mathbb{Z}})$ and $\underline{\pi}_*(S^V \wedge L_n)$, we can try to compute the connecting homomorphism and extensions, to figure out $\underline{\pi}_*(S^{V+n\lambda} \wedge H\underline{\mathbb{Z}})$.

(3) **Tate Diagram Method.** Consider the Tate diagram from [GM95], which is a homotopy pullback of ring spectra:

$$\begin{array}{ccc} X & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ X^h & \longrightarrow & X^t \end{array}$$

   Notations in the diagram will be explained in Section 3.3. In our computation $X = H\underline{\mathbb{Z}}$, and we can compute $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}}^h)$ and $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}}^t)$ by group cohomology. If we can compute $\underline{\pi}_\bigstar \tilde{X}$, then we can use homotopy pullback to compute $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}$, not only as $RO(G)$-graded Mackey functor, but as a $RO(G)$-graded Green functor, which includes its graded ring structure.

(4) **Duality Method**. There are two dualities in the homotopy category of $H\underline{\mathbb{Z}}$-modules, namely equivariant Anderson duality and Spanier-Whitehead duality. Equivariant Anderson duality, denoted by $I_{\mathbb{Z}}$, is constructed in [Ric16]. In our case it computes $\underline{\pi}_*(S^{2-\lambda-V} \wedge H\underline{\mathbb{Z}})$ from $\underline{\pi}_*(S^V \wedge H\underline{\mathbb{Z}})$ by a short exact sequence. The $H\underline{\mathbb{Z}}$-module Spanier Whitehead duality will send $S^V \wedge H\underline{\mathbb{Z}}$ to $S^{-V} \wedge H\underline{\mathbb{Z}}$, and there is a universal coefficient spectral sequence convergent to $\underline{\pi}_*(S^{-V} \wedge H\underline{\mathbb{Z}})$ with E_2-page

$$\underline{Ext}^{i,j}_{\underline{\mathbb{Z}}}(\underline{\pi}_*(S^V \wedge H\underline{\mathbb{Z}}), \underline{\mathbb{Z}})$$

   Therefore some homological algebra of $\underline{\mathbb{Z}}$-modules is required. We will show how to compute these $\underline{Ext}$ Mackey functors and use them to calculate the spectral sequence. These two dualities gives an explanation of structure of $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}$, especially an algebraic explanation of the "gap" phenomenon in [HHR16, Section 8].

The universal coefficient spectral sequence can be used reversely: We can compute $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}$ by other methods and use the spectral sequence to calculate homological algebra. This approach is taken in Section 6. We can compute various $\underline{Ext}$ Mackey functors and obtain pure algebra propositions by topological means.

In Section 2, we will review definitions and tools needed for our computation. We will go over four different methods of computation in Section 3. Section 4 is focused on the computation for $G = C_{p^2}$. In Section 5 we discuss some computation about large cyclic p-groups. In Section 6 we use our computation to explore some homological algebra of $\underline{\mathbb{Z}}$-modules.

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2. Definitions and Tools

In this section, we review some definitions and constructions that will be used in our computation.

2.1. Mackey Functors and $\underline{\mathbb{Z}}$-Modules.

The definition of Mackey functor and some properties are given by Thévenaz and Webb in [TW95] and Lewis in [Lew80]. We will use $Burn_G$ for the Burnside category, and $(Mack_G, \Box, \underline{A})$ for the category of Mackey functors, its tensor product(the box product) and its tensor unit(Burnside Mackey functor). All Mackey functors in this paper will be presented with an underline.

We use Lewis diagram to represent Mackey functors. Let $\underline{M}$ be a Mackey functor, We will put $\underline{M}(G/G)$ on the bottom and $\underline{M}(G/e)$ on the top. Thus restrictions are maps going downwards and transfers are maps going upwards. Weyl group action will be indicated by G-module structure on each $\underline{M}(X)$, since our group G is always abelian. For example, a Lewis diagram of a C_{p^2}-Mackey functor $\underline{M}$ is the following:

$$\begin{array}{c} \underline{M}(C_{p^2}/C_{p^2}) \\ {\scriptstyle Res^{p^2}_p} \downarrow \ \uparrow {\scriptstyle Tr^{p^2}_p} \\ \underline{M}(C_{p^2}/C_p) \\ {\scriptstyle Res^{p}_e} \downarrow \ \uparrow {\scriptstyle Tr^{p}_e} \\ \underline{M}(C_{p^2}/e) \end{array}$$

Now we give some examples of Mackey functors which will be used in our computation.

Definition 2.1. *Given a G-module M, the fixed point Mackey functor $\underline{M}$ is defined as $\underline{M}(G/H) = M^H$, the H-fixed point of M, as an G/H-module. Restrictions are inclusions of fixed point, and transfers are summations over cosets. $\underline{0}$ is the trivial Mackey functor and $\underline{\mathbb{Z}}$ is the constant Mackey functor of $\mathbb{Z}$.*

Definition 2.2. *A Mackey functor $\underline{M}$ is called a form of $\underline{\mathbb{Z}}$ if $\underline{M}(G/H) \cong \mathbb{Z}$ with trivial G-action for all $H \subset G$.*

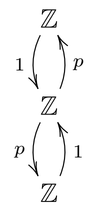
Remark 2.3. *Notice that by the double coset formula of Mackey functors, in a form of $\underline{\mathbb{Z}}$ composition of restriction and transfer between adjacent level in Lewis diagram is multiplication by p, thus one of them is isomorphism and another is multiplication by p. Therefore, for $G = C_{p^n}$, there are 2^n isomorphism classes of forms of $\underline{\mathbb{Z}}$.*

Definition 2.4. *Let $\underline{\mathbb{Z}}_{t_1,t_2,\ldots,t_n}$, where $t_i = 0$ or 1 be the form of $\underline{\mathbb{Z}}$ for C_{p^n} such that $Res^{p^i}_{p^{i-1}} = p^{t_i}$ for $1 \leq i \leq n$.*

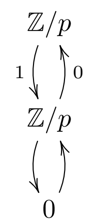
There is another class of Mackey functors that will appear in the computation:

Definition 2.5. *Let* $\underline{\mathbb{B}}_{t_0,t_1,\ldots,t_n}$ *be the cokernel of* $\underline{\mathbb{Z}}_{t_0,t_1,\ldots,t_n} \to \underline{\mathbb{Z}}$*, where the map is isomorphism on* G/e*-level.*

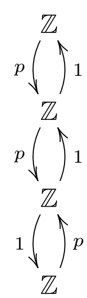
Example 2.6. $\underline{\mathbb{Z}}_{1,0}$ *for* C_{p^2} *has the following Lewis diagram*



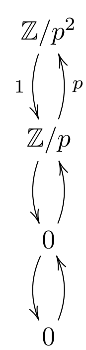
$\underline{\mathbb{B}}_{1,0}$ *for* C_{p^2} *has the following Lewis diagram*



Example 2.7. $\underline{\mathbb{Z}}_{0,1,1}$ *for* C_{p^3} *has the following Lewis diagram*



$\underline{\mathbb{B}}_{0,1,1}$ *for* C_{p^3} *has the following Lewis diagram*



Inside $Mack_G$, the category of G-Mackey functors, we will only be interested in $Mod_{\underline{\mathbb{Z}}}$, the category of $\underline{\mathbb{Z}}$-modules. The following proposition identifies $\underline{\mathbb{Z}}$-modules by a simple condition.

Proposition 2.8 ([TW95, Proposition 16.3]). *A Mackey functor $\underline{M}$ is a $\underline{\mathbb{Z}}$-module if and only if it is cohomological, i.e. $Tr^H_{H'}Res^H_{H'}(x) = [H : H']x$ for all $H' \subset H \subset G$ and all $x \in \underline{M}(G/H)$*

Notice that all $\underline{\mathbb{Z}}_{t_0,t_1,\ldots,t_n}$ and $\underline{\mathbb{B}}_{t_0,t_1,\ldots,t_n}$ are $\underline{\mathbb{Z}}$-modules.

The following is a simple lemma, but it is useful in computatoin.

Lemma 2.9. *Let $\underline{M}$ be a $\underline{\mathbb{Z}}$-module that $\underline{M}(G/e)$ is torsion, then $\underline{M}(G/H)$ is torsion for any orbit G/H.*

Proof. Let $x \in \underline{M}(G/H)$, then $|H|x = Tr^H_e(Res^H_e(x))$ is torsion by Proposition 2.8, so x is torsion. □

Since $\underline{\mathbb{Z}}$ is a commutative monoid under box product, we can define box product in $\underline{\mathbb{Z}}$-modules, $\Box_{\underline{\mathbb{Z}}}$ by the coequalizer diagram

$$\underline{M}\Box\underline{\mathbb{Z}}\Box\underline{N} \rightrightarrows \underline{M}\Box\underline{N} \to \underline{M}\Box_{\underline{\mathbb{Z}}}\underline{N}$$

Tensor unit of $\Box_{\underline{\mathbb{Z}}}$ is $\underline{\mathbb{Z}}$. It has a right adjoint, the internal hom of $\underline{\mathbb{Z}}$-modules, denoted by $\underline{Hom}_{\underline{\mathbb{Z}}}$.

Proposition 2.10. *The category $Mod_{\underline{\mathbb{Z}}}$ has enough projective and injective objects, and the set*

$$\{\underline{\mathbb{Z}[X]} |\ X \text{ a finite } G\text{-set}\}$$

forms enough projective objects.

Provided tensor product, internal Hom and enough projective and injective objects, it is a standard course to define derived functors used in homological algebra.

Definition 2.11. *For $\underline{N} \in Mod_{\underline{\mathbb{Z}}}$, we define $\underline{Ext}^i_{\underline{\mathbb{Z}}}(-, \underline{N})$ to be the i-th right derived functor of $\underline{Hom}_{\underline{\mathbb{Z}}}(-, \underline{N})$ and $\underline{Tor}^i_{\underline{\mathbb{Z}}}(-, \underline{N})$ to be the i-th left derived functor of $-\Box_{\underline{\mathbb{Z}}}\underline{N}$.*

A nice fact about homological algebra of $Mod_{\underline{\mathbb{Z}}}$ is that we know its global cohomological dimension:

Theorem 2.12 ([BSW17, Theorem 1.7] [Arn81]). *If G is cyclic and finite, then $Mod_{\underline{\mathbb{Z}}}$ has global cohomological dimension 3. More precisely, any $\underline{\mathbb{Z}}$-module has a projective resolution of length at most 3.*

Remark 2.13. *Even though this theorem is nice, we will make no use of it through the paper, since the minimal resolution might not be the easiest to compute. However, as we will see, this theorem gives very nice control of universal coefficient and Künneth spectral sequences.*

There is another kind of Hom and Ext functors that will be very useful, which is defined level-wisely using Hom and Ext of abelian groups. See [Ric16, Section 3].

Definition 2.14. *Given an abelian group A and a Mackey functor $\underline{M}$, The level-wise Hom functor, $Hom_L(\underline{M}, A)$ is the Mackey functor defined by the composition*

$$Burn_G \xrightarrow{\underline{M}} Ab \xrightarrow{Hom(-,A)} Ab$$

And the level-wise Ext functor, $Ext_L(\underline{M}, A)$ is the Mackey functor defined by the composition

$$Burn_G \xrightarrow{\underline{M}} Ab \xrightarrow{Ext(-,A)} Ab$$

Both subscript L stands for "level-wise", to distinguish this Hom and Ext from the "internal" one defined above. These two types of Hom and Ext functors behaves very differently: $\underline{Hom}_{\underline{\mathbb{Z}}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}$ but $Hom_L(\underline{\mathbb{Z}}, \mathbb{Z}) \cong \underline{\mathbb{Z}}^*$. We will see in Theorem 3.13 how these functors are related.

Definition 2.15. *Given a Mackey functor $\underline{M}$, we will use $\underline{M}^*$ for $Hom_L(\underline{M}, \mathbb{Z})$ and $\underline{M}^E$ for $Ext_L(\underline{M}, \mathbb{Z})$.*

For forms of $\underline{\mathbb{Z}}$, $\underline{\mathbb{Z}}^*_{t_0,t_1,\ldots,t_n} \cong \underline{\mathbb{Z}}_{1-t_0,1-t_1,\ldots,1-t_n}$.

2.2. **Indexing Groups.**

We need to address our indexing group of homotopy Mackey functor carefully. From a standard point of view, homotopy Mackey functors of a G-spectrum is indexed by $RO(G)$, the group of finite dimensional virtual $\mathbb{R}$-representations of G. For $G = C_{p^n}$, $RO(G)$ is generated by the trivial representation and $\mathbb{R}^2$-representations given by multiplication by a primitive p^i-th root of unity, for $1 \leq i \leq n$ (For $p = 2$, the representation given by multiplying -1 on $\mathbb{R}^2$ is isomorphic to sum of two copies of sign representation). We know from [Kaw80] that for different primitive p^i-th roots of unity γ and γ', their corresponding representation spheres are not stably equivalent. However, if we are interested in $H\underline{\mathbb{Z}}$, the following theorem of Hu and Kriz tells us that different choice of primitive p^i-th root of unity doesn't matter.

Theorem 2.16 ([HK, Lemma 1])**.** *If γ, γ' are primitive p^i-th roots of unity, then $S^\gamma \wedge H\underline{\mathbb{Z}} \simeq S^{\gamma'} \wedge H\underline{\mathbb{Z}}$.*

We give a proof in Section 3.1.

By this theorem, for $G = C_{p^n}$, we will not distinguish among different primitive p^i-th roots of unity, therefore we can think of our indexing group as direct sum of $n+1$ copies of $\mathbb{Z}$, generated by trivial representation and representation spheres corresponding to a primitive p^i-th root of unity for each $1 \leq i \leq n$. We will denote the representation by multiplying a primitive p^i-th root of unity on complex plane λ_{n-i} and set $\lambda := \lambda_0$. If $p = 2$, $\lambda_{n-1} = 2\sigma$, where σ is the sign representation on C_{2^n}.

In this paper, we will use $*$ for integer grading, and $\bigstar$ for $RO(G)$ grading.

2.3. **Equivariant Anderson Duality.**

The classical Anderson duality is defined in [And], and a C_2-equivariant version of it appears in [Ric16]. We need a version for C_{p^n}-equivariant spectra. However, all definition and properties in [Ric16, Section 3] will work with identical proof, so we will only list the definition and properties we need without giving proofs here.

Proposition 2.17. *Let A be an injective abelian group, then $X \mapsto Hom(\pi^G_{-*}(X), A)$ defines a cohomology theory.*

We will use I_A for the G-spectrum representing $Hom(\pi^G_{-*}(-), A)$.

Definition 2.18. *Let X be a G-spectrum, the Anderson dual of X, denoted by $I_{\mathbb{Z}}(X)$, is defined to be the homotopy fibre of $Fun(X, I_{\mathbb{Q}}) \to Fun(X, I_{\mathbb{Q}/\mathbb{Z}})$.*

Proposition 2.19. *Let E and X be G-spectra, then there is a short exact sequence of Mackey functors, natural in both X and E:*

$$0 \to Ext_L(E_{\bigstar-1}(X), \mathbb{Z}) \to I_{\mathbb{Z}}(E)^{\bigstar}(X) \to Hom_L(E_{\bigstar}(X), \mathbb{Z}) \to 0$$

Corollary 2.20. *(1) If $\underline{M}$ is a form of $\underline{\mathbb{Z}}$, then $I_{\mathbb{Z}}(H\underline{M}) \simeq H\underline{M}^*$.*
(2) If $\underline{M}$ is a $\underline{\mathbb{Z}}$-module with $\underline{M}(G/e) \cong 0$, then $I_{\mathbb{Z}}(H\underline{M}) \simeq \Sigma^{-1} H\underline{M}^E$.

Proof. (1) is a direct consequence of the short exact sequence of Anderson duality and uniqueness of Eilenberg-Mac Lane spectra.
For (2), by Lemma 2.9, all $\underline{M}(X)$ are torsion, so the result follows again from the short exact sequence. □

Corollary 2.21. *If R is a commutative ring spectrum and M is a R-module, then $I_{\mathbb{Z}}(M)$ is naturally an R-module. Furthermore, if M, N are R-modules, then*

$$Hom_R(M, I_{\mathbb{Z}}(N)) \simeq I_{\mathbb{Z}}(M \wedge_R N)$$

2.4. **Universal Coefficient and Künneth Spectral Sequences.**

The equivariant version of universal coefficient and Künneth spectral sequences are developed by Lewis and Mandell in [LM06]. $\underline{Ext}$ and $\underline{Tor}$ used in these spectral sequences are defined in Definition 2.11.

Theorem 2.22 ([LM06, Theorem 1.1])**.** *Let R be a commutative G-ring spectrum, X be a G-spcetrum and M be an R-module.*

1. *There is a natural strongly convergent homology spectral sequence of $\underline{R}_*$-modules*
$$\underline{E}^2_{s,t} = \underline{Tor}^{\underline{R}_*}_{s,t}(\underline{R}_* X, \underline{M}_*) \Rightarrow \underline{M}_{s+t} X$$
with differentials
$$d^r : \underline{E}^r_{s,t} \to \underline{E}^r_{s-r,t+r-1}$$
2. *There is a natural conditionally convergent cohomology spectral sequence of $\underline{R}_*$-modules*
$$\underline{E}^{s,t}_2 = \underline{Ext}^{s,t}_{\underline{R}_*}(\underline{R}_* X, \underline{M}_*) \Rightarrow \underline{M}^{s+t} X$$
with differentials
$$d_r : \underline{E}^{s,t}_r \to \underline{E}^{s+r,t-r+1}_r$$

We will mostly use the case when $R = H\underline{\mathbb{Z}}$, and $M = R$ for the universal coefficient spectral sequence, and $M = \Sigma^V H\underline{\mathbb{Z}}$ in the Künnneth spectral sequence.

Corollary 2.23. (1) *Given a G-spectrum X, there is a spectral sequence with $\underline{E}_2$-page*
$$\underline{Ext}^{i,j}_{\underline{\mathbb{Z}}}(\underline{H}_*(X; \underline{\mathbb{Z}}), \underline{\mathbb{Z}})$$
and is convergent to $\underline{H}^*(X; \underline{\mathbb{Z}})$.

(2) *Given G-spectra X, Y, there is a spectral sequence with $\underline{E}_2$-page*
$$\underline{Tor}^{\underline{\mathbb{Z}}}_{i,j}(\underline{H}_*(X; \underline{\mathbb{Z}}), \underline{H}_*(Y; \underline{\mathbb{Z}}))$$
and is convergent to $\underline{H}_*(X \wedge Y; \underline{\mathbb{Z}})$.

By Theorem 2.12, these spectral sequences are strongly convergent and can only have d_2, d_3 and potential extensions, making them very reasonable to compute. Notice that the version we need is the $\mathbb{Z}$-graded version rather than $RO(G)$-graded version, because if we are using the $RO(G)$-graded version, then the spectral sequence collapse at $\underline{E}_2$ and its input equals to its output.

2.5. **Some Elements in $H\underline{\mathbb{Z}}_\bigstar$.**

We define two families of elements in $H\underline{\mathbb{Z}}_\bigstar$, which plays a crucial role in our computation. More detail including proofs of this section is in [HHR17b, Section 3].

Definition 2.24. (1) *For an actual representation V with $V^G = 0$, let $a_V \in \underline{\pi}_{-V}(S^0)$ be the map $S^0 \to S^V$ which embeds S^0 to 0 and ∞ in S^V. We will also use a_V for its Hurewicz image in $\underline{\pi}_{-V}(H\underline{\mathbb{Z}})$.*

(2) *For an actual orientable representation W, let u_W be the generator of $\underline{H}_{|W|}(S^W;\underline{\mathbb{Z}})(G/G)$ which restricts to the choice of orientation in*

$$\underline{H}_{|W|}(S^W;\underline{\mathbb{Z}})(G/e) \cong H_{|W|}(S^{|W|};\mathbb{Z})$$

In homotopy grading, $u_W \in \underline{\pi}_{|W|-W}(H\underline{\mathbb{Z}})(G/G)$.

Proposition 2.25. *Elements $a_V \in \underline{\pi}_{-V}(H\underline{\mathbb{Z}})(G/G)$ and $u_W \in \underline{\pi}_{|W|-W}(H\underline{\mathbb{Z}})(G/G)$ satisfy the following:*

(1) $a_{V_1+V_2} = a_{V_1}a_{V_2}$ *and* $u_{W_1+W_2} = u_{W_1}u_{W_2}$.
(2) $Res^G_H(a_V) = a_{i^*_H(V)}$ *and* $Res^G_H(u_V) = u_{i^*_H(V)}$
(3) $|G/G_V|a_V = 0$, *where G_V is the isotropy subgroup of V.*
(4) ***The gold relation.*** *For V, W oriented representations of degree 2, with $G_V \subset G_W$,*

$$a_W u_V = |G_W/G_V| a_V u_W$$

In terms of oriented irreducible representations of C_{p^n}, the gold relation reads

$$\text{For } 0 \le i < j < n,\ a_{\lambda_j}u_{\lambda_i} = p^{i-j}a_{\lambda_i}u_{\lambda_j}$$

(5) *The subring consists of $\underline{\pi}_{i-V}(H\underline{\mathbb{Z}})(G/G)$ where V is an actual representation is*

$$\mathbb{Z}[a_{\lambda_i}, u_{\lambda_i}]/(p^{n-i}a_{\lambda_i} = 0, \text{gold relations}) \text{ for } 0 \le i, j < n$$

We will call this subring BB_G, standing for "basic block". We will use $\underline{BB}_G$ for the graded Green functor in the corresponding $RO(G)$-degree of BB_G. We will omit G if there is no ambiguity.

3. General Strategy

We have four different approaches to $H\underline{\mathbb{Z}}_\bigstar$ and this section serves as an introduction of them.

3.1. **Cellular Method.**

Since $H\underline{\mathbb{Z}}_V \cong H\underline{\mathbb{Z}}_0(S^{-V};\underline{\mathbb{Z}})$, and S^{-V} is a G-CW-complex, we can compute $H\underline{\mathbb{Z}}_\bigstar$ via cellular homology of all S^V. If we write $V = \sum_{i=0}^{n-1} a_i\lambda_i$, then $S^V \cong \bigwedge_{i=0}^{n-1} S^{a_i\lambda_i}$. Each $S^{a_i\lambda_i}$ has a cellular structure given in [HHR17a, Section 1.2]. And we can produce different filtrations by filtering along these cellular structures. In this way we will get $n-1$ different spectral sequences with different E_2-page but all converge to $\underline{H}_*(S^V;\underline{\mathbb{Z}})$. Then we can compare E_2-pages and try to figure out all differentials and extensions.

The following lemma is crucial in our computation, and it is easily seen from cellular structure of representation spheres.

Lemma 3.1. *Let $G = C_{p^n}$ and $H = C_p$ as the subgroup of G. Let $V_G = \Sigma_{i=1}^n a_i\lambda_i$ be a virtual G-representation with no copies of λ_0. Since V_G has no copies of λ_0, it factors through G/H. Let $V_{G/H}$ be the virtual G/H-representation that gives V_G by composition with the quotient map, then $\underline{H}_i(S^{V_G};\underline{\mathbb{Z}})$ can be computed by $\underline{H}_i(S^{V_{G/H}};\underline{\mathbb{Z}})$ by the following:*

$$\underline{H}_i(S^{V_G};\underline{\mathbb{Z}})(G/H') \cong \begin{cases} \underline{H}_i(S^{V_{G/H}};\underline{\mathbb{Z}})((G/H)/(H'/H)) & \text{for } H \subset H' \\ \underline{H}_i(S^{V_{G/H}};\underline{\mathbb{Z}})((G/H)/e) & \text{for } H = e \end{cases}$$

With $Res^H_e : \underline{H}_i(S^{V_G};\underline{\mathbb{Z}})(G/H) \to \underline{H}_i(S^{V_G};\underline{\mathbb{Z}})(G/e)$ the identity map and $Tr^H_e : \underline{H}_i(S^{V_G};\underline{\mathbb{Z}})(G/e) \to \underline{H}_i(S^{V_G};\underline{\mathbb{Z}})(G/H)$ multiplication by p.

Furthermore, $a_{V_{G/H}}$ (or $u_{V_{G/H}}$) maps to a_V (or u_V) under the above isomorphism. In terms of irreducible representations, a_{λ_i} (or u_{λ_i}) in G/H maps to $a_{\lambda_{i+1}}$ (or $u_{\lambda_{i+1}}$) in G.

Proof. Since all irreducible summands in V_G are from G/H, all G-cells of S^{V_G} are of the form $S^k \wedge G/H'_+$ for some nontrivial $H' \subset G$. In the $\underline{\mathbb{Z}}$-coefficient cellular chain complex, such a G-cell is corresponding to the Mackey functor $\underline{\mathbb{Z}[G/H']}$. For $V_{G/H}$, the corresponding G/H-cell is $S^k \wedge ((G/H)/(H'/H))_+$, and in chain complex it gives $\underline{\mathbb{Z}[(G/H)/(H'/H)]}$. Now we see that the algebraic description in the lemma is true in chain level with respect to all chain differentials. Therefore it is true in homology. □

Remark 3.2. *This lemma will be the heart of our induction. We can compute $\underline{\pi}_*(S^V \wedge H\underline{\mathbb{Z}})$ if V contains no copy of λ_0 by computing in quotient group. Then we can compute $\underline{\pi}_*(S^{V+n\lambda_0} \wedge H\underline{\mathbb{Z}})$ by different methods.*

As promised, we will give a proof of Theorem 2.16 here.

Proof of Theorem 2.16. We only need to show that $S^\gamma \wedge S^{-\gamma'} \wedge H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}$, where γ and γ' are two different primitive p^i-th roots of unity. Therefore, we only need to compute $\underline{\pi}_*(S^\gamma \wedge S^{\gamma'} \wedge H\underline{\mathbb{Z}})$. By [HHR17a, Section 1.2], the cellular chain complex of S^γ, $\underline{C}_*(S^\gamma)$ is

$$\underline{\mathbb{Z}} \leftarrow \underline{\mathbb{Z}[C_{p^n}/C_{p^i}]} \leftarrow \underline{\mathbb{Z}[C_{p^n}/C_{p^i}]}$$

Where all chain maps are determined by the fact that S^γ is S^2 when forget the group action. Similarly, $\underline{C}_*(S^{-\gamma'})$ is

$$\underline{\mathbb{Z}[C_{p^n}/C_{p^i}]} \leftarrow \underline{\mathbb{Z}[C_{p^n}/C_{p^i}]} \leftarrow \underline{\mathbb{Z}}$$

Now one can compute the total homology of $\underline{C}_*(S^\gamma)\Box_{\underline{\mathbb{Z}}}\underline{C}_*(S^{-\gamma})$ and see it is concentrated in degree 0 and $\underline{H}_0 \cong \underline{\mathbb{Z}}$. □

3.2. **Cofibre of a Method.**

Definition 3.3. *Let L_n be the cofibre of the map $a^n_{\lambda_0} : S^0 \wedge H\underline{\mathbb{Z}} \to S^{n\lambda_0} \wedge H\underline{\mathbb{Z}}$ for $n > 0$, and the fibre of the map $a^n_{\lambda_0} : S^{-\lambda_0} \wedge H\underline{\mathbb{Z}} \to S^0 \wedge H\underline{\mathbb{Z}}$. If $p = 2$, we define L'_n to be $\Sigma^\sigma L_n$ for all n.*

Lemma 3.4. *If $V \in RO(G)$ is orientable, then $\Sigma^V L_n \simeq \Sigma^{|V|} L_n$.*

Proof. We only need to prove this lemma for $V = \lambda_i$. The map $u_{\lambda_i} : S^2 \wedge S^{n\lambda_0} \to S^{\lambda_i} \wedge S^{n\lambda_0}$ induces isomorphism between $\underline{\pi}_*(\Sigma^2 L_n) \cong \underline{\pi}_*(\Sigma^{\lambda_i} L_n)$ by direct computation using cellular structure given in [HHR17a, Section 1.2]. □

Following this lemma, we consider cofibre sequences obtained by smashing the defining sequence of L_n with S^V, for V with no copies of λ_0:

$$S^V \wedge H\underline{\mathbb{Z}} \to S^{V+n\lambda_0} \wedge H\underline{\mathbb{Z}} \to S^V \wedge L_n$$

$\underline{\pi}_*(S^V \wedge H\underline{\mathbb{Z}})$ can be computed by Lemma 3.1 through induction, and $\underline{\pi}_*(S^V \wedge L_n) \cong \underline{\pi}_*(S^{|V|} \wedge L_n)$ is also known. Therefore, we need to understand the connecting homomorphisms and extensions to compute $\underline{\pi}_*(S^{V+n\lambda_0} \wedge H\underline{\mathbb{Z}})$.

The first nontrivial connecting homomorphism can be determined by the fact that in $\underline{\pi}_*(S^{V+n\lambda_0} \wedge H\underline{\mathbb{Z}})$ has only one form of $\underline{\mathbb{Z}}$ but $\underline{\pi}_* L_n$ has two, so the form of $\underline{\mathbb{Z}}$ of $\underline{\pi}_* L_n$ in the wrong dimension must kill the form of $\underline{\mathbb{Z}}$ in $\underline{\pi}_*(S^V \wedge H\underline{\mathbb{Z}})$. A large number of connecting homomorphism can be determined by the fact that the connecting homomorphism is a map of $H\underline{\mathbb{Z}}$-modules, therefore commutes with multiplication by a_V and u_V.

Some extensions can be computed by the following lemma.

Lemma 3.5. *(Lemma 4.2 in [HHR17b]) Let G be a finite cyclic 2-group with sign representation a_σ and index 2 subgroup G', and let X be a G-spectrum. Then in $\underline{\pi}_\bigstar X(G/G)$ the image of $Tr^G_{G'}$ is the kernel of multiplication by a_σ, and the kernel of $Res^G_{G'}$ is the image of multiplication by a_σ.*

Remark 3.6. *When $p > 2$, we can still make similar argument. The difference is that the 1-skeleton of $S^{\lambda_{n-1}}$ in C_{p^n} is not an element in the Picard group any more. We need to manually compute all maps from the smash product of S^V with this 1-skeleton into $H\underline{\mathbb{Z}}$. If we ignore the full multiplicative structure, the computation is similar to $p = 2$ in this section.*

This lemma tells us that if an element x maps to nonzero y under the connecting homomorphism but $a_\sigma x$ maps to 0, then there will be a nontrivial extension, as x must lies in the image of transfer. Similarly, if y is killed by connecting homomorphism but $a_\sigma y$ is not, then $a_\sigma y$ will support a nontrivial restriction in the extension.

As we will see in Section 4.3, the above argument determines all connecting homomorphisms and extensions for $G = C_{p^2}$.

3.3. Tate Diagram Method.

Consider the Tate diagram constructed in [GM95]:

Diagram 3.7.

$$\begin{array}{ccccc} X_h & \longrightarrow & X & \longrightarrow & \tilde{X} \\ \downarrow \simeq & & \downarrow & & \downarrow \\ X_h & \longrightarrow & X^h & \longrightarrow & X^t \end{array}$$

Where

$$\begin{aligned} \tilde{X} &= \tilde{EG} \wedge X = a_{\lambda_0}^{-1} X \\ X_h &= EG_+ \wedge X \\ X^h &= Fun(EG_+, X) \\ X^t &= \tilde{EG} \wedge X^h = a_{\lambda_0}^{-1} X^h \end{aligned}$$

And the right square is a homotopy pullback of ring spectra.

In our case $X = H\underline{\mathbb{Z}}$. Our computation goes as follows:

- Assume that we know $\underline{\pi}_*(S^V \wedge H\underline{\mathbb{Z}})$ for all V with no copies of λ_0. Since all such V factor through a quotient group, this can be done by induction of order of G by Lemma 3.1.
- For any V we can compute $\underline{\pi}_*(S^V \wedge H\underline{\mathbb{Z}}^h)$, $\underline{\pi}_*(S^V \wedge H\underline{\mathbb{Z}}_h)$ and $\underline{\pi}_*(S^V \wedge H\underline{\mathbb{Z}}^t)$ and maps between by the corresponding spectral sequences, they all collapse at E_2-page in our case, and maps between then are simply the corresponding maps in group homology and cohomology.
- For V with no copies of λ_0, we can compute $\underline{\pi}_*(S^V \wedge a_{\lambda_0}^{-1} H\underline{\mathbb{Z}})$ and the right vertical map by knowing all other five terms in Tate diagram.
- For $V + n\lambda_0$, notice that both $a_{\lambda_0}^{-1} H\underline{\mathbb{Z}}$ and $H\underline{\mathbb{Z}}^t$ are a_{λ_0}-periodic, therefore we can compute $\underline{\pi}_*(S^{V+n\lambda_0} \wedge H\underline{\mathbb{Z}}_h)$ and $\underline{\pi}_*(S^{V+n\lambda_0} \wedge a_{\lambda_0}^{-1} H\underline{\mathbb{Z}})$.
- In the fibre sequence $S^{V+n\lambda_0} \wedge H\underline{\mathbb{Z}}_h \to S^{V+n\lambda_0} \wedge H\underline{\mathbb{Z}} \to S^{V+n\lambda_0} \wedge a_{\lambda_0}^{-1} H\underline{\mathbb{Z}}$, we know the first and third terms. Connecting homomorphism and extension can be computed by comparison with the bottom row of Tate diagram, which we fully understand.

The main advantage of Tate diagram is that the homotopy pullback is a pullback of ring spectra, so we can compute the ring structure of $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}})$ by this method.

A very important technique in Tate diagram computation is tracking name of elements. We will use the following notation system:

Definition 3.8.
- *For elements in $\underline{\pi}_{*-V} H\underline{\mathbb{Z}}$, where V is an actual representation, we will use (5) of Proposition 2.25 to name them, that is, as monomials of a_{λ_i} and u_{λ_i}.*
- *For elements in $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}^h$, we will use a_V and u_V for their image under the middle vertical map of Tate diagram. In fact,*
$$\underline{\pi}_\bigstar(H\underline{\mathbb{Z}}^h) = \mathbb{Z}[u_{\lambda_i}^{\pm}, a_{\lambda_0}]/(p^n a_{\lambda_0} = 0)$$
and
$$a_{\lambda_i} = \frac{p^i a_{\lambda_0} u_{\lambda_i}}{u_{\lambda_0}}$$
by homotopy fixed point spectral sequence and the gold relation.
- *Since $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}^t = a_{\lambda_0}^{-1} \underline{\pi}_\bigstar(H\underline{\mathbb{Z}}^h) = \mathbb{Z}/p^n[u_{\lambda_i}^{\pm}, a_{\lambda_0}^{\pm}]$, we use the same names as in $H\underline{\mathbb{Z}}^h$.*
- *For elements in $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}}_h)$. If it is the generator of $\underline{\pi}_{|V|-V}(H\underline{\mathbb{Z}}_h)(G/G)$ for some $V \in RO(G)$ and let $V = V_1 - V_2$, where V_1 and V_2 are actual representations. Then it maps to $\frac{p^n u_{V_1}}{u_{V_2}}$. Therefore we will use $\frac{p^n u_{V_1}}{u_{V_2}}$ to name it. All other elements are from $\underline{\pi}_{\bigstar+1}(H\underline{\mathbb{Z}}^t)$ via connecting homomorphism(which is an isomorphism in that degree), therefore we name then as desuspension of their preimage. For example, the image of $\frac{1}{a_{\lambda_0}} \in \underline{\pi}_{\lambda_0}(H\underline{\mathbb{Z}}^t)$ will be named as $\Sigma^{-1}\frac{1}{a_{\lambda_0}}$ in $\underline{\pi}_{\lambda_0}(H\underline{\mathbb{Z}}_h)$.*
- *When $G = C_p$, for elements in $\underline{\pi}_{i+n\lambda}(H\underline{\mathbb{Z}})$ for $n > 0$, since they are all coming from $H\underline{\mathbb{Z}}_h$, we will use the same name as their preimage in $H\underline{\mathbb{Z}}_h$.*
- *For elements in $\underline{\pi}_{i-V}(a_{\lambda_0}^{-1} H\underline{\mathbb{Z}})$, first we assume V has no copies of λ_0. In this case, if $x \in \underline{\pi}_{i-V}(a_{\lambda_0}^{-1} H\underline{\mathbb{Z}})$ is from $\underline{\pi}_{i-V}(H\underline{\mathbb{Z}})$, then we will use the*

name in $\underline{\pi}_{i-V}(H\underline{\mathbb{Z}})$. If x maps nontrivially to $\underline{\pi}_{i-1-V}(H\underline{\mathbb{Z}}_h)$ under connecting homomorphism, then we will use the name of its image in $\underline{\pi}_{i-V}(H\underline{\mathbb{Z}}^t)$ to name it(the connecting homomorphism of the top row of Tate diagram factor through $H\underline{\mathbb{Z}}^t$). For a general V, we use a_{λ_0}-periodicity and names in those representations without λ_0 to name them.

- *For elements in $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}$, if the element comes from $H\underline{\mathbb{Z}}_h$, we use the name there. If it maps nontrivially to $a_{\lambda_0}^{-1}H\underline{\mathbb{Z}}$, we use the name of its image.*

Remark 3.9. *The way of naming elements in $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}(G/G)$ and $\underline{\pi}_\bigstar a_{\lambda_0}^{-1}H\underline{\mathbb{Z}}$ is actually an induction using Lemma 3.1. We will see very detailed examples in computation in Section 4.4 and 5. We need this naming system to give a description of the multiplicative structure of $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}$.*

3.4. **Duality Method.**

Consider $\underline{H}_*(S^{2-\lambda_0};\underline{\mathbb{Z}})$, by direct cellular computation, we see that it is concentrated in dimension 0, and $\underline{H}_0(S^{2-\lambda_0};\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}^*$. Combining with Corollary 2.20, we have the following lemma.

Lemma 3.10. *For $G = C_{p^n}$, $I_{\mathbb{Z}}(H\underline{\mathbb{Z}}) \simeq \Sigma^{2-\lambda_0}H\underline{\mathbb{Z}}$.*

By this lemma, we can think of Anderson duality as a duality with centre of rotation $\frac{2-\lambda_0}{2}$, while universal coefficient spectral sequence computing $Hom_{H\underline{\mathbb{Z}}}(-, H\underline{\mathbb{Z}})$ has centre of rotation 0. Therefore we can use these two dualities one followed by another, to compute $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}})$.

The computation goes as follows:

- Assume that we know $\underline{\pi}_*(S^V \wedge H\underline{\mathbb{Z}})$ for all V with no copies of λ_0, by Lemma 3.1.
- Compute $\underline{\pi}_*(I_{\mathbb{Z}}(S^V \wedge H\underline{\mathbb{Z}})) \cong \underline{\pi}_*(S^{2-\lambda_0-V} \wedge H\underline{\mathbb{Z}})$, using proposition 2.19.
- From $\underline{\pi}_*(S^{2-\lambda_0-V}) \wedge H\underline{\mathbb{Z}})$, we can use universal coefficient spectral sequence in 2.22 to compute $\underline{\pi}_*(S^{V+\lambda_0-2} \wedge H\underline{\mathbb{Z}})$.
- Repeat the procedure, and we can compute for all $V \in RO(G)$.

The following theorem is standard, but combining with Anderson duality, it gives a short cut in computation of universal coefficient spectral sequence.

Theorem 3.11. *The category of unbounded chain complexes of $\underline{\mathbb{Z}}$-modules with projective model structure is Quillen equivalent to the model category of $H\underline{\mathbb{Z}}$-modules.*

Proof. This follows directly from [SS03, Theorem 5.1.1]. The set of tiltors is $\{X_+ \wedge H\underline{\mathbb{Z}} | X \in Set_G\}$ and its endomorphism ringoid is $Mod_{\underline{\mathbb{Z}}}$, the category of $\underline{\mathbb{Z}}$-modules. □

Corollary 3.12. *$\underline{Ext}^i_{\underline{\mathbb{Z}}}(\underline{M}, \underline{N}) \cong \underline{\pi}_{-i}(Hom_{H\underline{\mathbb{Z}}}(H\underline{M}, H\underline{N}))$ and $\underline{Tor}_i^{\underline{\mathbb{Z}}}(H\underline{M}, H\underline{N}) \cong \underline{\pi}_i(H\underline{M} \wedge_{H\underline{\mathbb{Z}}} H\underline{N})$. Here $Hom_{H\underline{\mathbb{Z}}}$ and $\wedge_{H\underline{\mathbb{Z}}}$ are the derived internal Hom and smash product of $H\underline{\mathbb{Z}}$-modules.*

The following theorem and its corollary is essential to our computation.

Theorem 3.13. *If $\underline{M}$ is a $\underline{\mathbb{Z}}$-module with $\underline{M}(G/e) \cong 0$, then $\underline{Ext}^i_{\underline{\mathbb{Z}}}(\underline{M}, \underline{\mathbb{Z}})$ is concentrated in $i = 3$, and $\underline{Ext}^3_{\underline{\mathbb{Z}}}(\underline{M}, \underline{\mathbb{Z}}) \cong \underline{M}^E$.*

$\underline{M}^E$ is defined as the "level-wise" Ext^1 of $\underline{M}$ as in Definition 2.15.

We need a lemma for the proof.

Lemma 3.14. *Let $\underline{M}$ be a $\underline{\mathbb{Z}}$-module with $\underline{M}(G/e) = 0$, then $\Sigma^\lambda H\underline{M} \simeq H\underline{M}$.*

Proof. Consider $\underline{C}_*(S^\lambda)$, the cellular chain of S^λ, which is

$$\underline{\mathbb{Z}} \xleftarrow{\nabla} \underline{\mathbb{Z}[G]} \xleftarrow{1+\gamma} \underline{\mathbb{Z}[G]}$$

Now, $\underline{\pi}_*(\Sigma^\lambda H\underline{M}) = \underline{H}_*(C_*(S^\lambda)\Box_{\underline{\mathbb{Z}}}\underline{M})$. However,

$$\underline{\mathbb{Z}[G]}\Box_{\underline{\mathbb{Z}}}\underline{M}(X) \cong \underline{M}(X \times G) \cong 0$$

Since $X \times G$ is a free G-set and $\underline{M}$ evaluating on free G-set is 0 since $\underline{M}(G/e) \cong 0$. Therefore $\underline{\pi}_*\Sigma^\lambda H\underline{M}$ is concentrated in degree 0, and $\underline{\pi}_0 = \underline{\mathbb{Z}}\Box_{\underline{\mathbb{Z}}}\underline{M} = \underline{M}$. □

Proof of Theorem 3.13. By Lemma 2.9, $\underline{M}$ is level-wisely torsion. And by Corollary 3.12, $\underline{Ext}^i_{\underline{\mathbb{Z}}}(\underline{M}, \underline{\mathbb{Z}}) \cong \underline{\pi}_{-i}Hom_{H\underline{\mathbb{Z}}}(H\underline{M}, H\underline{\mathbb{Z}})$. Now by Lemma3.10, $H\underline{\mathbb{Z}} \cong \Sigma^{\lambda-2}I_{\mathbb{Z}}(H\underline{\mathbb{Z}})$, therefore

$$\begin{aligned} Hom_{H\underline{\mathbb{Z}}}(H\underline{M}, H\underline{\mathbb{Z}}) &\simeq \Sigma^{\lambda-2}Hom_{H\underline{\mathbb{Z}}}(H\underline{M}, I_{\mathbb{Z}}(H\underline{\mathbb{Z}})) \\ &\simeq \Sigma^{\lambda-2}I_{\mathbb{Z}}(H\underline{M}) && \text{by Corollary 2.21} \\ &\simeq \Sigma^{\lambda-3}H\underline{M}^E && \text{by Corollary 2.20} \\ &\simeq \Sigma^{-3}H\underline{M}^E && \text{by Lemma 3.14} \end{aligned}$$

□

Corollary 3.15.

$$\underline{Ext}^i_{\underline{\mathbb{Z}}}(\underline{\mathbb{Z}}_{t_0,t_1,\dots,t_n}, \underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{Z}} & i = 0 \\ \underline{\mathbb{B}}^E_{t_0,t_1,\dots,t_n} & i = 2 \\ \underline{0} & otherwise \end{cases}$$

Proof. Apply Theorem 3.13 to the $\underline{Ext}$ long exact sequence induced by the short exact sequence

$$0 \to \underline{\mathbb{Z}}_{t_0,t_1,\dots,t_n} \to \underline{\mathbb{Z}} \to \underline{\mathbb{B}}_{t_0,t_1,\dots,t_n} \to 0$$

□

Remark 3.16. *$\underline{Ext}^0_{\underline{\mathbb{Z}}}(\underline{\mathbb{Z}}_{t_0,t_1,\dots,t_n}, \underline{\mathbb{Z}}) \cong \underline{Hom}_{\underline{\mathbb{Z}}}(\underline{\mathbb{Z}}_{t_0,t_1,\dots,t_n}, \underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}$ can be seen by algebra: We can construct the generator starting by an isomorphism in G/e-level. Notice that the target's restrictions are isomorphisms. Therefore a map in C_{p^n}/C_{p^i}-level can always be lifted: If $t_i = 0$ then it lifts to the same map and if $t_i = 1$ it lifts to its p-multiple. By this procedure, the resulting map between Mackey functors always has an underlying isomorphism, thus the generator in $\underline{Hom}_{\underline{\mathbb{Z}}}(G/G)$ restricts to 1 in $\underline{Hom}_{\underline{\mathbb{Z}}}(G/e) \cong \underline{\mathbb{Z}}$. So we see that all restrictions are isomorphisms in the internal Hom. This procedure can effectively compute all internal Hom between forms of $\underline{\mathbb{Z}}$.*

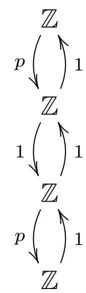
The following theorem is a first taste of the duality method.

Theorem 3.17. *If $\underline{M}$ is a form of $\underline{\mathbb{Z}}$, then $H\underline{M} \simeq S^V \wedge H\underline{\mathbb{Z}}$ for some $V \in RO(G)$.*

Proof. Consider $Res^{C_p}_e$ in $\underline{M}$, it is either 1 or p. If the restriction map is 1, then by Lemma 3.1, if the G/C_p-Mackey functor whose Lewis diagram is $\underline{M}$ with bottom cut is $S^{V'} \wedge H\underline{\mathbb{Z}}$ for some $V' \in RO(G/C_p)$, then $H\underline{M} \simeq S^V \wedge H\underline{\mathbb{Z}}$ where V is the composition of V' with the quotient map $G \to G/C_p$. If the restriction map is p, then the corresponding transfer map is 1. Therefore $Res^{C_p}_e$ in $\underline{M}^*$ is 1. And by the argument above, $H\underline{M}^* \cong S^V \wedge H\underline{\mathbb{Z}}$ for some $V \in RO(G)$. Then by Corollary 2.20, $H\underline{M} \cong S^{2-\lambda-V} \wedge H\underline{\mathbb{Z}}$. □

Remark 3.18.
(1) *This theorem tells us that in $\underline{\pi}_{\bigstar}(H\underline{\mathbb{Z}})$, every form of $\underline{\mathbb{Z}}$ will appear.*
(2) *Even though V is constructed implicitly in the proof, we can track down the induction and construct V explicitly. See the example below.*

Example 3.19. *For $G = C_{p^3}$, consider the Mackey functor $\underline{\mathbb{Z}}_{1,0,1}$, which has the Lewis diagram*



Then $H\underline{\mathbb{Z}}_{1,0,1} \simeq S^{-\lambda_0+\lambda_1-\lambda_2+2} \wedge H\underline{\mathbb{Z}}$.

Similarly, one can use the Künneth spectral sequence to compute $\underline{H}_*(S^{V_1+V_2}; \underline{\mathbb{Z}})$ from $\underline{Tor}^{\underline{\mathbb{Z}}}_{*,*}(\underline{H}_*(S^{V_1}; \underline{\mathbb{Z}}), \underline{H}_*(S^{V_2}; \underline{\mathbb{Z}}))$. This spectral sequence can help in tracking multiplicative structure, but has many nontrivial differentials and extensions. An interesting application is to do things reversely: We can compute $H\underline{\mathbb{Z}}_{\bigstar}$ by other method first, and its comparison with both universal spectral sequence and Künneth spectral sequence will tell us a lot about $\underline{Ext}_{\underline{\mathbb{Z}}}$ and $\underline{Tor}^{\underline{\mathbb{Z}}}$. This approach is taken in Section 6.

4. Computation of $\underline{\pi}_{\bigstar} H\underline{\mathbb{Z}}$ for $G = C_{p^2}$

In this section, we will apply all methods to compute $H\underline{\mathbb{Z}}_{\bigstar}$ for $G = C_{p^2}$. As discussed in section 2.2, our indexing group $RO(C_{P^2})$ is freely generated by $1, \lambda_1, \lambda_0$ for p odd and by $1, \sigma, \lambda_0$ for $p = 2$. The computation will be identical for all odd primes and orientable representations for $p = 2$. For non-orientable representations in C_4, we will use the cofibre sequence $C_4/C_{2+} \to S^0 \to S^\sigma$ to derive our result. We will use γ as generator of C_4.

We will use cellular method and cofibre method to compute several examples. A complete computation is done by Tate diagram, with a full description of multiplicative structure. Finally, we will use duality to explain the result and relate it to homological algebra.

4.1. $RO(G)$-graded Homotopy Mackey Functors of $H\underline{\mathbb{Z}}$ for C_{p^2}.

For $G = C_{p^2}$, $\underline{\pi}_{\bigstar}(H\underline{\mathbb{Z}})$ is described as follows, with names of Mackey functors in Definition 2.4, 2.5, 2.15:

Theorem 4.1. *Let $V = m\lambda_0 + n\lambda_1$. We describe $\underline{\pi}_{i-V}(H\underline{\mathbb{Z}})$ in several different cases:*

(1) *There are 3 Eilenberg-Mac Lane spectra of forms of* $\underline{\mathbb{Z}}$ *other than* $\underline{\mathbb{Z}}$ *itself:*

$$S^{-\lambda_0+2} \wedge H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}_{1,1}$$
$$S^{-\lambda_1+2} \wedge H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}_{0,1}$$
$$S^{-\lambda_0+\lambda_1} \wedge H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}_{1,0}$$

(2) *If* $m, n \geq 0$, *then*

$$\underline{\pi}_{i-V}(H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{B}}_{0,1} & 0 \leq i < 2n \text{ and } i \text{ is even} \\ \underline{\mathbb{B}}_{1,1} & 2n \leq i < 2(n+m) \text{ and } i \text{ is even} \\ \underline{\mathbb{Z}} & i = 2(n+m) \end{cases}$$

(3) *If* $m, n \leq 0$ *and it is not one of the Eilenberg-Mac Lane cases, then*

$$\underline{\pi}_{i-V}(H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{B}}_{0,1} & 2n-1 \leq i \leq -3 \text{ and } i \text{ is odd} \\ \underline{\mathbb{B}}_{1,1} & 2(m+n) < i < 2n-1 \text{ and } i \text{ is odd} \\ \underline{\mathbb{Z}}_{1,1} & i = 2(m+n) \text{ and } m < 0 \\ \underline{\mathbb{Z}}_{0,1} & i = 2(m+n) \text{ and } m = 0 \end{cases}$$

(4) *If* $m > 0$ *and* $n < 0$, *then* $\underline{\pi}_{i-V}(H\underline{\mathbb{Z}})$ *is the direct sum of two graded Mackey functors* $\underline{C}_{i,m,n}$ *and* $\underline{D}_{i,m,n}$, *where*

$$\underline{C}_{i,m,n} = \begin{cases} \underline{\mathbb{B}}_{1,1} & 2n < i < 2(n+m) \text{ and } i \text{ is even} \\ \underline{\mathbb{Z}} & i = 2(n+m) \end{cases}$$

$$\underline{D}_{i,m,n} = \begin{cases} \underline{\mathbb{B}}^{E}_{1,0} & i = n \\ \underline{\mathbb{B}}_{0,1} & 2n < i \leq -3 \text{ and } i \text{ is odd} \end{cases}$$

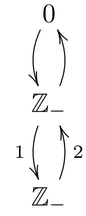
(5) *If* $m < 0$, $n > 0$ *and it is not one of the Eilenberg-Mac Lane cases, then* $\underline{\pi}_{i-V}(H\underline{\mathbb{Z}})$ *is the direct sum of two graded Mackey functors* $\underline{C}_{i,m,n}$ *and* $\underline{D}_{i,m,n}$, *where*

$$\underline{C}_{i,m,n} = \begin{cases} \underline{\mathbb{B}}_{0,1} & i = 2n-3 \\ \underline{\mathbb{B}}_{1,1} & 2(m+n) < i < n-3 \text{ and } i \text{ is odd} \\ \underline{\mathbb{Z}}_{1,1} & i = 2(m+n) \text{ and } m < -1 \\ \underline{\mathbb{Z}}_{1,0} & (i = 2(m+n) and m = -1) \end{cases}$$

$$\underline{D}_{i,m,n} = \underline{\mathbb{B}}_{0,1} \text{ where } 0 \leq i < 2n-3 \text{ and trivial otherwise.}$$

This theorem gives a complete description for $\underline{\pi}_{\bigstar}(H\underline{\mathbb{Z}})$ for C_{p^2} and $p > 2$. For $p = 2$, we still need to describe all the cases where V is non-orientable. In this case we need some more $\underline{\mathbb{Z}}$-modules in C_4:

Definition 4.2. (1) $\mathbb{Z}_-$ *is the* C_4*-module* $\mathbb{Z}$ *with action by multiplying* -1 *and* $\underline{\mathbb{Z}}_-$ *is its fixed point Mackey functor.* $\underline{\mathbb{Z}}_-$ *has Lewis diagram*



and let $\underline{\mathbb{Z}}^*_-$ *be* $Hom_L(\underline{\mathbb{Z}}_-, \mathbb{Z})$ *as in Definition 2.15.*

(2) *Let $\underline{\mathbb{B}}_-$ be the $\underline{\mathbb{Z}}$-module with Lewis diagram*

$$\begin{array}{c} 0 \\ \downarrow\uparrow \\ \mathbb{Z}/2 \\ \downarrow\uparrow \\ 0 \end{array}$$

(3) *Let $\underline{M} = \underline{\mathbb{Z}}_- or \underline{\mathbb{Z}}_-^*$, then $\dot{\underline{M}}$ is the Mackey functor which is isomorphic to $\underline{M}$ when the orbit is not C_4/C_4, and $\dot{\underline{M}}(C_4/C_4) = \mathbb{Z}/2$ is hit by the transfer. $\dot{\underline{\mathbb{Z}}}_-$ has Lewis diagram*

$$\begin{array}{c} \mathbb{Z}/2 \\ {\scriptstyle 0}\downarrow\uparrow{\scriptstyle 1} \\ \mathbb{Z}_- \\ {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} \\ \mathbb{Z}_- \end{array}$$

And $\dot{\underline{\mathbb{Z}}}_-^*$ has Lewis diagram

$$\begin{array}{c} \mathbb{Z}/2 \\ {\scriptstyle 0}\downarrow\uparrow{\scriptstyle 1} \\ \mathbb{Z}_- \\ {\scriptstyle 2}\downarrow\uparrow{\scriptstyle 1} \\ \mathbb{Z}_- \end{array}$$

Be careful that $\dot{\underline{\mathbb{Z}}}_-^*$ is not $(\dot{\underline{\mathbb{Z}}}_-)^*$.

Theorem 4.3. *Let $V = m\lambda_0 + n\sigma$. since $2\sigma = \lambda_1$, we can assume n is odd. $\underline{\pi}_{i-V}(H\underline{\mathbb{Z}})$ is the following:*

(1) *There are four Eilenberg-Mac Lane cases:*

$$\begin{aligned} S^{1-\sigma} \wedge H\underline{\mathbb{Z}} &\simeq H\underline{\mathbb{Z}}_- \\ S^{3-3\sigma} \wedge H\underline{\mathbb{Z}} &\simeq H\dot{\underline{\mathbb{Z}}}_- \\ S^{1-\lambda+\sigma} \wedge H\underline{\mathbb{Z}} &\simeq H\underline{\mathbb{Z}}_-^* \\ S^{3-\lambda-\sigma} \wedge H\underline{\mathbb{Z}} &\simeq H\dot{\underline{\mathbb{Z}}}_-^* \end{aligned}$$

(2) *If $m, n \geq 0$, then*

$$\underline{\pi}_{i-V}(H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{B}}_{0,1} & 0 \leq i < n \text{ and } i \text{ is even} \\ \underline{\mathbb{B}}_{0,1} & n \leq i < 2m+n \text{ and } i \text{ is even} \\ \underline{\mathbb{B}}_- & n \leq i < 2m+n \text{ and } i \text{ is odd} \\ \underline{\mathbb{Z}}_- & i = 2m+n \end{cases}$$

(3) *If $m, n \leq 0$, and is not one of the Eilenberg-Mac Lane cases, then*

$$\underline{\pi}_{i-V}(H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{B}}_{0,1} & n \leq i \leq -3 \text{ and } i \text{ is odd} \\ \underline{\mathbb{B}}_{0,1} & 2m+n < i < n-1 \text{ and } i \text{ is odd} \\ \underline{\mathbb{B}}_{-} & 2m+n < i < n-1 \text{ and } i \text{ is even} \\ \dot{\underline{\mathbb{Z}}}^{*}_{-} & i = 2m+n \text{ and } m < 0 \\ \dot{\underline{\mathbb{Z}}}_{-} & i = 2m+n \text{ and } m = 0 \end{cases}$$

(4) *If $m > 0$ and $n < 0$, then $\underline{\pi}_{i-V}(H\underline{\mathbb{Z}})$ is the direct sum of two graded Mackey functors $\underline{C}_{i,m,n}$ and $\underline{D}_{i,m,n}$.*

$$\underline{C}_{i,m,n} = \begin{cases} \underline{\mathbb{B}}_{0,1} & n < i < 2m+n \text{ and } i \text{ is even} \\ \underline{\mathbb{B}}_{-} & n < i < 2m+m \text{ and } i \text{ is odd} \\ \underline{\mathbb{Z}}_{-} & i = 2m+n \end{cases}$$

When $n < -1$,

$$\underline{D}_{i,m,n} = \begin{cases} \underline{\mathbb{B}}^{E}_{1,0} & i = n \\ \underline{\mathbb{B}}_{0,1} & n < i \leq -3 \text{ and } i \text{ is odd} \end{cases}$$

$\underline{D}_{i,m,-1} = \underline{\mathbb{B}}_{-}$ *when $i = -1$.*

(5) *If $m < 0$ and $n > 0$ and it is not one of the Eilienberg-Mac Lane cases, then $\underline{\pi}_{i-V}(H\underline{\mathbb{Z}})$ is the direct sum of three graded Mackey functors $\underline{C}_{i,m,n}$, $\underline{D}_{i,m,n}$ and $\underline{E}_{i,m,n}$.*

$$\underline{C}_{i,m,n} = \begin{cases} \underline{\mathbb{B}}_{0,1} & 2m+n < i < n-3 \text{ and } i \text{ is odd} \\ \underline{\mathbb{B}}_{-} & 2m+n < i < n-3 \text{ and } i \text{ is even} \\ \dot{\underline{\mathbb{Z}}}^{*}_{-} & i = 2m+n \end{cases}$$

$$\underline{D}_{i,m,n} = \underline{\mathbb{B}}_{0,1} \text{ for } 0 \leq i < n-3,\ i \text{ even and for all } m$$

$\underline{E}_{i,m,n} = \underline{\mathbb{B}}_{1,0}$ *for $i = n-3$ and for all m, and trvial in all other cases.*

Remark 4.4. (1) *All orientable Eilenberg-Mac Lane cases are direct consequences of Theorem 3.17 and the non-orientable one can be computed directly through cellular method.*

(2) *The part $\underline{C}_{i,m,n}$ are coming from $S^{m\lambda_0} \wedge H\underline{\mathbb{Z}}$, $\underline{D}_{i,m,n}$ are Mackey functors coming from $S^{n\lambda_1} \wedge H\underline{\mathbb{Z}}$ and $\underline{E}_{i,m,n}$ are coming from a exotic restriction between the former two. This can be seen clearly in cofibre of a computation in section 4.3.*

(3) *The graded Green functor structure, with names of elements in each Mackey functor above, is discussed in Section 4.4.*

The rest of this section is dedicated to computing and explaining the above result by four different methods in section 3. In cellular method and cofibre of a method, we will only compute certain examples instead of the full $RO(G)$-grading, to illustrate how these methods work. A full computation is given by the Tate diagram in Section 4.4.

In spectral sequences and charts, it is not convenient to put forms of $\underline{\mathbb{Z}}$ and $\underline{\mathbb{B}}$ in them. Therefore in the following tables we introduce a more compact way to present C_p and C_{p^2}-Mackey functors. These notations are first introduced in [HHR17b]. Generally speaking, a box means a form of $\underline{\mathbb{Z}}$, a circle shape means torsion, and overline means $\mathbb{Z}_{-}$.

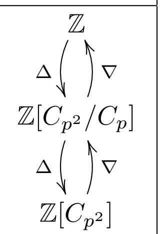
The following is the table of symbols for C_p-Mackey functors.

Name	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}_1 = \underline{\mathbb{Z}}^*$	$\underline{\mathbb{Z}}_-$	$\dot{\underline{\mathbb{Z}}}_-$	$\underline{\mathbb{B}}_1$
Symbol	$\square$	◩	$\overline{\square}$	$\dot{\overline{\square}}$	$\bullet$
Lewis Diagram	$\mathbb{Z}$ $1 \downarrow \uparrow p$ $\mathbb{Z}$	$\mathbb{Z}$ $p \downarrow \uparrow 1$ $\mathbb{Z}$	0 $\downarrow \uparrow$ $\mathbb{Z}_-$	$\mathbb{Z}/2$ $0 \downarrow \uparrow 1$ $\mathbb{Z}_-$	$\mathbb{Z}/p$ $\downarrow \uparrow$ 0

Notice that $\overline{\square}$ and $\dot{\overline{\square}}$ are only defined for $p = 2$.

The following table is symbols for C_{p^2}-Mackey functors.

Name	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}_{0,1}$	$\underline{\mathbb{Z}}_{1,1}$	$\underline{\mathbb{Z}}_{1,0}$	$\underline{\mathbb{Z}}_-$
Symbol	$\square$	◩	◪	⬓	$\overline{\square}$
Lewis Diagram	$\mathbb{Z}$ $1 \downarrow \uparrow p$ $\mathbb{Z}$ $1 \downarrow \uparrow p$ $\mathbb{Z}$	$\mathbb{Z}$ $p \downarrow \uparrow 1$ $\mathbb{Z}$ $1 \downarrow \uparrow p$ $\mathbb{Z}$	$\mathbb{Z}$ $p \downarrow \uparrow 1$ $\mathbb{Z}$ $p \downarrow \uparrow 1$ $\mathbb{Z}$	$\mathbb{Z}$ $1 \downarrow \uparrow p$ $\mathbb{Z}$ $p \downarrow \uparrow 1$ $\mathbb{Z}$	0 $\downarrow \uparrow$ $\mathbb{Z}_-$ $1 \downarrow \uparrow 2$ $\mathbb{Z}_-$
Name	$\dot{\underline{\mathbb{Z}}}_-$	$\underline{\mathbb{Z}}^*_-$	$\dot{\underline{\mathbb{Z}}}^*_-$	$\underline{\mathbb{Z}[C_{p^2}/C_p]}$	
Symbol	$\dot{\square}$	$\overline{◪}$	$\dot{◪}$	$\hat{\square}$	$\hat{\bullet}$
Lewis Diagram	$\mathbb{Z}/2$ $0 \downarrow \uparrow 1$ $\mathbb{Z}_-$ $1 \downarrow \uparrow 2$ $\mathbb{Z}$	0 $\downarrow \uparrow$ $\mathbb{Z}_-$ $2 \downarrow \uparrow 1$ $\mathbb{Z}_-$	$\mathbb{Z}/2$ $0 \downarrow \uparrow 1$ $\mathbb{Z}_-$ $2 \downarrow \uparrow 1$ $\mathbb{Z}_-$	$\mathbb{Z}$ $\triangle \downarrow \uparrow \triangledown$ $\mathbb{Z}[C_{p^2}/C_p]$ $1 \downarrow \uparrow p$ $\mathbb{Z}[C_{p^2}/C_p]$	$\mathbb{Z}/p$ $\triangle \downarrow \uparrow \triangledown$ $\mathbb{Z}/p[C_{p^2}/C_p]$ $\downarrow \uparrow$ 0
Name	$\underline{\mathbb{B}}_{1,1}$	$\underline{\mathbb{B}}_{0,1}$	$\underline{\mathbb{B}}_{1,0}$	$\underline{\mathbb{B}}^E_{1,0}$	$\underline{\mathbb{B}}_-$
Symbol	$\circ$	$\bullet$	$\blacktriangledown$	$\blacktriangle$	$\overline{\bullet}$
Lewis Diagram	$\mathbb{Z}/p^2$ $1 \downarrow \uparrow p$ $\mathbb{Z}/p$ $\downarrow \uparrow$ 0	$\mathbb{Z}/p$ $\downarrow \uparrow$ 0 $\downarrow \uparrow$ 0	$\mathbb{Z}/p$ $1 \downarrow \uparrow 0$ $\mathbb{Z}/p$ $\downarrow \uparrow$ 0	$\mathbb{Z}/p$ $0 \downarrow \uparrow p$ $\mathbb{Z}/p$ $\downarrow \uparrow$ 0	0 $\downarrow \uparrow$ $\mathbb{Z}/p$ $\downarrow \uparrow$ 0

Name	$\underline{\mathbb{Z}[C_{p^2}]}$
Symbol	$\hat{\hat{\square}}$
Lewis Diagram	

All Mackey functors involving $\mathbb{Z}_-$ are only defined for $p = 2$. Not all of Mackey functors in the table are named systematically.

4.2. **Cellular Method.**

We will focus on the example of $\underline{H}_*(S^{3\lambda_0-2\lambda_1};\underline{\mathbb{Z}})$, and all other $RO(G)$-grading can be computed in a similar way. We will use $\underline{C}_*(S^V)$ for cellular chain complex for S^V.

The cellular chain complex for $S^{3\lambda_0}$ is $\underline{C}_*(S^{3\lambda_0})$:

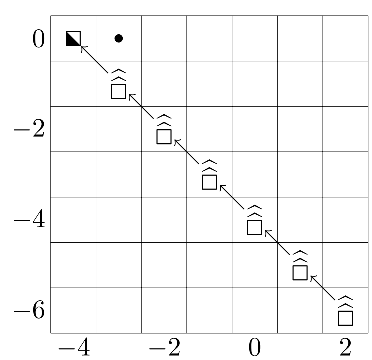
$$\underset{0}{\underline{\mathbb{Z}}} \longleftarrow \underset{1}{\underline{\mathbb{Z}[C_{p^2}]}} \longleftarrow \underset{2}{\underline{\mathbb{Z}[C_{p^2}]}} \longleftarrow \underset{3}{\underline{\mathbb{Z}[C_{p^2}]}} \longleftarrow \underset{4}{\underline{\mathbb{Z}[C_{p^2}]}} \longleftarrow \underset{5}{\underline{\mathbb{Z}[C_{p^2}]}} \longleftarrow \underset{6}{\underline{\mathbb{Z}[C_{p^2}]}}$$

Where all the maps are determined by the fact that in G/e-level the homology should be $\tilde{H}_*(S^6;\mathbb{Z})$.

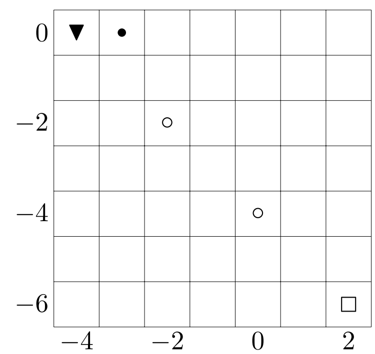
The cellular chain complex for $S^{-2\lambda_1}$ is $\underline{C}_*(S^{-2\lambda_1})$

$$\underset{-4}{\underline{\mathbb{Z}[C_{p^2}/C_p]}} \longleftarrow \underset{-3}{\underline{\mathbb{Z}[C_{p^2}/C_p]}} \longleftarrow \underset{-2}{\underline{\mathbb{Z}[C_{p^2}/C_p]}} \longleftarrow \underset{-1}{\underline{\mathbb{Z}[C_{p^2}/C_p]}} \longleftarrow \underset{0}{\underline{\mathbb{Z}}}$$

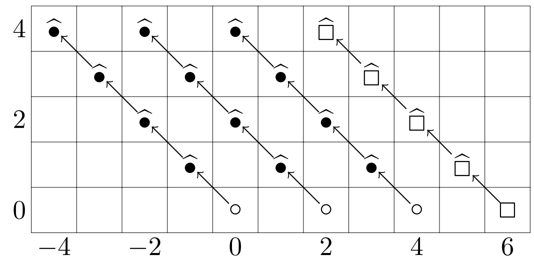
Now $\underline{H}_*(S^{3\lambda_0-2\lambda_1};\underline{\mathbb{Z}})$ can be computed by the total homology of $\underline{C}_*(S^{3\lambda_0})\square_{\underline{\mathbb{Z}}}\underline{C}_*(S^{-2\lambda_1})$. If we filter the double complex by $\underline{C}_*(S^{3\lambda_0})$, we will get a spectral sequence with following E_1-page and d_1:



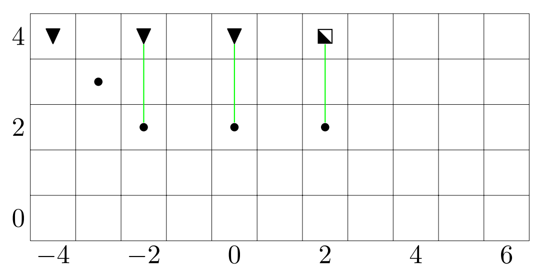
All d_1 is determined again by the fact that the underlying space is S^2. Thus we get the following E_2-page.



As we can see, there is a potential d_3 which cannot be determined. Now we filter the double complex by $\underline{C}_*(S^{-2\lambda_1})$. We get the following E_1 with d_1:



By comparison with the spectral sequence filtered by $\underline{C}_*(S^{3\lambda_0})$ above, we will eventually get to E_4 with no rooms for differential, here green vertical line means nontrivial restrictions:



From here, we know that $\underline{H}_{-2}(S^{3\lambda_0-2\lambda_1};\underline{\mathbb{Z}})$ is nontrivial, so the first spectral sequence has trivial d_3. We can summarize the result as follows:

$$\underline{H}_i(S^{3\lambda_0-2\lambda_1}) = \begin{cases} \underline{B}_{1,0} & \text{if } i = \text{-}4 \\ \underline{B}_{0,1} & \text{if } i = -3 \\ \underline{B}_{1,1} & \text{if } i = -2, 0 \\ \underline{\mathbb{Z}} & \text{if } i = 2 \\ \underline{0} & \text{otherwise} \end{cases}$$

4.3. **Cofibre of a Method.**

The original idea of this method is due to Doug Ravenel, and it is used in [HHR17b, Section 6].

We will assume $p = 2$ in this subsection, because we want to make direct use of 3.5.

Notice that when $p = 2$, $\lambda_1 = 2\sigma$. And indicated by the following lemma, we will use σ instead of λ_1 as part of our index. First, we need to compute $\underline{\pi}_* L_n$ and $\underline{\pi}_* L'_n$ defined in 3.3. Their homotopy Mackey functors are the following:

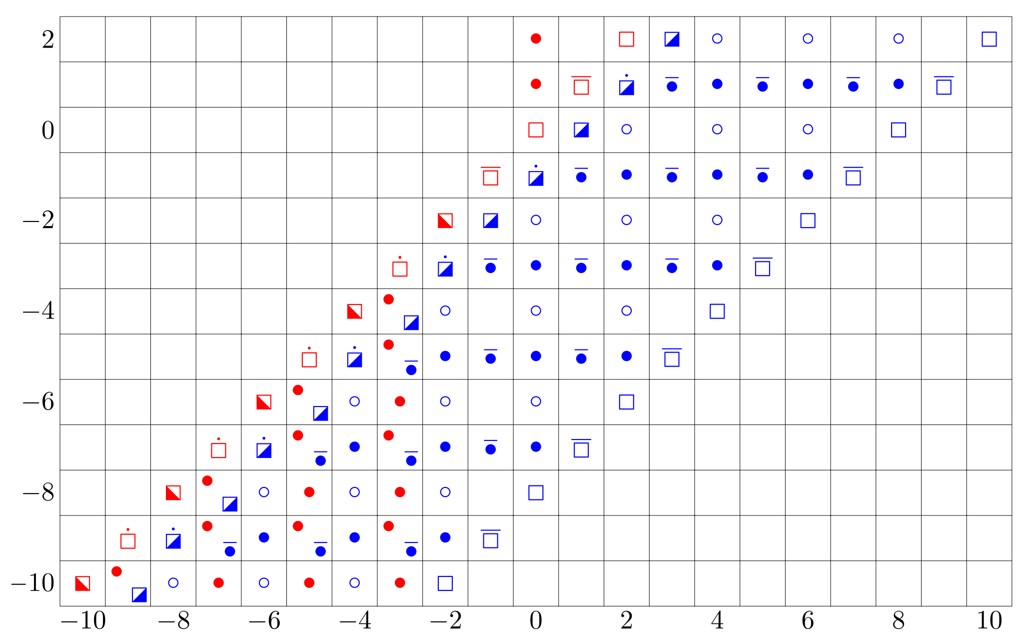
$$\underline{\pi}_i(L_n) = \begin{cases} \underline{\mathbb{Z}}_{1,1} & i = 1 \\ \underline{\mathbb{B}}_{1,1} & 1 < i < 2n \text{ and } i \text{ is even} \\ \underline{\mathbb{Z}} & i = 2n \\ \underline{0} & \text{otherwise} \end{cases}$$

$$\underline{\pi}_i(L'_n) = \begin{cases} \dot{\underline{\mathbb{Z}}}^*_- & i = 2 \\ \underline{\mathbb{B}}_- & 2 < i < 2n+1 \text{ and } i \text{ is odd} \\ \underline{\mathbb{B}}_{0,1} & 2 < i < 2n+1 \text{ and } i \text{ is even} \\ \underline{\mathbb{Z}}_- & i = 2n+1 \\ \underline{0} & \text{otherwise} \end{cases}$$

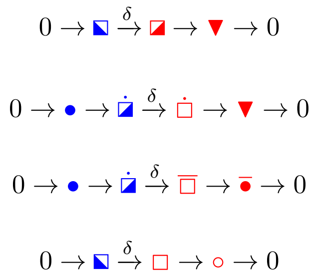
By Lemma 3.4, this determines $\underline{\pi}_\bigstar L_n$. Now we will fix $n = 4$, and consider the fibre sequences

$$S^{m\sigma} \wedge H\underline{\mathbb{Z}} \xrightarrow{a_{4\lambda_0}} S^{4\lambda_0 + m\sigma} \wedge H\underline{\mathbb{Z}} \to S^{m\sigma} \wedge L_4 \text{ for all } m \in \mathbb{Z}$$

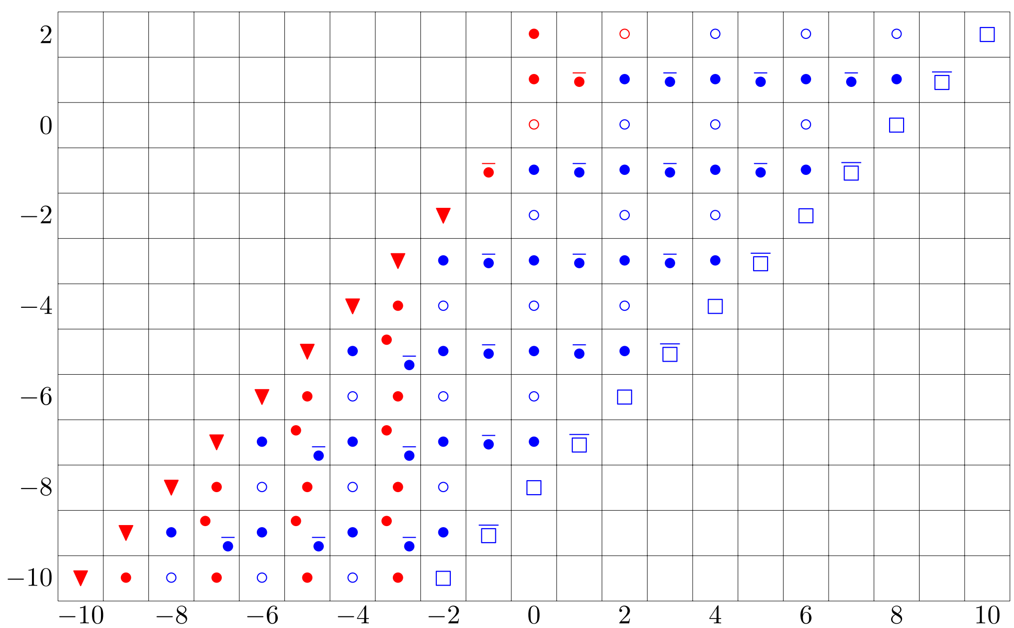
The following figure shows the long exact sequence in homotopy Mackey functors of this fibre sequence when m varies. The horizontal axis indicates $\underline{\pi}_i$ while the vertical axis indicates m in $S^{m\sigma}$. Elements named in red are those from $\underline{\pi}_*(S^{m\sigma} \wedge H\underline{\mathbb{Z}})$ and elements named in blue are those from $\underline{\pi}_* L_n$ and $\underline{\pi}_* L'_n$.



Now the connecting homomorphism δ has degree $(-1, 0)$, so goes from a blue element to the red element on its left. First, observe that after the connecting homomorphism, the result should be $\pi_*(S^{m+8} \wedge H\mathbb{Z})$ if we forget its C_4 structure, which means the left most δ must be isomorphism on G/e level (Since the only nontrivial homotopy group on G/e level in the result must be π_{m+8}). So, δ fits into one of four exact sequences below.



Also, there are potentially nontrivial connecting homomorphisms from • or ○ to •. But they are all trivial because δ commutes with a_σ multiplication and after multiple by a_σ once or twice, they cannot fit with each other. After applying δ, we obtain the following picture.



By Lemma 3.5 and a_σ-multiplicative structure, we see that all possible extensions are trivial. So we know $\underline{\pi}_{-4\lambda_0+m\lambda_1+*}H\underline{\mathbb{Z}}$. We can change the number of copies of λ_0 and apply the same method to compute all $H\underline{\mathbb{Z}}_\bigstar$. This method computes $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}})$ as a module over $\underline{\mathrm{BB}}$, but cannot tell about the full ring structure.

For $\underline{\pi}_*(S^V \wedge H\underline{\mathbb{Z}})$ with negative coefficient on λ_0, the computation is very similar and we will not repeat.

4.4. **Tate Diagram Method.**

In this subsection, we assume that all representations are orientable. This includes all representations for odd prime, and the index 2 subgroup $RO(G)$ generated by 2σ and λ_0 for C_4. We will use a_{λ_1} for $a_{2\sigma}$ in this subsection, to unify notations for all primes. When we are working between different groups, we will put the source of representation as a superscript. For example, $\lambda_0^{C_p}$ means the representation λ_0 on C_p as defined in Section 2.2.

We start with computation in C_p, to both illustrate the method and track name of elements. Computation in C_p is well known, and is written in [Gre, Section 2C].

First, by homotopy fixed point spectral sequence, we know that

$$\underline{\pi}_\bigstar(H\underline{\mathbb{Z}}^h)(G/G) = \mathbb{Z}[u_{\lambda_0}^{\pm}, a_{\lambda_0}]/(pa_{\lambda_0})$$

The Mackey functor structure is determined by $Res(u_{\lambda_0}) = 1$ and $Res(a_{\lambda_0}) = 0$, i.e. each form of $\underline{\mathbb{Z}}$ is the constant $\underline{\mathbb{Z}}$. Now, by inverting a_{λ_0}, we see that

$$\underline{\pi}_\bigstar(H\underline{\mathbb{Z}}^t)(G/G) = \mathbb{Z}/p[u_{\lambda_0}^{\pm}, a_{\lambda_0}^{\pm}]$$

and the underlying spectrum is contractible. The map $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}}^h) \to \underline{\pi}_\bigstar(H\underline{\mathbb{Z}}^t)$ is the a_{λ_0}-localization.

Now, $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}}_h)$ is the direct sum of kernel and cokernel of a_{λ_0}-localization, since there is no nontrivial extension by degree reason. Following Definition 3.8, we have the following description as a module over $\underline{BB}_{C_p}$

$$\underline{\pi}_\bigstar(H\underline{\mathbb{Z}}_h)(G/G) = \mathbb{Z}\langle pu_{\lambda_0}^i\rangle \oplus \mathbb{Z}/p\langle\Sigma^{-1}u_{\lambda_0}^i\frac{u_{\lambda_0}^j}{a_{\lambda_0}^j}\rangle \text{For } i\in\mathbb{Z} \text{ and } j>0$$

Where each $pu_{\lambda_0}^i$ generates $\underline{\mathbb{Z}}^*(G/G)$ instead of the constant $\underline{\mathbb{Z}}$. Specially, in the integer grading, we have

$$\underline{\pi}_*(H\underline{\mathbb{Z}}_h)(G/G) = \mathbb{Z}\langle p\rangle \oplus \mathbb{Z}/p\langle\Sigma^{-1}\frac{u_{\lambda_0}^i}{a_{\lambda_0}^i}\rangle \text{For } i>0$$

Now, we can compute $\underline{\pi}_*(a_{\lambda_0}^{-1}H\underline{\mathbb{Z}})$ as direct sum of kernel and cokernel of the map $\underline{\pi}_*(H\underline{\mathbb{Z}}_h) \to \underline{\pi}_*(H\underline{\mathbb{Z}})$. We have

$$\underline{\pi}_*(a_{\lambda_0}^{-1}H\underline{\mathbb{Z}})(G/G) = \mathbb{Z}/p[\frac{u_{\lambda_0}}{a_{\lambda_0}}]$$

Then by a_{λ_0}-periodicity,

$$\underline{\pi}_\bigstar(a_{\lambda_0}^{-1}H\underline{\mathbb{Z}})(G/G) = \mathbb{Z}/p[u_{\lambda_0}, a_{\lambda_0}^{\pm}]$$

Now we consider $\underline{\pi}_{i-m\lambda_0}$ of the Tate diagram.

If $m>0$, then

$$\underline{\pi}_{*-m\lambda_0}(H\underline{\mathbb{Z}}_h) = \mathbb{Z}\langle pu_{\lambda_0}^m\rangle \oplus \mathbb{Z}/p\langle\Sigma^{-1}\frac{u_{\lambda_0}^m u_{\lambda_0}^i}{a_{\lambda_0^i}}\rangle \text{For } i>0.$$

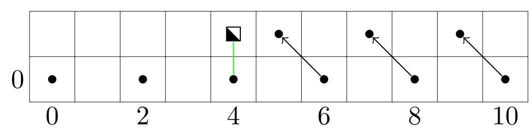
And

$$\underline{\pi}_{*-m\lambda_0}(a_{\lambda_0}^{-1}H\underline{\mathbb{Z}}) = \mathbb{Z}/p\langle\frac{a_{\lambda_0}^m u_{\lambda_0}^i}{a_{\lambda_0}^i}\rangle \text{For } i\geq 0.$$

Therefore ,the connnecting homomorphism maps those elements in $\underline{\pi}_{*-m\lambda_0}(a_{\lambda_0}^{-1}(H\underline{\mathbb{Z}}))$ who has positive powers of a_{λ_0} in the denominator into the corresponding desuspended elements in $\underline{\pi}_{*-m\lambda_0}(H\underline{\mathbb{Z}}_h)$, and in $\underline{\pi}_{2m-m\lambda_0}$, there is a nontrivial extension

$$0 \to \mathbb{Z}\langle pu_{m\lambda_0}\rangle \to \mathbb{Z}\langle u_{m\lambda_0}\rangle \to \mathbb{Z}/p\langle u_{m\lambda_0}\rangle \to 0$$

The following picture shows the case $m = 2$, with the bottom row $\underline{\pi}_{*-2\lambda_0}(a_{\lambda_0}^{-1}H\underline{\mathbb{Z}})$ and top row $\underline{\pi}_{*-2\lambda_0}(H\underline{\mathbb{Z}}_h)$. Arrows indicate connecting homomorphism, and green vertical line means an extension involving an exotic restriction.



So we know that for $m > 0$,

$$\underline{\pi}_{*-m\lambda_0}(H\underline{\mathbb{Z}})(G/G) = \mathbb{Z}\langle u_{\lambda_0}^m\rangle \oplus \mathbb{Z}/p\langle a_{\lambda_0}^i u_{\lambda_0}^{m-i}\rangle \text{For } 0 < i \leq m.$$

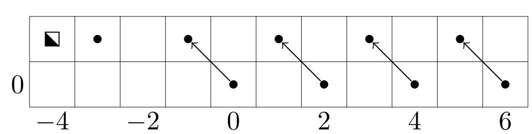
In this case, all torsion classes in $\underline{\pi}_{*-m\lambda_0}(H\underline{\mathbb{Z}})$ are coming from $\underline{\pi}_{*-m\lambda_0}(a_{\lambda_0}^{-1}H\underline{\mathbb{Z}})$.

For $m < 0$, the only difference is that $\underline{\pi}_{i-m\lambda_0}(H\underline{\mathbb{Z}}_h)$ can be nontrivial for $i < 0$ while $\underline{\pi}_{i-m\lambda_0}(a_{\lambda_0}^{-1}H\underline{\mathbb{Z}})$ can't. By similar computation, we see that in this case,

$$\underline{\pi}_{*-m\lambda_0}(H\underline{\mathbb{Z}})(G/G) = \mathbb{Z}\langle \frac{p}{u_{\lambda_0}^{|m|}}\rangle \oplus \mathbb{Z}/p\langle \Sigma^{-1}\frac{u_{\lambda_0}^i}{u_{\lambda_0}^{|m|}a_{\lambda_0}^i}\rangle \text{ for } 0 < i < |m|.$$

The awkward notation of $u_{\lambda_0}^{|m|}$ on the denominator is caused by the fact that u_W is only defined for actual representations.

For $m = -2$, the following picture shows the long exact sequence. It gives an interpretation of the gap, as the torsion class in dimension -1 is killed, and only classs in dimension≤ -3 can survive.



Now we start to do C_{p^2} computation. By Lemma 3.1 and computation above, we already know $\underline{\pi}_{*-n\lambda_1}(H\underline{\mathbb{Z}})$ for all n, with names of elements. The first step of computation is the homotopy orbit spectrum.

$$\underline{\pi}_\bigstar(H\underline{\mathbb{Z}}_h) = \mathbb{Z}\langle p^2 u_{\lambda_0}^m u_{\lambda_1}^n\rangle \oplus \mathbb{Z}/p^2\langle \Sigma^{-1}u_{\lambda_0}^m u_{\lambda_1}^n \frac{u_{\lambda_0}^i}{a_{\lambda_0}^i}\rangle \text{ for } m, n \in \mathbb{Z} \text{ and } i > 0.$$

Where each $p^2 u_{\lambda_0}^m u_{\lambda_1}^n$ generates a $\underline{\mathbb{Z}}_{1,1}(G/G)$, while each torsion summand is one of $\underline{\mathbb{B}}_{1,1}(G/G)$.

We compute two easier cases first:

If $m, n > 0$, then for computing $\underline{\pi}_{*-m\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}})$, we can start with $\underline{\pi}_{*-n\lambda_1}$ of the Tate diagram. First,

$$\underline{\pi}_{*-n\lambda_1}(H\underline{\mathbb{Z}}_h)(G/G) = \mathbb{Z}\langle p^2 u_{\lambda_1}^n\rangle \oplus \mathbb{Z}/p^2\langle \Sigma^{-1}u_{\lambda_1}^n \frac{u_{\lambda_0}^i}{a_{\lambda_0}^i}\rangle \text{ for } i > 0$$

While

$$\underline{\pi}_{*-n\lambda_1}(H\underline{\mathbb{Z}})(G/G) = \mathbb{Z}\langle u^n_{\lambda_1}\rangle \oplus \mathbb{Z}/p\langle a^i_{\lambda_1} u^{n-i}_{\lambda_1}\rangle \text{ for } 0 < i \leq n$$

Where each $\mathbb{Z}$ generates $\underline{\mathbb{Z}}$ and $\mathbb{Z}/p$ generates $\underline{\mathbb{B}}_{0,1}$.

Therefore, we see that in dimension $2n - n\lambda_1$, the map $H\underline{\mathbb{Z}}_h \to H\underline{\mathbb{Z}}$ is $\underline{\mathbb{Z}}_{1,1} \to \underline{\mathbb{Z}}$ which is an isomorphism in G/e-level. And in other dimension this map is trivial. So

$$\begin{aligned}\underline{\pi}_{*-n\lambda_1}(a^{-1}_{\lambda_0} H\underline{\mathbb{Z}})(G/G) = \mathbb{Z}/p\langle a^i_{\lambda_1} u^{n-i}_{\lambda_1}\rangle \qquad & \text{for } 0 < i \leq n \\ \oplus \mathbb{Z}/p^2\langle \Sigma^{-1} u^n_{\lambda_1} \frac{u^j_{\lambda_0}}{a^j_{\lambda_0}}\rangle \qquad & \text{for } 0 \leq j.\end{aligned}$$

Here all p-torsions are from $H\underline{\mathbb{Z}}$, and all p^2-torsions are from $H\underline{\mathbb{Z}}_h$ pulling back by connecting homomorphism.

Now we can move to $\underline{\pi}_{*-m\lambda_0-n\lambda_1}$. Homotopy Mackey functors of homotopy orbit can be read off directly:

$$\underline{\pi}_{*-m\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}}_h)(G/G) = \mathbb{Z}\langle p^2 u^m_{\lambda_0} u^n_{\lambda_1}\rangle \oplus \mathbb{Z}/p^2\langle \Sigma^{-1} u^m_{\lambda_0} u^n_{\lambda_1} \frac{u^i_{\lambda_0}}{a^i_{\lambda_0}}\rangle \text{ for } i > 0$$

And we can compute $a^{-1}_{\lambda_0} H\underline{\mathbb{Z}}$ by a_{λ_0}-periodicity:

$$\begin{aligned}\underline{\pi}_{*-m\lambda_0-n\lambda_1}(a^{-1}_{\lambda_0} H\underline{\mathbb{Z}})(G/G) = \mathbb{Z}/p\langle a^m_{\lambda_0} a^i_{\lambda_1} u^{n-i}_{\lambda_1}\rangle \qquad & \text{for } 0 < i \leq n \\ \oplus \mathbb{Z}/p^2\langle \Sigma^{-1} a^m_{\lambda_0} u^n_{\lambda_1} \frac{u^j_{\lambda_0}}{a^j_{\lambda_0}}\rangle \qquad & \text{for } 0 \leq j.\end{aligned}$$

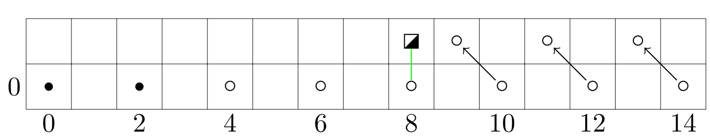
We see that classes of $H\underline{\mathbb{Z}}_h$ in dimension $i - m\lambda_0 - n\lambda_1$ for $i > 2(m+n)$ are killed by connecting homomorphism. In dimension $2(m+n) - m\lambda_0 - n\lambda_1$, there is a nontrivial extension

$$0 \to \underline{\mathbb{Z}}_{1,1} \to \underline{\mathbb{Z}} \to \underline{\mathbb{B}}_{1,1} \to 0$$

And connecting homomorphism or extension is trivial in other degrees. The conclusion is:

$$\begin{aligned}\underline{\pi}_{*-m\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}})(G/G) = \mathbb{Z}\langle u^m_{\lambda_0} u^n_{\lambda_1}\rangle \oplus \mathbb{Z}/p\langle a^i_{\lambda_1} a^m_{\lambda_0} u^{n-i}_{\lambda_1}\rangle \qquad & \text{for } 0 < i \leq n \\ \oplus \mathbb{Z}/p^2\langle a^j_{\lambda_0} u^{m-j}_{\lambda_0} u^n_{\lambda_1}\rangle \qquad & \text{for } 0 < j \leq m\end{aligned}$$

Where $\mathbb{Z}$ generates $\underline{\mathbb{Z}}$, $\mathbb{Z}/p^2$ generates $\underline{\mathbb{B}}_{1,1}$ and $\mathbb{Z}/p$ generates $\underline{\mathbb{B}}_{0,1}$. This is precisely $\underline{BB}_{C_{p^2}}$ The following picture describes this long exact sequence for $m = 2$ and $n = 2$.



If $m, n < 0$, we also start with $\underline{\pi}_{*-n\lambda_1}$ of the Tate diagram. We know that

$$\underline{\pi}_{*-n\lambda_1}(H\underline{\mathbb{Z}}_h)(G/G) = \mathbb{Z}\langle \frac{p^2}{u^{|n|}_{\lambda_1}}\rangle \oplus \mathbb{Z}/p^2\langle \Sigma^{-1} \frac{u^i_{\lambda_0}}{u^{|n|}_{\lambda_1} a^i_{\lambda_0}}\rangle \text{ for } i > 0$$

and

$$\underline{\pi}_{*-n\lambda_1}(H\underline{\mathbb{Z}})(G/G)=\mathbb{Z}\langle\frac{p}{u_{\lambda_1}^{|n|}}\rangle\oplus\mathbb{Z}/p\langle\Sigma^{-1}\frac{u_{\lambda_1}^i}{u_{\lambda_1}^{|n|}a_{\lambda_1}^i}\rangle \text{ for } 0<i<|n|$$

Here each $\mathbb{Z}/p$ is $\underline{\mathbb{B}}_{0,1}(G/G)$.

So we can compute $\underline{\pi}_{*-n\lambda_1}(a_{\lambda_0}^{-1}H\underline{\mathbb{Z}})$. In $\underline{\pi}_{2n-n\lambda_1}$, the map from $H\underline{\mathbb{Z}}_h$ to $H\underline{\mathbb{Z}}$ is $\underline{\mathbb{Z}}_{1,1}\to\underline{\mathbb{Z}}_{0,1}$, and underlies an isomorphism. Therefore it is injective and has cokernel $\underline{\mathbb{B}}_{1,0}^E$. On torsion classes, generators in $H\underline{\mathbb{Z}}_h$ are $\Sigma^{-1}\frac{u_{\lambda_0}^i}{u_{\lambda_1}^{|n|}a_{\lambda_0}^i}$, and those in $H\underline{\mathbb{Z}}$ are $\Sigma^{-1}\frac{u_{\lambda_1}^i}{u_{\lambda_1}^{|n|}a_{\lambda_1}^i}$. However, gold relation in Proposition 2.25 tells us

$$u_{\lambda_0}^i a_{\lambda_1}^i=p^i a_{\lambda_0}^i u_{\lambda_1}^i$$

Therefore $\Sigma^{-1}\frac{u_{\lambda_0}^i}{u_{\lambda_1}^{|n|}a_{\lambda_0}^i}$ sends to $p^i\Sigma^{-1}\frac{u_{\lambda_1}^i}{u_{\lambda_1}^{|n|}a_{\lambda_1}^i}=0$, since it is p-torsion and $i>0$. Thus, all connecting homomorphism on torsion classes are trivial. So we know that

$$\begin{aligned}\underline{\pi}_{*-n\lambda_1}(a_{\lambda_0}^{-1}H\underline{\mathbb{Z}})(G/G)=\mathbb{Z}/p\langle\frac{p}{u_{\lambda_0}^{|n|}}\rangle\oplus\mathbb{Z}/p\langle\Sigma^{-1}\frac{u_{\lambda_1}^i}{u_{\lambda_1}^{|n|}a_{\lambda_1}^i}\rangle &\quad \text{for } 0<i<|n|\\ \oplus\mathbb{Z}/p^2\langle\frac{u_{\lambda_0}^j}{u_{\lambda_1}^{|n|}a_{\lambda_0}^j}\rangle &\quad \text{for } j>0\end{aligned}$$

Moving to $\underline{\pi}_{*-m\lambda_0-n\lambda_1}$, we have

$$\underline{\pi}_{*-m\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}}_h)(G/G)=\mathbb{Z}\langle\frac{p^2}{u_{\lambda_0}^{|m|}u_{\lambda_1}^{|n|}}\rangle\oplus\mathbb{Z}/p^2\langle\Sigma^{-1}\frac{u_{\lambda_0}^i}{u_{\lambda_0}^{|m|}u_{\lambda_1}^{|n|}a_{\lambda_0}^i}\rangle \text{ for } i>0$$

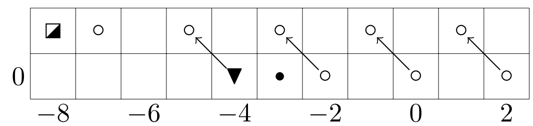
by reading from $RO(G)$-graded homotopy and

$$\begin{aligned}\underline{\pi}_{*-m\lambda_0-n\lambda_1}(a_{\lambda_0}^{-1}H\underline{\mathbb{Z}})(G/G)=\mathbb{Z}/p\langle a_{\lambda_0}^m\frac{p}{u_{\lambda_0}^{|n|}}\rangle\oplus\mathbb{Z}/p\langle\Sigma^{-1}a_{\lambda_0}^m\frac{u_{\lambda_1}^i}{u_{\lambda_1}^{|n|}a_{\lambda_1}^i}\rangle &\quad \text{for } 0<i<|n|\\ \oplus\mathbb{Z}/p^2\langle a_{\lambda_0}^m\frac{u_{\lambda_0}^j}{u_{\lambda_1}^{|n|}a_{\lambda_0}^j}\rangle &\quad \text{for } j>0\end{aligned}$$

from a_{λ_0}-periodicity. By comparing names of elements, we see that under connecting homomorphism, all $\mathbb{Z}/p^2$ in $a_{\lambda_0}^{-1}H\underline{\mathbb{Z}}$ maps isomorphically into corresponding $\mathbb{Z}/p^2$ in $H\underline{\mathbb{Z}}_h$, and in $\underline{\pi}_{2n-m\lambda_0-n\lambda_1}$, the map $\underline{\mathbb{B}}_{0,1}^E\to\underline{\mathbb{B}}_{1,1}$ is nontrivial, with cokernel $\underline{\mathbb{B}}_{0,1}$. All other $\mathbb{Z}/p$ in $a_{\lambda_0}^{-1}H\underline{\mathbb{Z}}$ has no target to hit. Therefore we draw the conclusion:

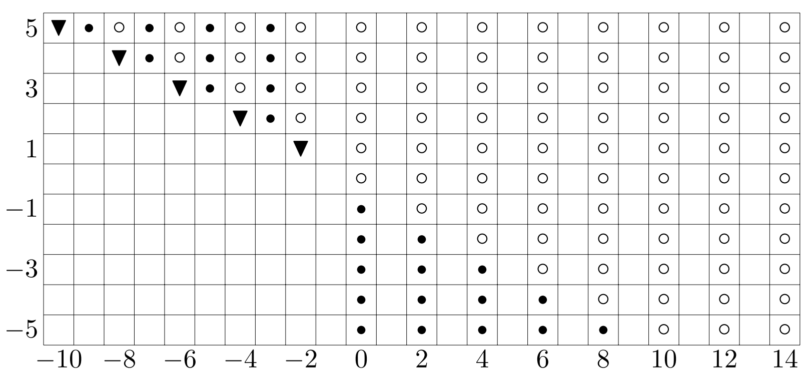
$$\begin{aligned}\underline{\pi}_{*-m\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}})(G/G)=\mathbb{Z}\langle\frac{p^2}{u_{\lambda_0}^{|m|}u_{\lambda_1}^{|n|}}\rangle\oplus\mathbb{Z}/p^2\langle\Sigma^{-1}\frac{u_{\lambda_0}^i}{u_{\lambda_0}^{|m|}u_{\lambda_1}^{|n|}a_{\lambda_0}^i}\rangle &\quad \text{for } 0<i<|m|\\ \oplus\mathbb{Z}/p\langle\Sigma^{-1}a_{\lambda_0}^m\frac{u_{\lambda_1}^j}{u_{\lambda_1}^{|n|}a_{\lambda_1}^j}\rangle &\quad \text{for } 0\le j<|n|\end{aligned}$$

For $m=n=-2$, the connecting homomorphism is described in the following picture:



Remark 4.5. *The above computation can be done by cellular method easier than by Tate diagram, since $S^{m\lambda_0+n\lambda_1}$ with m, n the same sign, has a very simple cellular structure. However, the Tate diagram method is better at tracking multiplicative structure, and easier to generalize to more complicated cases.*

Before we dive into the case where m, n has different signs, it is helpful to compile $\underline{\pi}_\bigstar(a_{\lambda_0}^{-1}H\underline{\mathbb{Z}})$ into a single chart. In the following chart we present $\underline{\pi}_{i+j\lambda_1}(a_{\lambda_0}^{-1}H\underline{\mathbb{Z}})$, with horizontal coordinate i and vertical coordinate j. We can omit λ_0 because the spectrum is a_{λ_0}-periodic. Generators of Mackey functors are described above.



Now we are ready for the rest of $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}})$. First we deal with the case $m > 0$ and $n < 0$. First, we read from the chart that

$$\underline{\pi}_{*-m\lambda_0-n\lambda_1}(a_{\lambda_0}^{-1}H\underline{\mathbb{Z}})(G/G) = \mathbb{Z}/p\langle a_{\lambda_0}^m \frac{2}{u_{\lambda_1}^{|n|}}\rangle \oplus \mathbb{Z}/p\langle \Sigma^{-1}a_{\lambda_0}^m \frac{u_{\lambda_1}^i}{u_{\lambda_1}^{|n|}a_{\lambda_1}^i}\rangle \quad \text{for } 0 < i < |n|$$

$$\oplus\, \mathbb{Z}/p^2\langle a_{\lambda_0}^m \frac{u_{\lambda_0}^j}{u_{\lambda_1}^{|n|}a_{\lambda_0}^j}\rangle \qquad \text{for } j > 0$$

and

$$\underline{\pi}_{*-m\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}}_h) = \mathbb{Z}\langle\frac{p^2u_{\lambda_0}^m}{u_{\lambda_0}^{|n|}}\rangle \oplus \mathbb{Z}/p^2\langle\Sigma^{-1}\frac{u_{\lambda_0}^m u_{\lambda_0}^i}{u_{\lambda_0}^{|n|}a_{\lambda_0}^i}\rangle \text{ for } i > 0$$

By the same method, we see that in this case, all torsion class in $H\underline{\mathbb{Z}}_h$ are killed by connecting homomorphism, and in $\underline{\pi}_{2(m+n)-m\lambda_0-n\lambda_1}$, there is a nontrivial extension of forms of $\underline{\mathbb{Z}}$:

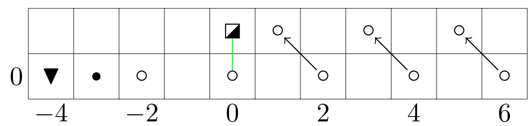
$$0 \to \underline{\mathbb{Z}}_{1,1} \to \underline{\mathbb{Z}} \to \underline{\mathbb{B}}_{1,1} \to 0$$

So we conclude that

$$\underline{\pi}_{*-m\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}}) = \mathbb{Z}\langle \frac{u^m_{\lambda_0}}{u^{|n|}_{\lambda_1}} \rangle \oplus \mathbb{Z}/p\langle a^m_{\lambda_0}\frac{2}{u^{|n|}_{\lambda_1}}\rangle \oplus \mathbb{Z}/p\langle \Sigma^{-1} a^m_{\lambda_0}\frac{u^i_{\lambda_1}}{u^{|n|}_{\lambda_1} a^i_{\lambda_1}}\rangle \quad \text{for } 0 < i < |n|$$

$$\oplus \mathbb{Z}/p^2\langle a^m_{\lambda_0}\frac{u^j_{\lambda_0}}{u^{|n|}_{\lambda_1} a^j_{\lambda_0}}\rangle \quad \text{for } 0 < j < m$$

The connecting homomorphism for $m = 2$ and $n = -2$ is described in the chart:



The last case is $m < 0$ and $n > 0$. We also start with $H\underline{\mathbb{Z}}_h$ and $a^{-1}_{\lambda_0}H\underline{\mathbb{Z}}$:

$$\underline{\pi}_{*-m\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}}_h)(G/G) = \mathbb{Z}\langle \frac{p^2 u^n_{\lambda_1}}{u^{|m|}_{\lambda_0}}\rangle \oplus \mathbb{Z}/p^2\langle \Sigma^{-1}\frac{u^n_{\lambda_1} u^i_{\lambda_0}}{u^{|m|}_{\lambda_0} a^i_{\lambda_0}}\rangle \text{ for } i > 0$$

and

$$\underline{\pi}_{*-m\lambda_0-n\lambda_1}(a^{-1}_{\lambda_0}H\underline{\mathbb{Z}})(G/G) = \mathbb{Z}/p\langle a^m_{\lambda_0} a^i_{\lambda_1} u^{n-i}_{\lambda_1}\rangle \quad \text{for } 0 < i \le n$$

$$\oplus \mathbb{Z}/p^2\langle \Sigma^{-1} a^m_{\lambda_0} u^n_{\lambda_1}\frac{u^j_{\lambda_0}}{a^j_{\lambda_0}}\rangle \quad \text{for } 0 \le j$$

We see that in this case, all $\mathbb{Z}/p^2$-torsion in $a^{-1}_{\lambda_0}H\underline{\mathbb{Z}}$ maps isomorphically into $H\underline{\mathbb{Z}}_h$. For $\mathbb{Z}/p$-torsions, gold relation will cause some subtlety. First we assume that $m < -1$,then the $\mathbb{Z}/p$ torsion of the highest integer degree is $a^m_{\lambda_0} a_{\lambda_1} u^{n-1}_{\lambda_1}$, and its target under connecting homomorphism is generated by $\Sigma^{-1}\frac{a^{m+1}_{\lambda_0} u^n_{\lambda_1}}{u_{\lambda_0}}$(If $m = -1$ this element doesn't exist). Gold relation tells us that

$$a_{\lambda_1} u_{\lambda_0} = p a_{\lambda_0} u_{\lambda_1}$$

Therefore the connecting homomorphism in this degree is $\mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2$, and in terms of Mackey functor, it is $\underline{\mathbb{B}}_{0,1} \to \underline{\mathbb{B}}_{1,1}$ with cokernel $\underline{\mathbb{B}}_{1,0}$. For other $\mathbb{Z}/p$-torsions, similar arguments tells us that the map is $\mathbb{Z}/p \xrightarrow{2^i} \mathbb{Z}/p^2$ for $i > 1$, therefore are trivial. The last problem is that in $\underline{\pi}_{2(m+n)-m\lambda_0-n\lambda_1}$, there is a potential extension if $m + n \ge 0$:

$$0 \to \underline{\mathbb{Z}}_{1,1} \to ? \to \underline{\mathbb{B}}_{0,1} \to 0$$

$\underline{\mathbb{Z}}_{1,1}(G/G)$ is generated by $\frac{p^2 u^n_{\lambda_1}}{u^{|m|}_{\lambda_0}}$, while the $\underline{\mathbb{B}}_{0,1}(G/G)$ is generated by $a^m_{\lambda_0} a^{-m}_{\lambda_1} u^{n+m}_{\lambda_1}$. Again, by gold relation, if $m = -1$, this extension is nontrivial, and the middle term is $\underline{\mathbb{Z}}_{1,0}$. If $m < -1$, the extension is trivial. So we conclude that, if $m < -1$,

$$\underline{\pi}_{*-m\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}}) = \mathbb{Z}\langle \frac{p^2 u^n_{\lambda_1}}{u^{|m|}_{\lambda_0}}\rangle \oplus \mathbb{Z}/p\langle \Sigma^{-1}\frac{a^{m+1}_{\lambda_0} u^n_{\lambda_1}}{u_{\lambda_0}}\rangle$$

$$\oplus \mathbb{Z}/p\langle a^m_{\lambda_0} a^i_{\lambda_1} u^{n-i}_{\lambda_1}\rangle \quad \text{for } 1 < i \le n$$

$$\oplus \mathbb{Z}/p^2\langle \Sigma^{-1}\frac{u^n_{\lambda_1} u^i_{\lambda_0}}{u^{|m|}_{\lambda_0} a^i_{\lambda_0}}\rangle \quad \text{for } 0 < i < |m| - 1$$

Where $\mathbb{Z}$ is $\underline{\mathbb{Z}}_{1,1}(G/G)$, $\mathbb{Z}/p\langle\Sigma^{-1}\frac{a_{\lambda_0}^{m+1}u_{\lambda_1}^n}{u_{\lambda_0}}\rangle$ is $\underline{\mathbb{B}}_{1,0}(G/G)$, all $\mathbb{Z}/p^2$ are $\underline{\mathbb{B}}_{1,1}(G/G)$ and all other $\mathbb{Z}/p$ are $\underline{\mathbb{B}}_{0,1}(G/G)$.

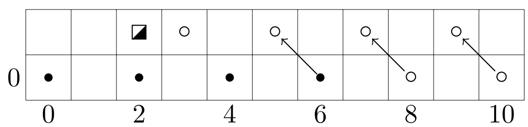
If $m=-1$,

$$\underline{\pi}_{*+\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}})=\mathbb{Z}\langle\frac{pu_{\lambda_1}^n}{u_{\lambda_0}}\rangle\oplus\mathbb{Z}/p\langle a_{\lambda_0}^{-1}a_{\lambda_1}^{i}u_{\lambda_1}^{n-i}\rangle \text{ for } 1<i\le n$$

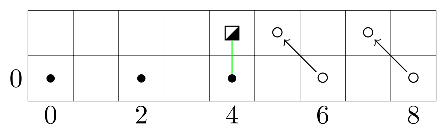
Where $\mathbb{Z}$ is $\underline{\mathbb{Z}}_{1,0}(G/G)$ and all $\mathbb{Z}/p$ are $\underline{\mathbb{B}}_{0,1}(G/G)$.

Remark 4.6. *When $m=-1$ and $n=1$, we see that $S^{-\lambda_0+\lambda_1}\wedge H\underline{\mathbb{Z}}\simeq H\underline{\mathbb{Z}}_{1,0}$. It is predicted by Anderson duality in Theorem 3.17. More generally, Anderson duality can give an interpretation of the appearance of $\underline{\mathbb{Z}}_{1,0}$ for $m=-1$, see Section 4.5.*

We also provide pictures for connecting homomorphism in this case. The following is for $m=-3$ and $n=4$:



The following is for $m=-1$ and $n=3$



An Exotic Multiplication. In the above computation, we see that $\frac{a_{\lambda_1}}{a_{\lambda_0}}=\frac{pu_{\lambda_1}}{u_{\lambda_0}}$ (A geometric argument is given in [HHR17a, Proposition 3.3]), that is, the generator of $\underline{\pi}_{\lambda_0-\lambda_1}(H\underline{\mathbb{Z}})(G/G)$ supports a nontrivial a_{λ_0}-multiplication:

$$a_{\lambda_0}\frac{pu_{\lambda_1}}{u_{\lambda_0}}=a_{\lambda_1}$$

Then if we think about the square of this generator, which deserves the name $\frac{p^2u_{\lambda_0}^2}{u_{\lambda_1}^2}$, then we see that it supports a nontrivial $a_{\lambda_0}^2$-multiplication:

$$a_{\lambda_0}^2\frac{p^2u_{\lambda_1}^2}{u_{\lambda_0}^2}=a_{\lambda_1}^2$$

However, the element $\frac{p^2u_{\lambda_1}^2}{u_{\lambda_0}^2}$, if we characterize as the image of transfer of a generator in G/e-level, will be killed by a_{λ_0}, by [HHR17b, Lemma 4.2]. Therefore $(\frac{pu_{\lambda_1}}{u_{\lambda_0}})^2$ cannot be in the image of transfer. Notice that $\underline{\pi}_{2\lambda_0-2\lambda_1}H\underline{\mathbb{Z}}\cong\underline{\mathbb{B}}_{0,1}\oplus\underline{\mathbb{Z}}_{1,1}$, where the torsion class is generated by $\frac{pa_{\lambda_1}^2}{a_{\lambda_0}^2}$, and it maps to an a_{λ_0}-tower in $a_{\lambda_0}^{-1}H\underline{\mathbb{Z}}$. Therefore, the only way of avoiding contradiction, is that $(\frac{pu_{\lambda_1}}{u_{\lambda_0}})^2=\frac{pa_{\lambda_1}^2}{a_{\lambda_0}^2}+\frac{p^2u_{\lambda_1}^2}{u_{\lambda_0}^2}$, the summation of p-torsion element and the element in the image of transfer. Notice that since $\frac{pa_{\lambda_1}^2}{a_{\lambda_0}^2}$ is killed by u_{λ_0} by gold relation, this exotic multiplication will only occur in $\underline{\pi}_{n(\lambda_0-\lambda_1)}(H\underline{\mathbb{Z}})$ but not other gradings where the Mackey functor is $\underline{\mathbb{B}}_{0,1}\oplus\underline{\mathbb{Z}}_{1,1}$. Summarizing above, we have the following.

Proposition 4.7.

$$(\frac{pu_{\lambda_1}}{u_{\lambda_0}})^n = \frac{pa^n_{\lambda_1}}{a^n_{\lambda_0}} + \frac{p^n u^n_{\lambda_1}}{u^n_{\lambda_0}}$$

That means, the n-th power of the generator of $\underline{\pi}_{\lambda_0-\lambda_1}(H\underline{\mathbb{Z}})(G/G)$ is the summation of the p-torsion class and p^{n-2}-multiple of $Tr_1^{p^2}(1)$ in the corresponding degree.

After all the work, what we have done is actually describe $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}})(G/G)$ as a ring, therefore describe $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}})$ as a $RO(G)$-graded Green functor. The folllowing theorem concludes this section.

Theorem 4.8. (1) *As a module over $BB_{C_{p^2}}$, $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}})(G/G)$ is*

- *If $m, n \geq 0$,*

$$\begin{aligned}\underline{\pi}_{*-m\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}})(G/G) = \mathbb{Z}\langle u^m_{\lambda_0}u^n_{\lambda_1}\rangle &\oplus \mathbb{Z}/p\langle a^i_{\lambda_1}a^m_{\lambda_0}u^{n-i}_{\lambda_1}\rangle \quad \text{for } 0 < i \leq n\\ &\oplus \mathbb{Z}/p^2\langle a^j_{\lambda_0}u^{m-j}_{\lambda_0}u^n_{\lambda_1}\rangle \quad \text{for } 0 < j \leq m\end{aligned}$$

- *If $m = 0$ and $n < 0$,*

$$\underline{\pi}_{*-n\lambda_1}(H\underline{\mathbb{Z}})(G/G) = \mathbb{Z}\langle \frac{p}{u^{|m|}_{\lambda_1}}\rangle \oplus \mathbb{Z}/p\langle \Sigma^{-1}\frac{u^i_{\lambda_1}}{u^{|m|}_{\lambda_1}a^i_{\lambda_1}}\rangle \text{ for } 0 < i < |n|.$$

- *If $m < 0$ and $n \leq 0$,*

$$\begin{aligned}\underline{\pi}_{*-m\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}})(G/G) = \mathbb{Z}\langle \frac{p^2}{u^{|m|}_{\lambda_0}u^{|n|}_{\lambda_1}}\rangle &\oplus \mathbb{Z}/p^2\langle \Sigma^{-1}\frac{u^i_{\lambda_0}}{u^{|m|}_{\lambda_0}u^{|n|}_{\lambda_1}a^i_{\lambda_0}}\rangle \quad \text{for } 0 < i < |m|\\ &\oplus \mathbb{Z}/p\langle \Sigma^{-1}a^m_{\lambda_0}\frac{u^j_{\lambda_1}}{u^{|n|}_{\lambda_1}a^j_{\lambda_1}}\rangle \quad \text{for } 0 \leq j < |n|\end{aligned}$$

- *If $m > 0$ and $n < 0$,*

$$\begin{aligned}\underline{\pi}_{*-m\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}}) = \mathbb{Z}\langle \frac{u^m_{\lambda_0}}{u^{|n|}_{\lambda_1}}\rangle \oplus \mathbb{Z}/p\langle a^m_{\lambda_0}\frac{p}{u^{|n|}_{\lambda_1}}\rangle &\oplus \mathbb{Z}/p\langle \Sigma^{-1}a^m_{\lambda_0}\frac{u^i_{\lambda_1}}{u^{|n|}_{\lambda_1}a^i_{\lambda_1}}\rangle \quad \text{for } 0 < i < |n|\\ &\oplus \mathbb{Z}/p^2\langle a^m_{\lambda_0}\frac{u^j_{\lambda_0}}{u^{|n|}_{\lambda_1}a^j_{\lambda_0}}\rangle \quad \text{for } 0 < j < m\end{aligned}$$

- *If $m = -1$ and $n > 0$,*

$$\underline{\pi}_{*+\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}}) = \mathbb{Z}\langle \frac{pu^n_{\lambda_1}}{u_{\lambda_0}}\rangle \oplus \mathbb{Z}/p\langle a^{-1}_{\lambda_0}a^i_{\lambda_1}u^{n-i}_{\lambda_1}\rangle \text{ for } 1 < i \leq n$$

- *If $m < -1$ and $n > 0$,*

$$\begin{aligned}\underline{\pi}_{*-m\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}}) = \mathbb{Z}\langle \frac{p^2u^n_{\lambda_1}}{u^{|m|}_{\lambda_0}}\rangle &\oplus \mathbb{Z}/p\langle \Sigma^{-1}\frac{a^{m+1}_{\lambda_0}u^n_{\lambda_1}}{u_{\lambda_0}}\rangle\\ &\oplus \mathbb{Z}/p\langle a^m_{\lambda_0}a^i_{\lambda_1}u^{n-i}_{\lambda_1}\rangle \quad \text{for } 1 < i \leq n\\ &\oplus \mathbb{Z}/p^2\langle \Sigma^{-1}\frac{u^n_{\lambda_1}u^i_{\lambda_0}}{u^{|m|}_{\lambda_0}a^i_{\lambda_0}}\rangle \quad \text{for } 0 < i < |m| - 1\end{aligned}$$

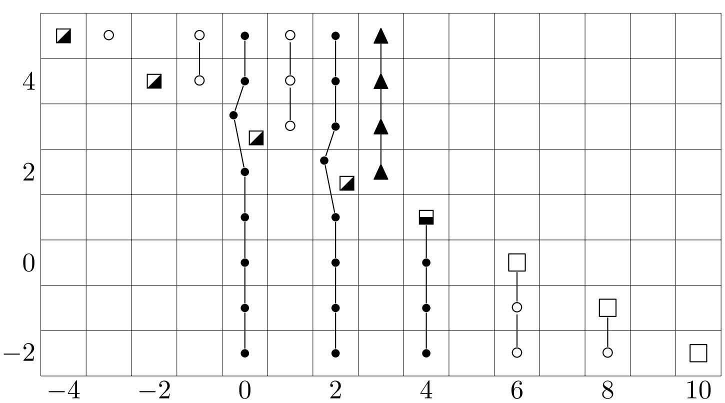
(2) *The ring structure of $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}})(G/G)$ is determined by the following:*

- *Multiplication between elements without desuspension sign is determined by their names from Tate diagram, with one execption as Proposition 4.7.*

- *All elements with desuspension sign is a square zero extension of the above ring.*

Proof. We only need to prove the square zero extension part. Assume $x, y \in \underline{\pi}_{i-m\lambda_0-n\lambda_1}(H\underline{\mathbb{Z}})(G/G)$ are elements with desuspension sign. First, if x, y are torsions in $m, n \leq 0$, then $xy = 0$ since all torsions there are in odd dimension. Now if at least one of x, y is from the part $m > 0$ and $n < 0$, then by the module structure, $x = a^i_{\lambda_0}x'$ and $y = a^j_{\lambda_0}y'$, where x' and y' are torsions in $m, n \leq 0$, therefore $xy = a^{i+j}_{\lambda_0}x'y' = 0$. If x is from the part where $m < -1$ and $n > 0$, then it is a_{λ_0}-torsion. However, any torsion in even degree is in the subalgebra of $\mathbb{Z}[u^{\pm}_{\lambda_0}, u^{\pm}_{\lambda_1}, a^{\pm}_{\lambda_0}]/(p^2a_{\lambda_0})$, where no element is killed by a_{λ_0}. Therefore x kills all elements in odd degree. □

Remark 4.9. *All p-torsions in $\underline{\pi}_{*-V}(H\underline{\mathbb{Z}})(G/G)$ where V an actual representation or the opposite of one, except the family $a_{\lambda_1}a^i_{\lambda_0}u^j_{\lambda_1}$, are a_{λ_0} ∞-divisible, and not a_{λ_0}-torsion. Therefore they support a_{λ_0} tower of length ∞ going both ways. This phenomenon is not seen in C_p. The following figure is $\underline{\pi}_{m\lambda_0-3\lambda_1+i}(H\underline{\mathbb{Z}})$, with horizontal coordinate i, and vertical coordinate m. Black vertical lines indicates multiplication by a_{λ_0}. The only exotic multiplication of Proposition 4.7 in this figure is in* $(0, 3)$.



4.5. **Duality Method.**

In this section, we will use Anderson duality and universal coefficient spectral sequence to compute $\underline{\pi}_{\bigstar}(H\underline{\mathbb{Z}})$, and to present its rich structure. This method is not perfect: there is a certain d_3 and extension problem the author does not know how to solve inside this method, however by comparing to the Tate diagram, we can resolve these problems easily. Therefore, we will use results from the previous section freely and this section serves more as an exploration of relations between $\underline{\pi}_{\bigstar}(H\underline{\mathbb{Z}})$ and homological algebra.

As a warm-up, we compute $\underline{\pi}_{\bigstar}(H\underline{\mathbb{Z}})$ for C_p by this method.

The starting point is $\underline{\pi}_*(S^0 \wedge H\underline{\mathbb{Z}})$, which is $\underline{\mathbb{Z}}$ concentrated in dimension 0. Applying Anderson dual to it, we get $I_{\mathbb{Z}}(H\underline{\mathbb{Z}}) \simeq H\underline{\mathbb{Z}}^* \simeq S^{2-\lambda_0} \wedge H\underline{\mathbb{Z}}$ by 3.10, therefore $\underline{\pi}_*(S^{-\lambda_0} \wedge H\underline{\mathbb{Z}})$ is concentrated in dimension -2, and is $\underline{\mathbb{Z}}^*$. Now we apply universal coefficient spectral sequence of Theorem 2.22 to $S^{-\lambda_0} \wedge H\underline{\mathbb{Z}}$. The

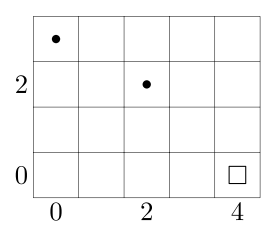
E_2-page is exactly $\underline{Ext}^i_{\underline{\mathbb{Z}}}(\underline{\mathbb{Z}}^*, \underline{\mathbb{Z}})$ without any differentials. Thus by Corollary 3.15, we know that

$$\underline{\pi}_i(S^{\lambda_0} \wedge H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{B}}_1 & i = 0 \\ \underline{\mathbb{Z}} & i = 2 \\ \underline{0} & \text{otherwise} \end{cases}$$

Now we apply Anderson dual again to $S^{\lambda_0} \wedge H\underline{\mathbb{Z}}$. Now $I_{\mathbb{Z}}(S^{\lambda_0} \wedge H\underline{\mathbb{Z}}) \simeq S^{2-2\lambda_0} \wedge H\underline{\mathbb{Z}}$ by Lemma 3.10, and by Proposition 2.19, we know that

$$\underline{\pi}_i(S^{-2\lambda_0} \wedge H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{B}}_1^E \cong \underline{\mathbb{B}}_1 & i = -3 \\ \underline{\mathbb{Z}}^* & i = -4 \\ \underline{0} & \text{otherwise} \end{cases}$$

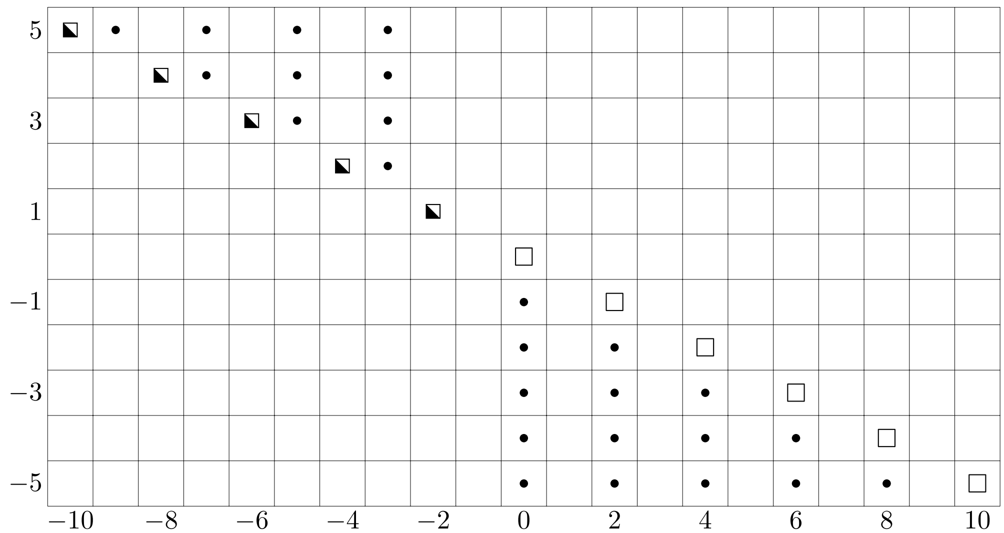
Apply universal coefficient spectral sequence to $S^{-2\lambda_0} \wedge H\underline{\mathbb{Z}}$, we have the following E_2 page by Theorem 3.13 and Corollay 3.15 in Adams index, converging to $\underline{\pi}_i(S^{2\lambda_0} \wedge H\underline{\mathbb{Z}})$:



We see that there is no room for differential, and read

$$\underline{\pi}_i(S^{2\lambda_0} \wedge H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{B}}_1 & i = 0, 2 \\ \underline{\mathbb{Z}} & i = 4 \\ \underline{0} & \text{otherwise} \end{cases}$$

By repeat using Anderson dual and universal coefficient spectral sequence, we can compute $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}})$. The result is in the following chart, where horizontal coordinate is copies of 1 and vertical coordinate is copies of λ_0. It is essentially a sub-chart of the chart in [Gre, pp. 6], only consisting of orientable representations, but with Mackey functor value instead of abelian group value:



We can see intuitively in this chart how Anderson duality and universal coefficient spectral sequence are working. Anderson duality is a duality with centre of symmetry $(-1, \frac{\lambda_0}{2})$ and it rotates the whole chart by 180-degree and change Mackey functors to their level-wise dual, with a 1 shift on torsion Mackey functors. Universal coefficient spectral sequence is a duality with centre of symmerty $(0,0)$, but there is a gap of length 3 in $\underline{Ext}_{\underline{\mathbb{Z}}}$ of torsion classes, so after rotation by 180-degree about the origin, we need to shift all torsion classes by 3. Also, in the universal coefficient spectral sequence, if the form of $\underline{\mathbb{Z}}$ is not $\underline{\mathbb{Z}}$ itself, it will produce a nontrivial $\underline{Ext}^2_{\underline{\mathbb{Z}}}$ term in duality.

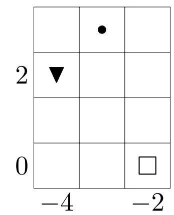
Now we start to compute in C_{p^2}. We will compute $\underline{\pi}_*(S^{n\lambda_0 \pm 2\lambda_1} \wedge H\underline{\mathbb{Z}})$, to compare to previous sections. The starting point is $\underline{\pi}_*(S^{-2\lambda_1} \wedge H\underline{\mathbb{Z}})$, which can be read from the chart above by Lemma 3.1.

$$\underline{\pi}_i(S^{-2\lambda_1} \wedge H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{B}}_{0,1} & i = -3 \\ \underline{\mathbb{Z}}_{0,1} & i = -4 \\ 0 & \text{otherwise} \end{cases}$$

Now we apply Anderson duality in C_{p^2} to it. By the standard short exact sequence argument, we compute $\underline{\pi}_*(S^{-\lambda_0+2\lambda_1} \wedge H\underline{\mathbb{Z}})$:

$$\underline{\pi}_i(S^{-\lambda_0+2\lambda_1} \wedge H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{B}}_{0,1} & i = 0 \\ \underline{\mathbb{Z}}_{1,0} & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

The next step is applying universal coefficient spectral sequence to $\underline{\pi}_*(S^{-\lambda_0+2\lambda_1} \wedge H\underline{\mathbb{Z}})$. We get a spectral sequence convergent to $\underline{\pi}_*(S^{\lambda_0-2\lambda_1} \wedge H\underline{\mathbb{Z}})$ with E_2-page



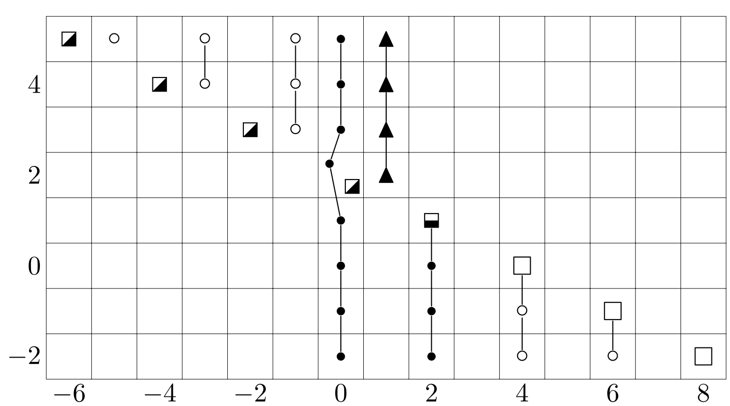
As we can see, there is a potential d_3 from $\underline{\mathbb{Z}}$ in $(-2,0)$ to $\underline{\mathbb{B}}_{0,1}$ in $(-3,3)$. However, by computation from Tate diagram method, we know that $\frac{u_{\lambda_0}}{u^n_{\lambda_1}}$ exist for all $n \geq 0$, so the generator of $\underline{\mathbb{Z}}(G/G)$ is a permanent cycle, therefore d_3 is trivial. So our conclusion is the following:

$$\underline{\pi}_*(S^{\lambda_0-2\lambda_1} \wedge H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{B}}^E_{1,0} & i=-4 \\ \underline{\mathbb{B}}_{0,1} & i=-3 \\ \underline{\mathbb{Z}} & i=-2 \\ 0 & \text{otherwise} \end{cases}$$

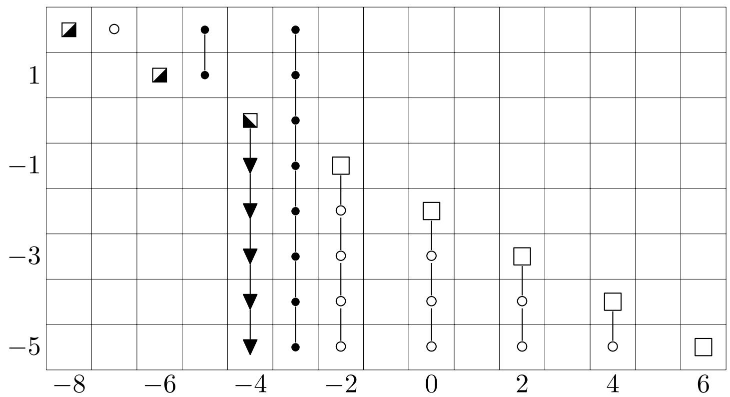
We can therefore apply Anderson duality and universal coefficient spectral sequence repeatedly to compute $\underline{\pi}_*(S^{n\lambda_0 \pm 2\lambda_1})$. Since $\frac{u_{\lambda_0}}{u^n_{\lambda_1}}$ exist for all $n \geq 0$, all potential d_3 are trivial in this situation. The following two charts shows $\underline{\pi}_{n\lambda_0+2\lambda_1+m}(H\underline{\mathbb{Z}})$ and $\underline{\pi}_{n\lambda_0-2\lambda_1+m}(H\underline{\mathbb{Z}})$, the horizontal coordinate is m and the vertical coordinate is n.

Notice that Anderson duality and universal coefficient spectral sequence are p-resenting two kinds of duality between these two charts. We should think of these two charts as two 2 dimensional slices in a 3 dimensional chart, since our indexing group is free of rank 3.

$\underline{\pi}_{m\lambda_0-2\lambda_1+*}H\underline{\mathbb{Z}}$. Horizontal coordinate is $*$ and vertical coordinate is m.



$\underline{\pi}_{m\lambda_0+2\lambda_1+*}H\underline{\mathbb{Z}}$. Horizontal coordinate is $*$ and the vertical coordinate is m.



5. $G = C_{p^n}$ for $n > 2$

For $n > 2$, all methods above only works partially. Cellular method becomes hard to compute, since we need to compare more spectral sequences, and comparison cannot guarantee the result. In cofibre of a method, there will be potential connecting homomorphisms which cannot be resolved by a_σ-multiplication. In Tate diagram method, the only problem is that when we compute $\underline{\pi}_*(S^V \wedge a_{\lambda_0}^{-1}H\underline{\mathbb{Z}})$ for those V without λ_0, there might be nontrivial extension of $\underline{\mathbb{Z}}$-modules which the author cannot resolve. In duality method, the main problem is that we have no direct argument for the possible d_3 and extension between filtration 3 and 2.

However, some very interesting phenomenon can be seen in attempt to answer the following question: "Fix $G = C_{p^n}$, Given two actual representation V_1, V_2, what's the smallest k such that $\frac{p^k u_{V_1}}{u_{V_2}}$ is defined in $\underline{\pi}_\bigstar(H\underline{\mathbb{Z}})$?" In the case of C_{p^2}, we have the following from Section 4:

Proposition 5.1. *For $G = C_{p^2}$*

(1) $\frac{u_{\lambda_0}}{u^i_{\lambda_1}}$ *exist for all $i \geq 0$. Therefore $\frac{u^j_{\lambda_0}}{u^i_{\lambda_1}}$ exists for all $i, j \geq 0$.*
(2) $\frac{p}{u^i_{\lambda_1}}$ *exist for all $i \geq 1$ and cannot be divided by p.*
(3) $\frac{pu_{\lambda_1}}{u_{\lambda_0}}$ *exists and cannot be divided by p.*
(4) $\frac{p^2 u^i_{\lambda_1}}{u^j_{\lambda_0}}$ *exist for all $i \leq 0$ and $j \geq 2$ and cannot be divided by p.*

When the order of group becomes larger, the answer is still computable, but becomes more complicated. For example, in general, the following proposition gives a first look at this complexity:

Proposition 5.2. *For $G = C_{p^n}$,*

(1) $\frac{u_{\lambda_{k_0}}}{u_{\lambda_{k_1}}}$ *exists if $k_0 < k_1$.*
(2) $\frac{p^{n-k-2}u_{\lambda_k}}{u^i_{\lambda_{k+1}}}$ *exists for $i \geq 2$ and are not divisible by p.*

Proof. (1) By Theorem 3.17 we know $S^{\lambda_{k_1}-\lambda_{k_0}} \wedge H\underline{\mathbb{Z}} \simeq H\underline{M}$ for $\underline{M}$ a form of $\underline{\mathbb{Z}}$. Therefore, the universal coefficient spectral sequence convergent to

$\underline{\pi}_*(S^{\lambda_{k_0}-\lambda_{k_1}} \wedge H\underline{\mathbb{Z}})$ collapses at E_2-page, and the corresponding form of $\underline{\mathbb{Z}}$ is $\underline{Ext}^0_{\underline{\mathbb{Z}}}(\underline{M}, \underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}$. Therefore $\frac{u_{\lambda_{k_0}}}{u_{\lambda_{k_1}}}$ exists.

(2) We can assume $k = 0$, since if $k > 0$ we can compute inside the quotient group C_{p^n}/C_{p^k} and λ_k on C_{p^n} is λ_0 on C_{p^n}/C_{p^k}. We use Tate diagram of 3.7 in this proof.

Consider the map $\underline{\pi}_*(S^{-i\lambda_1} \wedge H\underline{\mathbb{Z}}_h) \to \underline{\pi}_*(S^{-i\lambda_1} \wedge H\underline{\mathbb{Z}})$. When $* = -2i+1$, the map is $\underline{\mathbb{B}}_{1,1,\ldots,1} \to \underline{\mathbb{B}}_{0,1,1,\ldots,1}$. On G/G-level, the source is generated by $\Sigma^{-1}\frac{u_{\lambda_0}}{a_{\lambda_0}u^i_{\lambda_1}}$ and the target is generated by $\Sigma^{-1}\frac{1}{a_{\lambda_1}u^i_{\lambda_1}}$. So by gold relation, this map on G/G-level is multiplication by p:$\mathbb{Z}/p^n \xrightarrow{p} \mathbb{Z}/p^{n-1}$. Therefore $\underline{\pi}_{-2i+2}(S^{-i\lambda_1} \wedge a^{-1}_{\lambda_0} H\underline{\mathbb{Z}})(G/G) \cong \mathbb{Z}/(p^2)$ is generated by $\frac{p^{n-2}u_{\lambda_0}}{a_{\lambda_1}u^i_{\lambda_1}}$. Therefore, when we shift to $S^{\lambda_0-i\lambda_1}$, in dimension $-2i+2$ and G/G-level, the Tate diagram gives the following:

$$\begin{array}{ccccc}
\mathbb{Z} & \longrightarrow & \underline{\pi}_{-2i+2}(S^{\lambda_0-i\lambda_1} \wedge H\underline{\mathbb{Z}})(G/G) & \longrightarrow & \mathbb{Z}/(p^2) \\
\downarrow \cong & & \downarrow & & \downarrow p^{n-2} \\
\mathbb{Z} & \xrightarrow{p^n} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/(p^n)
\end{array}$$

The bottom extension force the top extension to be nontrivial, with the middle vertical map $\mathbb{Z} \xrightarrow{p^{n-2}} \mathbb{Z}$. Since we know in $H\underline{\mathbb{Z}}^h$, the corresponding generator is $\frac{u_{\lambda_0}}{u^i_{\lambda_1}}$, the generator of $\underline{\pi}_{-2i+2}(S^{\lambda_0-i\lambda_1} \wedge H\underline{\mathbb{Z}})(G/G)$ should be $\frac{p^{n-2}u_{\lambda_0}}{u_{\lambda^i_1}}$.

□

6. Homological Algebra of $\underline{\mathbb{Z}}$-modules

As we see, homological algebra of $\underline{\mathbb{Z}}$-modules appears to be very different from the classical homological algebra. Computation in previous sections can be interpreted as $RO(G)$-graded homological algebra $\underline{Ext}^\bigstar_{\underline{\mathbb{Z}}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}})$. From a more classical point of view, $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}$ tells us a lot about $\underline{Ext}_{\underline{\mathbb{Z}}}$ which is more difficult to compute using projective or injective resolutions.

Theorem 6.1. *Let $\underline{M}$, $\underline{N}$ be two forms of $\underline{\mathbb{Z}}$, such that $H\underline{M} \simeq S^{V_1} \wedge H\underline{\mathbb{Z}}$ and $H\underline{N} \simeq S^{V_2} \wedge H\underline{\mathbb{Z}}$ (See Theorem 3.17), then $\underline{Ext}^*_{\underline{\mathbb{Z}}}(\underline{M}, \underline{N}) \cong \underline{\pi}_{-*}(S^{V_2-V_1} \wedge H\underline{\mathbb{Z}})$*

Proof. By Corollary 3.12, we have

$$\begin{aligned}
\underline{Ext}^*_{\underline{\mathbb{Z}}}(\underline{M}, \underline{N}) &\cong \underline{\pi}_{-*}(Hom_{H\underline{\mathbb{Z}}}(H\underline{M}, H\underline{N})) \\
&\cong \underline{\pi}_{-*}(Hom_{H\underline{\mathbb{Z}}}(S^{V_1} \wedge H\underline{\mathbb{Z}}, S^{V_2} \wedge H\underline{\mathbb{Z}})) \\
&\cong \underline{\pi}_{-*}(S^{V_2-V_1} \wedge H\underline{\mathbb{Z}})
\end{aligned}$$

□

Example 6.2. *If $G = C_{p^2}$,*

$$\underline{Ext}^i_{\underline{\mathbb{Z}}}(\underline{\mathbb{Z}}_{1,0}, \underline{\mathbb{Z}}_{0,1}) = \begin{cases} \underline{\mathbb{Z}} & i = 0 \\ \underline{\mathbb{B}}_{0,1} & i = 1 \\ \underline{\mathbb{B}}^E_{1,0} & i = 2 \\ \underline{0} & \text{otherwise} \end{cases}$$

Proof. Since $\underline{\mathbb{Z}}_{1,0} \simeq S^{-\lambda_0+\lambda_1} \wedge H\underline{\mathbb{Z}}$ and $\underline{\mathbb{Z}}_{0,1} \simeq S^{2-\lambda_1} \wedge H\underline{\mathbb{Z}}$, by the above theorem, $\underline{Ext}^i_{\underline{\mathbb{Z}}}(\underline{\mathbb{Z}}_{1,0}, \underline{\mathbb{Z}}_{0,1}) \cong \underline{\pi}_*(S^{\lambda_0-2\lambda_1+2} \wedge H\underline{\mathbb{Z}})$. And $\underline{\pi}_*(S^{\lambda_0-2\lambda_1+2} \wedge H\underline{\mathbb{Z}})$ is computed in Section 4.5. □

Remark 6.3. *This example shows something counter-intuitive: There is a non-trivial extension of $\underline{\mathbb{Z}}$-modules*

$$0 \to \underline{\mathbb{Z}}_{0,1} \to \underline{M} \to \underline{\mathbb{Z}}_{1,0} \to 0$$

However, in C_p, by similar computation,

$$\underline{Ext}^1_{\underline{\mathbb{Z}}}(\underline{\mathbb{Z}}_0, \underline{\mathbb{Z}}_1) \cong \underline{Ext}^1_{\underline{\mathbb{Z}}}(\underline{\mathbb{Z}}_1, \underline{\mathbb{Z}}_0) \cong \underline{0}$$

So, if we only look at adjacent parts of $\underline{M}$, it splits. Such an extension can be constructed as follows:

Let $\underline{M}$ be the $\underline{\mathbb{Z}}$-module with Lewis diagram

$$\begin{array}{c} \mathbb{Z} \oplus \mathbb{Z} \\ \downarrow\uparrow \\ \mathbb{Z} \oplus \mathbb{Z} \\ \downarrow\uparrow \\ \mathbb{Z} \oplus \mathbb{Z} \end{array}$$

We call generators of $\mathbb{Z}\oplus\mathbb{Z}$ in C_{p^2}/C_{p^i}-level a_i and b_i for $0 \le i \le 2$, then restrictions and transfers are defined as follows:

$$Res^{p^2}_p(a_2) = pa_1,\ Res^{p^2}_p(b_2) = a_1 + b_1 \quad Tr^{p^2}_p(a_1) = a_2,\ Tr^{p^2}_p(b_1) = -a_2 + pb_2$$
$$Res^p_1(a_1) = a_0,\ Res^p_1(b_1) = a_0 + pb_0 \quad Tr^p_1(a_0) = pa_1,\ Tr^p_1(b_0) = -a_1 + b_1$$

Now, $\underline{\mathbb{Z}}_{0,1} \to \underline{M}$ maps generator in each $\mathbb{Z}$ to the corresponding a_i and $\underline{M} \to \underline{\mathbb{Z}}_{1,0}$ maps a_i to 0 and b_i to generators in each level. Now, we try to construct a section $\underline{\mathbb{Z}}_{1,0} \to \underline{M}$. We start with C_{p^2}/e-level, by sending 1 to $ta_0 + b_0$ for some $t \in \mathbb{Z}$. This forces 1 in C_{p^2}/C_p-level going to $(tp-1)a_1 + b_1$, and therefore p in C_{p^2}/C_{p^2}-level goes to $(tp-2)a_2 + pb_2$. But this element is not divisible by p if $p > 2$. So the section cannot exist.

The following theorem says that we can exchange two variables of $\underline{Ext}_{\underline{\mathbb{Z}}}$ via Anderson duality, when they both are torsion.

Theorem 6.4. *Let $\underline{M}$ and $\underline{N}$ be $\underline{\mathbb{Z}}$-modules that $\underline{M}(G/e)$ and $\underline{N}(G/e)$ are torsion, then $\underline{Ext}^*_{\underline{\mathbb{Z}}}(\underline{M}, \underline{N}^E) \cong \underline{Ext}^*_{\underline{\mathbb{Z}}}(\underline{M}^E, \underline{N})$.*

Proof. Since both $\underline{M}$ and $\underline{N}$ are torsion by Lemma2.9, we know that $I_{\mathbb{Z}}(H\underline{M}) \cong \Sigma^{-1}H\underline{M}^E$ and $I_{\mathbb{Z}}(H\underline{N}) \cong \Sigma^{-1}H(\underline{N}^E)$. Therefore,

$$\begin{aligned} Hom_{H\underline{\mathbb{Z}}}(H\underline{M}, H\underline{N}^E) &\cong Hom_{H\underline{\mathbb{Z}}}(H\underline{M}, \Sigma(I_{\mathbb{Z}}(H\underline{N}))) \\ &\cong \Sigma Hom_{H\underline{\mathbb{Z}}}(H\underline{M}, I_{\mathbb{Z}}(H\underline{N})) \\ &\cong \Sigma I_{\mathbb{Z}}(H\underline{M} \Box_{\underline{\mathbb{Z}}} H\underline{N}) \\ &\cong \Sigma I_{\mathbb{Z}}(H\underline{N} \Box_{\underline{\mathbb{Z}}} H\underline{M}) \\ &\cong \Sigma Hom_{H\underline{\mathbb{Z}}}(H\underline{N}, I_{\mathbb{Z}}(H\underline{M})) \\ &\cong Hom_{H\underline{\mathbb{Z}}}(H\underline{N}, H\underline{M}^E) \end{aligned}$$

□

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