

EXAMPLES OF CHROMATIC REDSHIFT IN ALGEBRAIC K -THEORY

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ABSTRACT. We give a simple argument to detect chromatic redshift in the algebraic K -theory of \mathbb{E}_∞ -ring spectra and give two applications: we show for $n \geq 1$ that $K(E_n)$, the algebraic K -theory of any height n Lubin-Tate theory, has nontrivial $T(n+1)$ -localization, and that $K^{(n)}(k)$, the n -fold iterated algebraic K -theory of a field k of characteristic different from p , has nontrivial $T(n)$ -localization.

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1. INTRODUCTION

Chromatic homotopy theory organizes generalized cohomology theories on spaces by *chromatic height*. Roughly, the chromatic height of a theory measures the complexity of the torsion seen by the theory. Height 0 theories, such as rational cohomology, see no torsion information. At height 1, one sees more sophisticated cohomology theories such as real or complex K -theory; nevertheless, these theories remain relatively well understood. At heights 2 and above, the situation rapidly becomes more complicated: one has elliptic cohomology and topological modular forms at height 2, and Lubin-Tate theories E_n and topological automorphic forms at higher height, but these theories are computationally more difficult and elude concrete geometric understanding [LRS95, GH04, Lur18, BL10].

This paper is concerned with a phenomenon in which cohomology theories of higher height arise from theories of lower height: the chromatic redshift in algebraic K -theory. Motivated by finding ring spectrum analogues of the Lichtenbaum-Quillen conjectures, Ausoni and Rognes computed the $(\text{mod } (p, v_1))$ algebraic K -theory of (the Adams summand of) connective K -theory [AR02]. From these computations, they observed that algebraic K -theory tends to increase chromatic complexity by one: that is, the algebraic K -theory of a height n ring spectrum is often of height $n+1$. This became known as the *chromatic redshift philosophy*.

Chromatic redshift has since been studied extensively: in height 1 examples such as ku_p^\wedge, KU_p^\wedge and ku/p by [BM08, Aus10, AR02, AR12]; computationally in [AK17, AKQ21, Vee18] and via theoretical considerations [Wes17, CSY21]; additionally, questions of descent were addressed in

the foundational papers [CMNN20a, CM19]. This work builds upon recent progress by Land-Mathew-Meier-Tamme [LMMT20] and Clausen-Mathew-Naumann-Noel [CMNN20b]: to motivate it, we sample a result of [CMNN20b] which generalizes a theorem of Mitchell [Mit90] in the case $n = 1$:

Theorem 1.1 (Theorem A, [CMNN20b]). Let R be an \mathbb{E}_∞ -ring spectrum and let $n \geq 0$. If $L_{T(n)}R = 0$, then $L_{T(n+1)}K(R) = 0$.

Remark 1.2. Here, $L_{T(n)}$ denotes Bousfield localization at a height n telescope $T(n)$; the non-vanishing of this localization is a sense in which a spectrum is “supported at height n .” We will say an \mathbb{E}_∞ -ring R is *of height* $n \geq 0$ if $L_{T(n)}R \not\approx 0$ and $L_{T(n+1)}R \simeq 0$; in fact, in this situation, a theorem of Hahn [Hah16] implies that $L_{T(j)}R \simeq 0$ for all $j > n$.

Roughly speaking, Theorem 1.1 asserts that the algebraic K -theory of a height n \mathbb{E}_∞ -ring is of height *at most* $n + 1$. Following this result, the question remained whether height shifting actually occurs; that is, when is the algebraic K -theory of a height n ring is of height *exactly* $n + 1$? In this paper, we give a simple and non-computational explanation for this height shifting in a range of examples. We show:

Theorem A. Let k be a perfect field of characteristic p , let \mathbb{G}_0 be a 1-dimensional formal group over k of height $n \geq 1$, and let $E_n = E_{k, \mathbb{G}_0}$ denote the associated Lubin-Tate theory. Then $L_{T(n+1)}K(E_n)$ is nonzero.

Theorem B. Let $n \geq 0$ and let $K^{(n)}$ denote the n -fold iterate of algebraic K -theory. Then, for any field k of characteristic different from p , the spectrum $L_{T(n)}K^{(n)}(k)$ is nonzero.

We remark that the question of height shifting has recently been independently addressed by Hahn and Wilson [HW20] in the case of truncated Brown-Peterson spectra $BP\langle n \rangle$; their approach is more computational in nature, and further proves Lichtenbaum-Quillen-type statements in this case. The nonvanishing of $L_{T(n+1)}K(BP\langle n \rangle)$ is closely related to Theorem A, though we do not know if either result implies the other. Theorem A was previously known in the case $n = 1, p \geq 5$ by [Aus10, AR02, BM08], and the cases $p = 2, 3$ also follow from [HW20].

Theorem B addresses another case of interest in Rognes’s redshift conjectures, that of iterated K -theory (of fields), in a form stated by Barwick [Bar15]. There has been previous work on this case by [Vee18, AK17, LSW20, BCD10], and the theorem answers the K -theory variant of a question of Hesselholt-Madsen about the chromatic complexity of $\mathrm{TC}^{(n)}(\mathbb{F}_p)$ [HM97, §4].

The n -fold iterated K -theory of a field k is related to an $(n - 1)$ -fold categorification of the category of k -vector spaces (cf. [Rog14]) – for instance, when $n = 2$, the work of Baas, Dundas, Richter and Rognes expresses $K(K(k))$ as a classifying space for certain k -linear categories [BDRR11, Theorem 1.1]. Moreover, motivated by the work of Ausoni-Rognes [AR02], Baas-Dundas-Rognes describe the cohomology theory $K(ku)$ (which is $T(2)$ -locally equivalent to $K(K(\mathbb{C}))$) geometrically via virtual 2-vector bundles [BDR04]. Theorem B asserts that this relatively tangible and geometric categorification procedure produces cohomology theories of arbitrary height. In addition, we remark that the proof of Theorem B is at least in part “effective,” as it describes a ring map from $K^{(n)}(\mathbb{Q})$ to rings constructed out of arbitrary height Lubin-Tate theories; in particular, at height 2, one obtains \mathbb{E}_∞ -ring maps from $K(K(\mathbb{Q}))$ to certain elliptic cohomology theories.

Remark 1.3. The results in this paper hold equally well with $L_{T(n)}$ replaced by the Bousfield localization functors $L_{K(n)}$ at a Morava K -theory at height n . The distinction between $T(n)$ and $K(n)$ is invisible to our discussion because a ring spectrum is $T(n)$ -acyclic if and only if it is $K(n)$ -acyclic, essentially by the Devinatz-Hopkins-Smith nilpotence theorem (cf. [LMMT20, Lemma 2.3], [CSY18, §4.4]).

1.1. Outline of methods. Suppose E is a $T(n)$ -local \mathbb{E}_∞ -ring. Then one can consider the \mathbb{E}_∞ -ring E^{tC_p} , the Tate cohomology of C_p acting trivially on E . A theorem of Kuhn (cf. Section 2) implies that $L_{T(n)}E^{tC_p} \simeq 0$, so E^{tC_p} is an \mathbb{E}_∞ -ring of height at most $n - 1$ (a phenomenon dubbed “chromatic blueshift”). The foundation for our examples is the following result, which asserts that *chromatic redshift* occurs for rings of the form E^{tC_p} :

Theorem C. Let E be an \mathbb{E}_∞ -ring spectrum such that $L_{T(n)}E$ is nonzero. Then $L_{T(n)}K(E^{tC_p})$ is nonzero.

The proof uses the following elementary observation:

Remark 1.4. Note that the zero ring admits no nonzero modules. Thus, if $A \rightarrow B$ is a map of ring spectra and B is nonzero, then A is nonzero. In particular, in order to show a ring R is $T(n)$ -locally nonzero, it suffices to produce a ring map from R to another ring which is $T(n)$ -locally nonzero. We will apply this observation throughout this paper.

We now outline the proof of Theorem C, with the full proof to be given in Section 3:

- (1) First, we reduce to the case when E is connective.
- (2) The Dennis trace provides a map of rings $K(E^{tC_p}) \rightarrow \mathrm{THH}(E^{tC_p})^{hS^1}$.
- (3) Using the universal property of THH in \mathbb{E}_∞ -rings, we produce a map of \mathbb{E}_∞ -rings

$$\mathrm{THH}(E^{tC_p})^{hS^1} \rightarrow (E^{tC_p})^{hS^1/C_p}.$$

Here, the action of S^1/C_p on E^{tC_p} is the residual action, arising because the trivial action of C_p on E can be extended to a trivial action of S^1 .

- (4) Since E is connective by (1), the “Tate orbit lemma” of Nikolaus-Scholze [NS18] provides a p -complete equivalence $(E^{tC_p})^{hS^1/C_p} \simeq_p E^{tS^1}$. We will see that because $L_{T(n)}E \neq 0$ by assumption, it follows that $L_{T(n)}E^{tS^1} \neq 0$ (for instance, when E is complex oriented, E is a summand of E^{tS^1}). This step is where one can “see the redshift happen,” and it will be explained in detail in Section 2.
- (5) Combining the above steps and applying Remark 1.4, we conclude that $K(E^{tC_p})$ is $T(n)$ -locally nontrivial, since it admits a ring map to a $T(n)$ -locally nontrivial ring.

Applying Remark 1.4, Theorem C can be leveraged to deduce chromatic redshift for rings mapping to E^{tC_p} for appropriately chosen E .

For the case of Lubin-Tate theories (Theorem A), the idea is to attempt to construct a ring map from $E_n \rightarrow E_{n+1}^{tC_p}$ for a height $n + 1$ Lubin-Tate theory E_{n+1} . It turns out that this can be done “up to étale extension,” which then suffices by applying the descent results of Clausen-Mathew [CM19]; we thank Akhil Mathew for explaining this point to us. We give the details in Section 4.

We prove Theorem B in Section 5. For the case $k = \mathbb{Q}$ (and in fact, $k = \mathbb{Q}(\zeta_p)$), one roughly uses the n -fold iterate of the proof of Theorem C. The proof then uses an argument of Akhil Mathew to extend to the general case: we are grateful to him for explaining this to us and allowing us to reproduce his arguments, which appear in Section 5.3.

1.2. Acknowledgments. The author would like to thank Andrew Blumberg, Robert Burklund, Shachar Carmeli, Jeremy Hahn, Jacob Lurie, Akhil Mathew, and Arpon Raksit for numerous helpful suggestions and conversations related to this material. Special thanks are due to Akhil Mathew for generously contributing the ideas in Lemma 4.5 and Section 5.3 and allowing us to use his arguments in this paper, and to Jeremy Hahn and Arpon Raksit for asking about the case of iterated K -theory. The author was supported in part by NSF grant DMS-2002029.

2. RECOLLECTIONS ON THE TATE ORBIT LEMMA

Roughly, our approach to demonstrating chromatic redshift is by showing that any blueshift (i.e., lowering of chromatic height) arising from Tate cohomology is reversed by algebraic K -theory. The purpose of this (entirely expository) section is to explain the mechanism for this “un-blueshift.” The key idea is the Tate orbit lemma of Nikolaus-Scholze [NS18], which we review in Section 2.2.

2.1. Tate cohomology and blueshift. Let X be a spectrum with the action of a finite group G . Then one can consider X^{tG} , the Greenlees-May Tate cohomology of G with coefficients in X [GM95b], which is defined via the cofiber sequence:

$$X_{hG} \xrightarrow{\text{tr}} X^{hG} \rightarrow X^{tG}.$$

Here, the map tr is the additive transfer, which can be thought of as taking an orbit \bar{x} to its sum $\sum_g gx$. For us, the group G will most often be C_p , the cyclic group of p elements, and unless otherwise stated, the operation $(-)^{tG}$ will be taken with respect to the trivial action. We remark also that $(-)^{tG}$ is lax symmetric monoidal [NS18, Theorem I.3.1], and thus takes \mathbb{E}_∞ -rings to \mathbb{E}_∞ -rings.

From work of Greenlees, Hovey, Sadofsky, and Kuhn [GS96, HS96, Kuh04], it became understood that Tate cohomology exhibited a “blueshift” phenomenon: it tends to shift chromatic height down. More precisely, Kuhn proved the following theorem:

Theorem 2.1 (Kuhn [Kuh04]). Let $n \geq 0$ and let X be a $T(n)$ -local spectrum with an action of C_p . Then $L_{T(n)}X^{tC_p} \simeq 0$.

Remark 2.2. Suppose that E is a complex oriented ring spectrum such that $[p](x)$, the p -series of its associated formal group, is not a zero divisor in $E_*[[x]]$. Then, one can compute the homotopy groups of E^{tC_p} as:

$$\pi_* E^{tC_p} \cong E_*((x))/[p](x), \quad |x| = -2.$$

This description of E^{tC_p} can be used to understand Kuhn’s theorem more concretely in some cases, as the following example illustrates:

Example 2.3. If E is the p -complete complex K -theory spectrum KU_p , then while KU_p is $T(1)$ -local, we have

$$\pi_* KU_p^{tC_p} \simeq \frac{\mathbb{Z}_p[\beta^{\pm 1}]((x))}{\beta^{-1}((1 + \beta x)^p - 1)} \simeq \mathbb{Q}_p(\zeta_p)((x)),$$

so $KU_p^{tC_p}$ is rational (the key point being that, using the relation, inverting x also inverts p). The last identification comes from noting that $1 + \beta x$ becomes a primitive p th root of unity ζ_p .

2.2. The Tate orbit lemma and reversing blueshift. Let X be a spectrum. We saw in the previous section that we may form X^{tC_p} , where X is given the trivial C_p action. However, the trivial action of C_p extends to a trivial action of the circle S^1 : it follows that we can regard X as having trivial S^1 -action and regard X^{tC_p} as the Tate cohomology with respect to the subgroup $C_p \subset S^1$. As such, X^{tC_p} acquires a residual action of the quotient S^1/C_p (which is isomorphic to S^1).

This residual action of S^1/C_p is critical to our approach: we will see that in the case when X is a connective ring spectrum, while X^{tC_p} can be of lower height than X , the homotopy fixed points $(X^{tC_p})^{hS^1/C_p}$ are always of the same height as X (cf. Remark 2.7). We first illustrate this with an example:

Example 2.4. Consider the case $X = \mathbb{Z}$; then

$$\pi_* \mathbb{Z}^{tC_p} = \mathbb{F}_p((x)), \quad |x| = -2.$$

It will be instructive to think of the ring \mathbb{Z}^{tC_p} as having height -1 because $v_0 := p$ vanishes. Note that there is a canonical S^1/C_p -equivariant map $\mathbb{Z}^{hC_p} \rightarrow \mathbb{Z}^{tC_p}$, which induces a map

$$(\mathbb{Z}^{hC_p})^{hC_{p^2}/C_p} \rightarrow (\mathbb{Z}^{tC_p})^{hC_{p^2}/C_p}. \quad (1)$$

But the source can be identified with $\mathbb{Z}^{hC_{p^2}}$, which has homotopy groups $\mathbb{Z}[[x]]/p^2x$, $|x| = -2$. Comparing the homotopy fixed point spectral sequences in (1), one deduces that the spectral sequence in the target adds a class y (coming from $H^2(C_{p^2}/C_p; \mathbb{F}_p)$) which satisfies the equation $y = px$ (via an additive extension in the spectral sequence). It follows that

$$\pi_* (\mathbb{Z}^{tC_p})^{hC_{p^2}/C_p} \simeq \mathbb{Z}/p^2((x)), \quad |x| = -2.$$

Thus, while $\mathbb{Z} \mapsto \mathbb{Z}^{tC_p}$ sends $v_0 = p$ to zero, taking homotopy fixed points by C_{p^2}/C_p starts to add back the powers of p . In fact, one can do a similar computation for the subgroups $C_{p^k}/C_p \subset S^1/C_p$ and S^1/C_p itself:

$$\pi_* (\mathbb{Z}^{tC_p})^{hC_{p^k}/C_p} = \mathbb{Z}/p^k((x)), \quad |x| = -2$$

$$\pi_* (\mathbb{Z}^{tC_p})^{hS^1/C_p} = \mathbb{Z}_p((x)), \quad |x| = -2.$$

Hence, taking fixed points for the action of S^1/C_p passes from characteristic p back to characteristic 0, which we can view as height shifting from height -1 to height 0.

In fact, the phenomenon observed in the above example is a special case of the following result of Nikolaus-Scholze:

Proposition 2.5 (Tate orbit lemma, [NS18], Lemma II.4.2). Let X be a bounded below spectrum with S^1 -action. Then there is a natural map $X^{tS^1} \rightarrow (X^{tC_p})^{hS^1/C_p}$ which exhibits the target as the p -completion of the source.

We refer the reader to [NS18, Lemma II.4.2] for the proof. The idea is that the statement commutes with limits of Postnikov towers, so one reduces to the case when X is discrete, in which case the statement can be understood directly in the spirit of Example 2.4.

Our interest in Proposition 2.5 stems from the fact that unlike $(-)^{tC_p}$, the functor $(-)^{tS^1}$ tends to *not* lower chromatic height (as illustrated in Example 2.4):

Proposition 2.6. Suppose E is a homotopy commutative ring spectrum such that $L_{T(n)}E \not\cong 0$. Then $L_{T(n)}E^{tS^1} \not\cong 0$.

Proof. Since there is a ring map $E \rightarrow E \otimes K(n)$, it suffices to show that $L_{T(n)}(E \otimes K(n))^{tS^1}$ is nonzero. But by Remark 1.3, we have $L_{T(n)}E \simeq 0$ iff $L_{K(n)}E \simeq 0$ iff $L_{K(n)}(E \otimes K(n)) \simeq 0$ iff $L_{T(n)}(E \otimes K(n)) \simeq 0$. Thus, we may assume without loss of generality that E is complex oriented by replacing E with $E \otimes K(n)$.

In the special case that E is complex oriented, we have

$$\pi_* E^{hS^1} \cong E^*(BS^1) \cong E_*[[x]], \quad |x| = -2$$

and $E^{tS^1} \simeq E^{hS^1}[x^{-1}]$; it follows that E^{tS^1} has E as an E -module summand, which completes the proof. \square

Remark 2.7. Proposition 2.5 and Proposition 2.6 are ultimately “where the redshift happens” in our arguments. To illustrate this, let E be a $K(n)$ -local ring spectrum and let $e \rightarrow E$ denote its connective cover. Then, by Kuhn’s theorem (Theorem 2.1) and Lemma 3.2, we have

$$L_{T(n)}e^{tC_p} \simeq L_{T(n)}E^{tC_p} \simeq 0,$$

so e^{tC_p} is of height at most $n - 1$. On the other hand, by Proposition 2.5 and Proposition 2.6, $(e^{tC_p})^{hS^1/C_p}$ is equivalent to the p -completion of e^{tS^1} , and therefore has height n . Thus, the operation $(-)^{hS^1/C_p}$ shifts the height of e^{tC_p} up by one.

Remark 2.8. In fact, passing to the connective cover (to satisfy the bounded below hypothesis in Proposition 2.5) is essential. We saw in Example 2.3 that $KU_p^{tC_p}$ is rational, and since rational spectra are closed under limits, it follows that $(KU_p^{tC_p})^{hS^1/C_p}$ is also rational (and in particular, not equivalent to $KU_p^{tS^1}$). On the other hand, if one uses *connective* K -theory ku_p , then $\pi_0 ku_p^{tC_p} \cong \mathbb{Z}_p[\zeta_p]$, which is not rational.

3. PROOF OF THEOREM C

Here, we prove Theorem C following the outline in the introduction:

Theorem C. Let E be an \mathbb{E}_∞ -ring spectrum such that $L_{T(n)}E$ is nonzero. Then $L_{T(n)}K(E^{tC_p})$ is nonzero.

Proof. We begin by reducing to the connective case. The main ingredient in the reduction is the following landmark result of Land-Mathew-Meier-Tamme [LMMT20] and Clausen-Mathew-Noel-Naumann [CMNN20b]:

Theorem 3.1 ([LMMT20], Theorem A). Let $n \geq 1$, and let $A \rightarrow B$ be a map of \mathbb{E}_1 -ring spectra which is a $T(n-1) \oplus T(n)$ -equivalence. Then $K(A) \rightarrow K(B)$ is a $T(n)$ -equivalence.

We also note the following lemma, which also appears in [CMNN20b, Lemma 4.7]; we reproduce their proof for the reader’s convenience:

Lemma 3.2. Let X be a coconnective spectrum. Then X^{tC_p} is $T(n)$ -acyclic for all $n \geq 0$.

Proof. Because $(-)^{tC_p}$ commutes with filtered colimits on coconnective spectra, one can reduce to the case when X is concentrated in a single degree. In this case, X^{tC_p} is a p -torsion \mathbb{Z} -module, and thus $T(n)$ -acyclic for any $n \geq 0$. \square

Combining this lemma with Theorem 3.1, we obtain:

Corollary 3.2.1. Let A be an \mathbb{E}_1 -ring spectrum and $a \rightarrow A$ be its connective cover. Then for any $n \geq 1$, the natural map $K(a^{tC_p}) \rightarrow K(A^{tC_p})$ is a $T(n)$ -equivalence.

Hence, we are free to replace E by its connective cover $e = \tau_{\geq 0}E$. To finish the proof, we will use THH and the Dennis trace map, so we first recall the following fact about THH:

Recollection 3.3. Let A be an \mathbb{E}_∞ -ring spectrum and let B be an \mathbb{E}_∞ -ring spectrum with an action of S^1 . Then $\mathrm{THH}(A)$ is given by the colimit over S^1 of the constant diagram at A in \mathbb{E}_∞ -rings. Hence, restriction along $A \rightarrow \mathrm{THH}(A)$ in the diagram

$$\begin{array}{ccc} A & & \\ \downarrow & \searrow & \\ \mathrm{THH}(A) & \dashrightarrow & B \end{array}$$

induces a bijection between homotopy classes of S^1 -equivariant \mathbb{E}_∞ -ring maps $\mathrm{THH}(A) \rightarrow B$ and homotopy classes of (non-equivariant) \mathbb{E}_∞ -ring maps $A \rightarrow B$.

We apply this in the case when $A = B = e^{tC_p}$, and where B is given a *nontrivial* S^1 -action as follows: we saw in the previous section that e^{tC_p} acquires an (interesting) residual S^1/C_p action, and we give $B = e^{tC_p}$ an S^1 -action by choosing an isomorphism $S^1 \cong S^1/C_p$. Then, by the above universal property of THH, the identity map $e^{tC_p} \rightarrow e^{tC_p}$ extends canonically to an S^1 -equivariant \mathbb{E}_∞ map $\mathrm{THH}(e^{tC_p}) \rightarrow e^{tC_p}$ (for this nontrivial S^1 action on the target). Taking fixed points, this determines an \mathbb{E}_∞ -ring map $\mathrm{THH}(e^{tC_p})^{hS^1} \rightarrow (e^{tC_p})^{hS^1/C_p}$. Composing this with the (circle invariant) Dennis trace yields a map of \mathbb{E}_∞ -rings

$$K(e^{tC_p}) \rightarrow \mathrm{THH}(e^{tC_p})^{hS^1} \rightarrow (e^{tC_p})^{hS^1/C_p}.$$

Since e is connective, the Tate orbit lemma (Proposition 2.5) identifies the target ring with the p -completion of e^{tS^1} , which has nontrivial $T(n)$ -localization by Proposition 2.6. Hence, $K(e^{tC_p})$ is $T(n)$ -locally nontrivial because it admits a ring map to a $T(n)$ -locally nontrivial ring. \square

Remark 3.4. Asaf Horev has pointed out that A is an S^1 -equivariant module over $\mathrm{THH}(A)$ for any *framed* \mathbb{E}_2 -algebra A , not just for \mathbb{E}_∞ -rings. Given this, one can wonder if Theorem C holds in some greater generality.

4. LUBIN-TATE THEORIES

Let E_n denote the Lubin-Tate theory associated to a formal group \mathbb{G}_0 of height n over a perfect field k of characteristic p . The goal of this section will be to show that $K(E_n)$ is an \mathbb{E}_∞ -ring spectrum of height $n + 1$; that is, that $L_{T(n+1)}K(E_n)$ is nonzero.

One way to prove this would be to exhibit an \mathbb{E}_∞ -ring A such that $L_{T(n+1)}K(A) \not\cong 0$ together with an \mathbb{E}_∞ -ring map $E_n \rightarrow A$. In fact, by work of Goerss-Hopkins-Miller [GH04] and Lurie [Lur18], Lubin-Tate theories are characterized by a universal property which makes them amenable to mapping out of. In order to state this universal property, we first recall a definition:

Recollection 4.1. Let R be a complex orientable ring spectrum. Then for each $n \geq 1$, there is a canonical n th Landweber ideal $\mathcal{I}_n \subset \pi_*(R)$, generated by (any choice of) p, v_1, \dots, v_{n-1} . In the following, we will be working only with 2-periodic complex orientable rings, and we will denote the corresponding ideal of $\pi_0(R)$ by $I_n \subset \pi_0(R)$.

Theorem 4.2 (Goerss-Hopkins-Miller [GH04], Lurie [Lur18] Theorem 5.0.2). Let E_n denote the Lubin-Tate theory associated to a height n formal group \mathbb{G}_0 over a perfect field k of characteristic p , let R be a 2-periodic complex orientable $K(n)$ -local \mathbb{E}_∞ -ring spectrum, and let $\mathbb{G}_R = \mathrm{Spf} R^0(\mathbb{C}P^\infty)$ denote the canonical Quillen formal group over $\pi_0(R)$.

Then there is a canonical homotopy equivalence between:

- (1) The space $\mathrm{Hom}_{\mathrm{CAlg}}(E_n, R)$ of \mathbb{E}_∞ -ring maps from E_n to R .
- (2) The set of pairs (f, α) , where $f : k \rightarrow \pi_0(R)/I_n$ is a ring homomorphism, and $\alpha : f^*\mathbb{G}_0 \cong (\mathbb{G}_R)_{\pi_0(R)/I_n}$ is an isomorphism of formal groups over $\pi_0(R)/I_n$.

Given a $K(n)$ -local \mathbb{E}_∞ -ring A , this theorem provides an algebraic method for producing \mathbb{E}_∞ -maps from E_n to A . The remaining question is how to choose the \mathbb{E}_∞ -ring A . By Theorem C, an example of an \mathbb{E}_∞ -ring A which exhibits redshift is $E_{n+1}^{tC_p}$, where E_{n+1} is a Lubin-Tate theory of corresponding to a height $n+1$ formal group over k . Unfortunately, while the formal group of $E_{n+1}^{tC_p}$ does have height n at certain residue fields, these fields are generally finite extensions of Laurent series rings over k ; in general, the corresponding formal groups are complicated, and not isomorphic to formal groups defined over k (cf. [AMS98] for a more detailed discussion).

However, one does not quite need to produce such a map; roughly speaking, we will see that the étale descent results of Clausen-Mathew-Naumann-Noel [CMNN20a] imply that one only needs to produce this map after a sequence of étale extensions; this allows one to reduce to the case of a separably closed residue field, where all formal groups of a given height *are* isomorphic. The key ingredient powering this reduction is the following theorem, which asserts that the functor $L_{T(n)}K(-)$ satisfies étale descent:

Theorem 4.3 (Theorem A.4 [CMNN20a]). Let R be an \mathbb{E}_∞ -ring spectrum and $n \geq 0$. Then the construction $Y \mapsto L_{T(n)}K(\mathrm{Perf}(Y))$ defines an étale sheaf on R .

The author would like to thank Akhil Mathew for explaining how to apply this theorem, and especially for explaining Lemma 4.5 (of course, any errors in the argument produced here are solely the responsibility of the author).

4.1. Proof of redshift for E_n . We now give the details of the argument.

Construction 4.4. Let R be an \mathbb{E}_∞ -ring spectrum and $\mathfrak{p} \subset \pi_0(R)$ be a prime ideal. Then one can produce a ring $\pi_0(R)_{\mathfrak{p}}^{\mathrm{sh}}$, known as the *strict henselization* of $\pi_0(R)$ at \mathfrak{p} , with the following properties (see, for instance, [Sta21, Tag 0BSK]):

- (1) $\pi_0(R)_{\mathfrak{p}}^{\mathrm{sh}}$ is strictly henselian; in particular, it is a local ring with separably closed residue field.
- (2) $\pi_0(R)_{\mathfrak{p}}^{\mathrm{sh}}$ is a filtered colimit of étale $\pi_0(R)$ -algebras.
- (3) $\mathfrak{p} \pi_0(R)_{\mathfrak{p}}^{\mathrm{sh}}$ is the maximal ideal of $\pi_0(R)_{\mathfrak{p}}^{\mathrm{sh}}$.

By [Lur16, Theorem 7.5.0.6], this construction can be lifted (essentially uniquely) to produce an \mathbb{E}_∞ -ring $R_{\mathfrak{p}}^{\mathrm{sh}}$ which is a filtered colimit of étale extensions of R and such that π_0 of the map $R \rightarrow R_{\mathfrak{p}}^{\mathrm{sh}}$ is identified with $\pi_0(R) \rightarrow \pi_0(R)_{\mathfrak{p}}^{\mathrm{sh}}$.

Lemma 4.5. Let R be an \mathbb{E}_∞ -ring spectrum, and suppose that $L_{T(n)}K(R) \not\cong 0$ for some integer $n \geq 0$. Then there exists a prime ideal $\mathfrak{p} \subset \pi_0(R)$ such that $L_{T(n)}K(R_{\mathfrak{p}}^{\mathrm{sh}}) \not\cong 0$.

Proof. Suppose the contrary, that $L_{T(n)}K(R_{\mathfrak{p}}^{\mathrm{sh}})$ vanishes for every prime $\mathfrak{p} \subset \pi_0(R)$. Since the strict henselizations of R are the local rings in the étale topology, we may write $R_{\mathfrak{p}}^{\mathrm{sh}}$ as a filtered colimit

$$R_{\mathfrak{p}}^{\mathrm{sh}} = \mathrm{colim}_{i \in I} R_{\mathfrak{p}}^{(i)},$$

where each $R_{\mathfrak{p}}^{(i)}$ is an étale neighborhood of \mathfrak{p} . Since $K(-)$ commutes with filtered colimits [BGT13, Theorem 1.1], we conclude that the $L_{T(n)} \operatorname{colim}_{i \in I} K(R_{\mathfrak{p}}^{(i)}) \simeq 0$, or equivalently that

$$\operatorname{colim}_{i \in I} K(R_{\mathfrak{p}}^{(i)}) \otimes T(n) \simeq 0.$$

But we can choose $T(n)$ to be an \mathbb{E}_1 -ring spectrum, so this is a filtered colimit of rings; thus, it vanishes if and only if it vanishes at some finite stage. It follows that $L_{T(n)} K(R_{\mathfrak{p}}^{(i)}) \simeq 0$ for some $i \in I$. Thus, we have shown that every $\mathfrak{p} \in \operatorname{Spec} R$ admits an étale neighborhood on which $L_{T(n)} K(-)$ vanishes. Since this defines an étale sheaf by Theorem 4.3, we conclude that $L_{T(n)} K(R)$ must have been zero. \square

We are now ready to prove that $K(E_n)$ has height $n + 1$.

Proof of Theorem A. Choose any formal group \mathbb{G}'_0 over k of height $n + 1$, and let $E_{n+1} = E_{\mathbb{G}'_0, k}$ denote the corresponding Lubin-Tate theory. By Theorem C, the ring $R = E_{n+1}^{tC_p}$ has the property that $L_{T(n+1)} K(R) \not\simeq 0$. Therefore, applying Lemma 4.5, we may choose a prime ideal $\mathfrak{p} \in \pi_0(R)$ such that $L_{T(n+1)} K(R_{\mathfrak{p}}^{\operatorname{sh}}) \not\simeq 0$. Since $L_{T(n+1)} R = 0$ (for instance, by Theorem 2.1), it follows from Theorem 3.1 that $L_{T(n+1)} K(L_{T(n)} R_{\mathfrak{p}}^{\operatorname{sh}}) \simeq L_{T(n+1)} K(R_{\mathfrak{p}}^{\operatorname{sh}})$ is also nonzero. In fact, since $K(n)$ and $T(n)$ localization coincide for BP -modules ([Hov95, Corollary 1.10]), we also have $L_{T(n+1)} K(L_{K(n)} R_{\mathfrak{p}}^{\operatorname{sh}}) \not\simeq 0$. Thus, in order to see that $K(E_n)$ has height $n + 1$, we will exhibit a map of \mathbb{E}_{∞} -rings $E_n \rightarrow L_{K(n)} R_{\mathfrak{p}}^{\operatorname{sh}}$.

Let $\Gamma = \operatorname{Spf}(E_{n+1}^{tC_p})^0(\mathbb{C}P^{\infty})$ denote the canonical formal group on $\pi_0(R_{\mathfrak{p}}^{\operatorname{sh}}) = \pi_0(R)_{\mathfrak{p}}^{\operatorname{sh}}$. By Theorem 4.2, in order to produce an \mathbb{E}_{∞} -ring map $E_n = E_{k, \mathbb{G}_0} \rightarrow L_{K(n)} R_{\mathfrak{p}}^{\operatorname{sh}}$, it suffices to exhibit a ring homomorphism $k \rightarrow \pi_0(L_{K(n)} R_{\mathfrak{p}}^{\operatorname{sh}})/I_n$ and an isomorphism between Γ and \mathbb{G}_0 over $\pi_0(L_{K(n)} R_{\mathfrak{p}}^{\operatorname{sh}})/I_n$. In fact, since the ideal I_n is already defined in $\pi_0(R_{\mathfrak{p}}^{\operatorname{sh}})$, it suffices to produce $f : k \rightarrow \pi_0(R_{\mathfrak{p}}^{\operatorname{sh}})/I_n$ and an isomorphism $f^* \mathbb{G}_0 \cong \Gamma_{\pi_0(R_{\mathfrak{p}}^{\operatorname{sh}})/I_n}$ of formal groups.

Since $\pi_0(R) = \pi_0(E_{n+1}^{tC_p})$ is naturally an algebra over the Witt vectors $W(k)$ and $p \in I_n$, there is a canonical inclusion $k \subset \pi_0(R)/I_n$; composing this with the map $\pi_0(R)/I_n \rightarrow \pi_0(R_{\mathfrak{p}}^{\operatorname{sh}})/I_n$, we define the ring homomorphism $f : k \rightarrow \pi_0(R_{\mathfrak{p}}^{\operatorname{sh}})/I_n$. It remains to show the requisite isomorphism of formal groups.

Lemma 4.6. Let $\kappa(\mathfrak{p}) = \pi_0(R_{\mathfrak{p}}^{\operatorname{sh}})/\mathfrak{p}$ denote the residue field at \mathfrak{p} and let $\Gamma_{\kappa(\mathfrak{p})}$ denote the reduction of the formal group Γ modulo \mathfrak{p} . Then $\Gamma_{\kappa(\mathfrak{p})}$ has height n .

Proof of Lemma. $\Gamma_{\kappa(\mathfrak{p})}$ has height at most n because the ring $L_{K(n+1)} R = L_{K(n+1)} E_{n+1}^{tC_p} = 0$ (for instance by Theorem 2.1). If $\Gamma_{\kappa(\mathfrak{p})}$ had height at most $n - 1$, then some v_i , $0 \leq i \leq n - 1$, would be invertible in $\kappa(\mathfrak{p})$. Since $\pi_0(R)_{\mathfrak{p}}^{\operatorname{sh}}$ is a local ring, this means that this v_i is invertible in $\pi_0(R_{\mathfrak{p}}^{\operatorname{sh}})$, which implies that $R_{\mathfrak{p}}^{\operatorname{sh}}$ would be $T(n) \vee T(n + 1)$ -acyclic. By Theorem 3.1, this would imply $L_{T(n+1)} K(R_{\mathfrak{p}}^{\operatorname{sh}}) \simeq 0$ which contradicts our choice of \mathfrak{p} . \square

It follows that the ideal $I_n \subset \pi_0(R_{\mathfrak{p}}^{\operatorname{sh}})$ is contained in \mathfrak{p} . Consequently, the ring $\pi_0(R_{\mathfrak{p}}^{\operatorname{sh}})/I_n$ is strictly henselian (with maximal ideal \mathfrak{p}), and $\Gamma_{\pi_0(R_{\mathfrak{p}}^{\operatorname{sh}})/I_n}$ is a formal group of height exactly n over it. Recall that any two formal group laws of height n over a strictly henselian ring are isomorphic (this is essentially a result of Lazard; see, for instance, [Lur10, Lecture 13, Theorem 11]). In particular, $\Gamma_{\pi_0(R_{\mathfrak{p}}^{\operatorname{sh}})/I_n}$ and $f^* \mathbb{G}_0$ are isomorphic, as desired. \square

5. ITERATED K -THEORY

Notation. For a functor G , we will use the superscript $G^{(n)}$ to denote the n th iterate of G .

In this section, we prove Theorem B, which we restate here for the reader's convenience:

Theorem B. Let $n \geq 0$ and let $K^{(n)}$ denote the n -fold iterate of algebraic K -theory. Then, for any field k of characteristic different from p , the spectrum $L_{T(n)}K^{(n)}(k)$ is nonzero.

For the purposes of exposition, we will start by treating the case $k = \mathbb{Q}$ in Section 5.1 and prove the general case in the following sections. The idea for extending to the general case and the arguments in Section 5.3 are due to Akhil Mathew: we are grateful to him for allowing us to include his work in this paper; however, any errors are solely the responsibility of the author.

 5.1. The case $k = \mathbb{Q}$.

Notation. Let $t_+ : \mathrm{Sp} \rightarrow \mathrm{Sp}$ be the functor given by the formula

$$t_+ X = (\tau_{\geq 0} X)^{tC_p}.$$

As usual, the operation $(-)^{tC_p}$ is taken with respect to the trivial action of C_p .

Let E_n be a height n Lubin-Tate theory and set $R := t_+^{(n)} E_n$. The nonvanishing of $L_{T(n)}K^{(n)}(\mathbb{Q})$ follows from the following two propositions about R :

Proposition 5.1. There exists a canonical \mathbb{E}_∞ -ring map $L_{T(n)}K^{(n)}(\mathbb{Q}) \rightarrow L_{T(n)}K^{(n)}(R)$.

Proposition 5.2. The iterated algebraic K -theory spectrum $K^{(n)}(R)$ is not $T(n)$ -acyclic.

The proof of Proposition 5.1 relies on the theorem of [LMMT20] (Theorem 3.1), which implies that one can replace R by its localization at $T(0) \vee T(1) \vee \cdots \vee T(n)$, which we denote by $L_n^{p,f} R$. Essentially by the blueshift phenomenon for Tate cohomology, this localization $L_n^{p,f} R$ is rational, and thus receives a ring map from \mathbb{Q} . The proof of Proposition 5.2 uses the iterated Dennis trace map and an iterated variant of Proposition 2.6. We now give the full proofs:

Proof of Proposition 5.1. First, observe that by the theorem of Land-Mathew-Meier-Tamme (Theorem 3.1), the map

$$L_{T(n)}K^{(n)}(R) \rightarrow L_{T(n)}K(L_{T(n) \vee T(n-1)}K^{(n-1)}(R))$$

is an equivalence. Applying Theorem 3.1 in this manner repeatedly, we conclude that the natural map

$$L_{T(n)}K^{(n)}(R) \rightarrow L_{T(n)}K^{(n)}(L_n^{p,f} R)$$

is an equivalence. Thus, it suffices to produce a map $\mathbb{Q} \rightarrow L_n^{p,f} R$. This follows immediately from the following lemma, which implies that $L_n^{p,f} R \simeq L_{T(0)}R$, and is therefore rational (as everything in sight is p -local):

Lemma 5.3. The ring R is $T(1) \vee T(2) \vee \cdots \vee T(n)$ -acyclic.

Proof. This lemma can be approached computationally, but we will give a more hands-off proof using genuine equivariant homotopy theory; the reader is referred to [LMSM86, MNN17] for additional background. Let $G = (C_p)^{\times n}$, and regard E_n as a genuine G -equivariant spectrum such that it is Borel-equivariant (i.e., the natural map $E_n^H \rightarrow E_n^{hH}$ is an equivalence for all $H \subset G$), and the underlying G -action is trivial. Let e_n denote the equivariant connective cover

of the genuine equivariant spectrum E_n (in the usual Mackey t -structure), which has the feature that the natural map $e_n \rightarrow E_n$ induces connective covers

$$e_n^H \simeq \tau_{\geq 0}(E_n^H) \rightarrow E_n^H$$

on genuine fixed points for all $H \subset G$.

Remark 5.4. For a genuine G -spectrum X , recall that one can consider its geometric fixed point spectrum $\Phi^G X$, which admits the following description (for any finite abelian group G): it fits into a cofiber sequence

$$Y \rightarrow X^G \rightarrow \Phi^G X$$

where the spectrum Y is given by the formula

$$Y \simeq \operatorname{colim}_{H \in \operatorname{Sub}(G)^-} (X^H)_{hG/H}.$$

Here, $\operatorname{Sub}(G)^-$ denotes the poset of proper subgroups of G ordered by inclusion, and the maps in the diagram are the genuine additive transfer maps; accordingly, the map $Y \rightarrow X^G$ is induced by the genuine additive transfer.

We first note the following variant of Lemma 3.2:

Lemma 5.5. The natural map $L_{T(j)}\Phi^G e_n \rightarrow L_{T(j)}\Phi^G E_n$ is an equivalence for $j \geq 1$.

Proof. By Remark 5.4, it suffices to check that the natural map

$$L_{T(j)}(e_n^H)_{hG/H} \rightarrow L_{T(j)}(E_n^H)_{hG/H}$$

is an equivalence for all $H \subset G$. But the cofiber of the map $e_n^H \rightarrow E_n^H$ is the coconnective spectrum $\tau_{< 0}(E_n^{hH})$. Since $L_{T(j)}$ and homotopy orbits commute with colimits, the conclusion follows from the fact that $T(j)$ -localization annihilates coconnective spectra. \square

On the other hand, we claim that there is a map of \mathbb{E}_∞ -rings $\Phi^G e_n \rightarrow R = t_+^{(n)} E_n$. This is because for each $1 \leq k \leq n$, there is a natural map

$$\Phi^{C_p^{\times k}} e_n \simeq \Phi^{C_p} \Phi^{C_p^{\times(k-1)}} e_n \rightarrow (\Phi^{C_p^{\times(k-1)}} e_n)^{tC_p}.$$

Since $\Phi^{C_p^{\times k}} e_n$ is connective (for instance, by the above description), the map factors through the connective cover $\tau_{\geq 0}(\Phi^{C_p^{\times(k-1)}} e_n)^{tC_p}$. Composing these for $1 \leq k \leq n$, we obtain the desired map.

Combining this with Lemma 5.5, we obtain an \mathbb{E}_∞ -ring map

$$L_{T(j)}\Phi^G E_n \rightarrow L_{T(j)}t_+^{(n)} E_n = L_{T(j)}R.$$

But by [BHN⁺19, Theorem 3.5], which computes the blueshift for geometric fixed points of finite abelian groups on Lubin-Tate theories, the spectrum $\Phi^G E_n$ is rational (this also goes back to [HKR00], cf. Section 5.2). In particular, for any $1 \leq j \leq n$, we have $L_{T(j)}\Phi^G E_n \simeq 0$, which implies that $L_{T(j)}R \simeq 0$ as well. \square

\square

We now prove Proposition 5.2 (and with it, the $k = \mathbb{Q}$ case of Theorem B) – the proof is a relatively straightforward adaptation of the proof of Theorem C, so we will assume some familiarity with that proof.

Proof of Proposition 5.2. By iterating the Dennis trace map, one obtains a ring map

$$K^{(n)}(R) \rightarrow \mathrm{THH}^{(n)}(R)$$

which is $(S^1)^{\times n}$ -equivariant for the trivial action on the source and the natural action on the target. It therefore induces a map of rings

$$K^{(n)}(R) \rightarrow \mathrm{THH}^{(n)}(R)^{h(S^1)^{\times n}}.$$

Remark 5.6. We have seen that the spectrum $t_+X = (\tau_{\geq 0}X)^{tC_p}$ acquires a residual action of S^1/C_p . Accordingly, $t_+^{(k)}X$ acquires a canonical action of $(S^1/C_p)^{\times k}$; to distinguish the iterates, we will label them by writing

$$t_+^{(k)}X = (\tau_{\geq 0}(\cdots(\tau_{\geq 0}X)^{tC_p^{(1)}} \cdots))^{tC_p^{(k)}},$$

which admits an action of $S^1/C_p^{(1)} \times \cdots \times S^1/C_p^{(k)} \cong (S^1)^{\times k}$.

Using (the $(S^1)^{\times n}$ analogue of) Recollection 3.3 as in the proof of Theorem C, we see that the identity map of $R = t_+^{(n)}E_n$ determines an \mathbb{E}_∞ -ring map

$$\mathrm{THH}^{(n)}(R)^{h(S^1)^{\times n}} \rightarrow R^{h(S^1/C_p)^{\times n}}.$$

Thus, to show that $L_{T(n)}K^{(n)}(R) \not\cong 0$, it suffices (by Remark 1.4), to show that:

Lemma 5.7. The ring $R^{h(S^1/C_p)^{\times n}}$ has nontrivial $T(n)$ -localization.

Proof. In fact, we will show by induction that for any $k \geq 0$, the ring $(t_+^{(k)}E_n)^{h(S^1/C_p)^{\times k}}$ has nontrivial $T(n)$ -localization. The base case $k = 0$ is immediate. For the inductive step $k \geq 1$, referring to the notation in Remark 5.6, we write

$$t_+^{(k)}E_n = t_+(t_+^{(k-1)}E_n) = (\tau_{\geq 0}t_+^{(k-1)}E_n)^{tC_p^{(k)}}.$$

Then, applying the Tate orbit lemma (Proposition 2.5), we have

$$(t_+^{(k)}E_n)^{hS^1/C_p^{(k)}} = ((\tau_{\geq 0}t_+^{(k-1)}E_n)^{tC_p^{(k)}})^{hS^1/C_p^{(k)}} \simeq_p (\tau_{\geq 0}t_+^{(k-1)}E_n)^{tS^1},$$

where \simeq_p denotes equivalence after p -completion. Since $\tau_{\geq 0}t_+^{(k-1)}E_n$ is complex orientable (as it receives a ring map from the complex orientable ring $\tau_{\geq 0}E_n$), it is a retract of its own S^1 -Tate construction. It follows that

$$(t_+^{(k)}E_n)^{h(S^1/C_p)^{\times k}} \simeq ((t_+^{(k)}E_n)^{hS^1/C_p^{(k)}})^{h(S^1/C_p)^{\times k-1}}$$

has $(\tau_{\geq 0}t_+^{(k-1)}E_n)^{h(S^1/C_p)^{\times k-1}}$ as a retract. By the inductive hypothesis, this latter spectrum has nontrivial $T(n)$ -localization, which completes the proof. □

□

5.2. Adding a primitive p th root of unity. Here, we make an additional observation about the proof of Proposition 5.1 which allows us to extend the results of the previous section to $k = \mathbb{Q}(\zeta_p)$. Recall that E_n denotes a Lubin-Tate theory of height n and residue field k , equipped with a genuine $C_p^{\times n}$ equivariant structure via the Borel equivariant trivial action, and e_n denotes the equivariant connective cover of E_n . Our goal will be to prove the following:

Proposition 5.8. Suppose that the residue field k of E_n is algebraically closed. Then the ring $\pi_0(\Phi_p^{C_p^{\times n}} e_n) \otimes \mathbb{Q}$ contains a primitive p th root of unity.

We first recall some computational facts:

Remark 5.9. Fixing a complex orientation on E_n , we have an isomorphism (cf. [HKR00, §5.4])

$$\pi_* E_n^{hC_p^{\times n}} \cong (E_n)_* [[\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n]] / ([p](\tilde{x}_1), [p](\tilde{x}_2), \dots, [p](\tilde{x}_n)),$$

where $\tilde{x}_i \in E_n^2(BC_p^{\times n})$ is the first Chern class of the line bundle corresponding to the i th projection $C_p^{\times n} \rightarrow C_p$, and $[p]$ denotes the p -series of (the chosen coordinate on) the formal group $\mathrm{Spf} E_n^*(CP^\infty)$ over $\pi_*(E_n)$. Choosing a periodicity element $u \in \pi_2 E_n$ and setting $x_i = u\tilde{x}_i$, we obtain an isomorphism

$$\pi_0 E_n^{hC_p^{\times n}} \simeq \pi_0(E_n)[[x_1, x_2, \dots, x_n]] / ([p]_0(x_1), [p]_0(x_2), \dots, [p]_0(x_n)), \quad (2)$$

where $[p]_0$ denotes the p -series shifted into degree zero, so that $[p]_0(x_i) = u[p](\tilde{x}_i)$. In other words, $[p]_0$ is the p -series of the formal group $\mathrm{Spf} E_n^0(CP^\infty)$ over $\pi_0(E_n)$, which we will denote by \mathbb{G}_{E_n} . Note that, essentially by definition of equivariant connective cover, there is an isomorphism of rings $\pi_0 E_n^{C_p^{\times n}} \cong \pi_0 E_n^{hC_p^{\times n}}$.

By [GM95a, Proposition 3.20], the ring $\pi_0 \Phi^{C_p^{\times n}} E_n$ is obtained from the ring $\pi_0 E_n^{hC_p^{\times n}}$ by inverting the Chern classes of the line bundles corresponding to the nontrivial characters of $C_p^{\times n}$. These are the elements

$$[a_1]_0(x_1) +_{\mathbb{G}_{E_n}} [a_2]_0(x_2) +_{\mathbb{G}_{E_n}} \cdots +_{\mathbb{G}_{E_n}} [a_n]_0(x_n)$$

for $0 \leq a_i \leq p-1$.

We then have the following variant of Proposition 5.8:

Proposition 5.10. Suppose that the residue field k of E_n is algebraically closed. Then the ring $\pi_0 \Phi^{C_p^{\times n}} E_n$ contains a primitive p th root of unity.

Proof. By [HKR00, Proposition 6.2] and the description of $\pi_0 \Phi^{C_p^{\times n}} E_n$ above, there is an isomorphism of finite flat group schemes

$$(\underline{C}_p^{\times n})_{\pi_0 \Phi^{C_p^{\times n}} E_n} \simeq (\mathbb{G}_{E_n}[p])_{\pi_0 \Phi^{C_p^{\times n}} E_n} \quad (3)$$

between the constant group scheme at $C_p^{\times n}$ and the p -torsion scheme of the formal group \mathbb{G}_{E_n} (in fact, $\pi_0 \Phi^{C_p^{\times n}} E_n$ is the universal $\pi_0 E_n$ -algebra with such an isomorphism).

Remark 5.11. Let R be a commutative ring and let H be a finite flat commutative group scheme over R . Then Hopkins and Lurie show that, for $d \geq 0$, one can associate to H its *group scheme of alternating maps* $\mathrm{Alt}_H^{(d)}$ [HL13, Construction 3.2.11], which can be thought of as classifying alternating maps $H^{\times d} \rightarrow \mathbb{G}_m$ (cf. [HL13, §3] for details). In the special case that $H = \Gamma[p^t]$ is the p^t -torsion in a p -divisible group Γ of height n and dimension 1, the schemes $\mathrm{Alt}_{\Gamma[p^t]}^{(d)}$ are also finite flat commutative group schemes, and their Cartier duals (denoted \mathbf{D}) form a directed system

$$\mathbf{D}(\mathrm{Alt}_{\Gamma[1]}^{(d)}) \rightarrow \mathbf{D}(\mathrm{Alt}_{\Gamma[p]}^{(d)}) \rightarrow \mathbf{D}(\mathrm{Alt}_{\Gamma[p^2]}^{(d)}) \rightarrow \cdots$$

which defines a p -divisible group of height $\binom{n}{d}$ and dimension $\binom{n-1}{d-1}$ [HL13, Corollary 3.5.5]. We will denote this p -divisible group by $\wedge^d \Gamma$ and refer to it as the *d th exterior power* of Γ .

We record two additional features of this construction:

- (1) For a p -divisible group Γ of height n and dimension 1, the “top” exterior power $\wedge^n \Gamma$ has height 1 and dimension 1. In the special case where $\Gamma = \mathbb{G}_{E_n}[p^\infty]$ is the p -divisible group corresponding to the Lubin-Tate theory E_n of height n with algebraically closed residue field k , there is an isomorphism of group schemes $\wedge^n \mathbb{G}_{E_n}[p] \cong \mu_p$ over $\pi_0 E_n$ [HL13, Proposition 5.3.30].
- (2) For the constant group scheme $\underline{C}_p^{\times n}$ over a commutative ring R , there are isomorphisms

$$\wedge^n \underline{C}_p^{\times n} := \mathbf{D}(\mathrm{Alt}_{\underline{C}_p^{\times n}}^{(n)}) \cong \mathbf{D}(\mu_p) \cong \underline{C}_p$$

by [HL13, Remark 3.2.21].

Thus, applying \wedge^n to the isomorphism (3) of group schemes and using the above remark, we find that the finite flat group schemes $\wedge^n \underline{C}_p^{\times n} \cong \underline{C}_p$ and $\wedge^n \mathbb{G}_{E_n}[p] \cong \mu_p$ are isomorphic over $\pi_0 \Phi^{C_p^{\times n}} E_n$. It follows that the ring $\pi_0 \Phi^{C_p^{\times n}} E_n$ contains a primitive p th root of unity. \square

We are now ready to prove Proposition 5.8.

Proof of Proposition 5.8. By Proposition 5.10, $\pi_0 \Phi^{C_p^{\times n}} E_n$ contains a primitive p th root of unity. Hence, it will be enough to exhibit a ring homomorphism fitting into the square:¹

$$\begin{array}{ccc} \pi_0 E_n^{hC_p^{\times n}} & \xrightarrow{\cong} & \pi_0 e_n^{C_p^{\times n}} \\ \downarrow & & \downarrow \\ \pi_0 \Phi^{C_p^{\times n}} E_n & \dashrightarrow & \pi_0(\Phi^{C_p^{\times n}} e_n) \otimes \mathbb{Q}. \end{array} \quad (4)$$

Here, the vertical maps are the canonical maps from genuine fixed points to geometric fixed points (composed with rationalization on the right). Recalling the descriptions of these groups given in Remark 5.9, the left vertical map is given by inverting elements $\sum_{\mathbb{G}_{E_n}} [a_i]_0(x_i)$, where $0 \leq a_i \leq p-1$. Thus, to produce the desired factorization, it suffices to see that these elements are invertible in $\pi_0(\Phi^{C_p^{\times n}} e_n) \otimes \mathbb{Q}$. In fact, by symmetry, it suffices to see that the image of x_1 in $\pi_0(\Phi^{C_p^{\times n}} e_n) \otimes \mathbb{Q}$ is invertible.

Note that by definition of geometric fixed points, the map $e_n^{C_p^{\times n}} \rightarrow \Phi^{C_p^{\times n}} e_n$ is nullhomotopic when precomposed with additive transfers from any proper subgroup of $C_p^{\times n}$: in particular, if we let $\mathrm{tr}^{(1)}$ denote the composite

$$\mathrm{tr}^{(1)} : e \xrightarrow{\mathrm{tr}} e^{C_p} \xrightarrow{\mathrm{proj}_1^*} e^{C_p^{\times n}}$$

(where proj_1^* denotes the map induced by projection $C_p^{\times n} \rightarrow C_p$ onto the first component), then the image of $\mathrm{tr}^{(1)}$ in $\pi_0 e_n^{C_p^{\times n}}$ is in the kernel of the map $\pi_0 e_n^{C_p^{\times n}} \rightarrow \pi_0 \Phi^{C_p^{\times n}} e_n$. By [HKR00, Remark 6.15], we have the formula

$$\mathrm{tr}^{(1)}(1) = [p](\tilde{x}_1)/\tilde{x}_1 = [p]_0(x_1)/x_1,$$

where $1 \in \pi_0 e_n$ denotes the unit. Thus, $[p]_0(x_1)/x_1 = \mathrm{tr}^{(1)}(1)$ vanishes in $\pi_0(\Phi^{C_p^{\times n}} e_n) \otimes \mathbb{Q}$. But the series $[p]_0(x_1)/x_1$ has the form $p + x_1 g(x_1)$ for some series $g \in (e_n)_0[[x_1]]$ and p is invertible in $\pi_0(\Phi^{C_p^{\times n}} e_n) \otimes \mathbb{Q}$, from which it follows that x_1 is also invertible, as desired. \square

¹In fact, the map we construct turns out to be an equivalence, but we will not need this.

5.3. The case of an arbitrary field k . Following arguments of Akhil Mathew, we now extend the results of Section 5.1 to prove Theorem B for all fields k . The proof rests on the work of Clausen-Mathew-Naumann-Noel [CMNN20b] on Galois descent for telescopically localized K -theory and work of Barthel-Carmeli-Schlack-Yanovski [BCSY] on $T(n)$ -local homotopy theory. We refer the reader to [Rog08] for background on the notion of $(T(n)$ -local) G -Galois extensions of ring spectra in the sense of Rognes.

Theorem 5.12. Let G be a finite p -group and let $R \rightarrow R'$ be a map of \mathbb{E}_∞ -algebras over \mathbb{Q} which is a G -Galois extension in the sense of [Rog08]. Then for $n \geq 0$, the natural map

$$L_{T(n)}K^{(n)}(R) \rightarrow L_{T(n)}K^{(n)}(R')$$

is a $T(n)$ -local G -Galois extension. In particular, it induces an equivalence $L_{T(n)}K^{(n)}(R) \rightarrow L_{T(n)}K^{(n)}(R')^{hG}$.

Proof. We proceed by induction. The case $n = 0$ is true by assumption, so suppose $n \geq 1$ and assume that the map

$$L_{T(n-1)}K^{(n-1)}(R) \rightarrow L_{T(n-1)}K^{(n-1)}(R')$$

is a $T(n-1)$ -local G -Galois extension. Then, by [CMNN20b, Corollary 4.13], the induced map

$$L_{T(n)}K(L_{T(n-1)}K^{(n-1)}(R)) \rightarrow (L_{T(n)}K(L_{T(n-1)}K^{(n-1)}(R')))^{hG} \quad (5)$$

is an equivalence. On the other hand, by [CMNN20b] (Theorem 1.1), the $T(n)$ -localization of $K^{(n-1)}(A)$ vanishes for any rational \mathbb{E}_∞ -algebra A , and hence by [LMMT20] (Theorem 3.1), there is a natural equivalence

$$L_{T(n)}K^{(n)}(A) \simeq L_{T(n)}K(L_{T(n) \vee T(n-1)}K^{(n-1)}(A)) = L_{T(n)}K(L_{T(n-1)}K^{(n-1)}(A)).$$

Combining this with (5), it follows that the natural map

$$L_{T(n)}K^{(n)}(R) \rightarrow (L_{T(n)}K^{(n)}(R'))^{hG}$$

is an equivalence. The inductive step then follows from the following theorem of [BCSY]. \square

Theorem 5.13 (Barthel-Carmeli-Schlack-Yanovski). Let G be a finite p -group and let E be a $T(n)$ -local \mathbb{E}_∞ -ring spectrum with an action of G . Then the natural map $E^{hG} \rightarrow E$ is a $T(n)$ -local G -Galois extension.

This theorem will be a consequence of the results in the forthcoming work [BCSY], but we indicate some of the proof for the reader's convenience:

Proof outline. It suffices to check that the natural map

$$L_{T(n)}(E \otimes_{E^{hG}} E) \rightarrow \prod_G E$$

(induced by the G -action and multiplication) is an equivalence. But the map is an equivalence after applying the functor $(-)^{hG}$: this is because by Kuhn's blueshift theorem (Theorem 2.1), $T(n)$ -local orbits and fixed points coincide, so

$$\begin{aligned} (L_{T(n)}(E \otimes_{E^{hG}} E))^{hG} &\simeq L_{T(n)}(E \otimes_{E^{hG}} E)_{hG} \simeq L_{T(n)}(L_{T(n)}E_{hG} \otimes_{E^{hG}} E) \\ &\simeq L_{T(n)}(E^{hG} \otimes_{E^{hG}} E) \simeq E. \end{aligned}$$

The conclusion then follows from the fact that $(-)^{hG}$ is conservative on $T(n)$ -local spectra – this statement, whose $K(n)$ -local analogue is [HL13, Corollary 5.4.4], will appear in [BCSY]. \square

We are now ready to complete the proof of Theorem B.

Proof of Theorem B. We make a series of reductions:

- (1) By mapping to an algebraic closure, it suffices to demonstrate the statement for an algebraically closed field k (of characteristic different from p).
- (2) In this case, by work of Suslin [Sus83, Sus84], one has that $L_{T(1)}K(k) \simeq KU_p^\wedge$. On the other hand, by [BCM20, Theorem 1.4], there is an equivalence of \mathbb{E}_∞ -rings

$$L_{T(1)}K(\mathbb{Q}(\zeta_{p^\infty})) \simeq L_{T(1)}(KU_p^\wedge \otimes K(\mathbb{Q})).$$

Since this receives a ring map from KU_p^\wedge , we are reduced to the case $k = \mathbb{Q}(\zeta_{p^\infty})$.

- (3) In fact, because K -theory commutes with filtered colimits, we have that

$$T(n) \otimes K^{(n)}(\mathbb{Q}(\zeta_{p^\infty})) \simeq \operatorname{colim}_j (T(n) \otimes K^{(n)}(\mathbb{Q}(\zeta_{p^j}))).$$

We may choose $T(n)$ to be an \mathbb{E}_1 -ring spectrum, making this colimit a filtered colimit of rings. As a filtered colimit of rings vanishes if and only if it does at a finite stage, we are reduced to showing that $L_{T(n)}K^{(n)}(\mathbb{Q}(\zeta_{p^j})) \neq 0$ (as j ranges over the natural numbers).

- (4) The map of rings $\mathbb{Q}(\zeta_p) \rightarrow \mathbb{Q}(\zeta_{p^j})$ is a Galois extension with Galois group $C_{p^{j-1}}$, and therefore by Theorem 5.12, we have an equivalence

$$L_{T(n)}K^{(n)}(\mathbb{Q}(\zeta_p)) \simeq L_{T(n)}K^{(n)}(\mathbb{Q}(\zeta_{p^j}))^{hC_{p^{j-1}}}.$$

Therefore, we are reduced to the case of the field $k = \mathbb{Q}(\zeta_p)$.

The case of $\mathbb{Q}(\zeta_p)$ follows by using the results of Section 5.2 to strengthen the proof of Proposition 5.1. Namely, the ring R (from the proof of Proposition 5.1) receives a ring map from $\Phi^{C_p^{\times n}}e_n$, and therefore by Proposition 5.8, $\pi_0(R \otimes \mathbb{Q})$ contains a primitive p th root of unity. Since the map $\mathbb{Q} \rightarrow \mathbb{Q}(\zeta_p)$ is étale, the map $\mathbb{Q} \rightarrow L_{T(0)}R \simeq R \otimes \mathbb{Q}$ in the proof of Proposition 5.1 extends to a ring map $\mathbb{Q}(\zeta_p) \rightarrow L_{T(0)}R$ and one obtains an \mathbb{E}_∞ -ring map

$$L_{T(n)}K^{(n)}(\mathbb{Q}(\zeta_p)) \rightarrow L_{T(n)}K^{(n)}(R).$$

The rest of the argument in Section 5.1 shows that the target of this map is nonzero, so $L_{T(n)}K^{(n)}(\mathbb{Q}(\zeta_p)) \neq 0$, and we are done. \square

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