

**$RO(C_2)$ -GRADED COHOMOLOGY OF C_2 -EQUIVARIANT
EILENBERG-MAC LANE SPACES**

UĞUR YİĞİT

ABSTRACT. In this paper, we calculate $RO(C_2)$ -graded cohomology of C_2 -equivariant Eilenberg-Mac Lane spaces $K(\underline{\mathbb{Z}/2}, n + \sigma)$ for $n \geq 0$. These can be used to give the relation between equivariant lambda algebra and equivariant Adams resolution and equivariant unstable Adams spectral sequence, which are defined in author's dissertation.

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1. INTRODUCTION

An ordinary cohomology theory $H_G^*(- : \underline{M})$ on G -spaces with Mackey functor \underline{M} coefficients and graded by real orthogonal representations is defined by Lewis, May and McClure [8]. In this paper, we compute the $RO(C_2)$ -graded cohomology of the C_2 -equivariant Eilenberg-Mac Lane spaces with the constant Mackey functor $\underline{M} = \underline{\mathbb{Z}/2}$ coefficients, which are crucial to give the relation between the equivariant lambda algebra and the equivariant unstable Adams resolution and equivariant unstable Adams spectral sequence, which is given by Mahowald [12] in the classical case. Throughout this paper, $H^*(-)$ denotes the ordinary $RO(C_2)$ -graded cohomology of a C_2 -space with the constant Mackey functor coefficients $\underline{\mathbb{Z}/2}$.

To compute the $RO(C_2)$ -graded cohomology of the C_2 -equivariant Eilenberg-Mac Lane spaces with the constant Mackey functor $\underline{M} = \underline{\mathbb{Z}/2}$ coefficients, we use Borel theorem 17 for the path-space fibration

$$\Omega K(\underline{\mathbb{Z}/2}, V) \longrightarrow P(K(\underline{\mathbb{Z}/2}, V)) \longrightarrow K(\underline{\mathbb{Z}/2}, V).$$

for $V = \sigma + n$, where $n \geq 0$.

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If we knew $H^*(K(\underline{Z}/2, n\sigma))$ for $n \geq 2$, one could use the Eilenberg-Moore spectral sequence [4, Chapter 5], the Borel theorem, and the $RO(G)$ -graded Serre spectral sequence of Kronholm [7, Theorem 1.2.] for the path-space fibration

$$\Omega K(\underline{Z}/2, V) \longrightarrow P(K(\underline{Z}/2, V)) \longrightarrow K(\underline{Z}/2, V).$$

This paper is organized as follows. In section 2, we provide the basic equivariant topology tools, and C_2 -equivariant cohomology $M_2^{C_2}$ of a point, and equivariant connectivity of G -spaces. In section 3, we describe equivariant Steenrod squares, C_2 -equivariant Steenrod algebra \mathcal{A}_{C_2} and axioms of it. In section 4, we give the definition of the equivariant Eilenberg-Mac Lane spaces with some properties, and the fixed point sets of the equivariant Eilenberg-Mac Lane spaces that is very useful to compute the cohomology of them. In section 5, we compute the $RO(C_2)$ -graded C_2 -equivariant cohomology of some C_2 -equivariant Eilenberg-Mac Lane spaces K_V for real orthogonal representations $V = \sigma + n$, $n \geq 0$. Also, we give some conjectures and future directions for the other cases.

Notation. We provide here notation used in this paper for convenience.

- $V = r\sigma + s$, a real orthogonal representation of C_2 , which is a sum of r -copy of the sign representation σ and s -copy of the trivial representation 1.
- $\rho = \sigma + 1$, the regular representation of C_2 .
- $RO(C_2)$, the real representation ring of C_2 .
- S^V , the equivariant sphere which is the one-point compactification of V .
- $\pi_V^{C_2}(X)$, the V -th C_2 -equivariant homotopy group of a topological C_2 -space X .
- $\pi_{r\sigma+s}^S$, the C_2 -equivariant stable homotopy groups of spheres.
- $\Sigma^\sigma(X)$, the σ -th suspension of X .
- $\Omega^\sigma(X)$, all continuous functions from S^σ to X .
- $H_G^*(-; \underline{M})$, $RO(G)$ -graded ordinary equivariant cohomology with Mackey functor \underline{M} coefficients.
- $M_2^{C_2}$, $RO(C_2)$ -graded C_2 -equivariant cohomology of a point.
- \mathcal{A}_{C_2} , C_2 -equivariant Steenrod algebra.
- $K(\underline{M}, V)$ or shortly K_V , the V th equivariant Eilenberg-Mac Lane space with a Mackey functor \underline{M} .
- $\underline{\pi}_V^G(X)$, C_2 -equivariant homotopy of a G -space X as a Mackey functor.
- $Sq_{C_2}^k$, C_2 -equivariant Steenrod squaring operations for $k \geq 0$.
- RP_{tw}^∞ , the space of lines in the complete universe $\mathcal{U} = (\mathbb{R}^\rho)^\infty$, which is equivalent to $K(\underline{Z}/2, \sigma)$.

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2. PRELIMINARIES

In this section we give the main tools that are used in the rest of the article. Let X be a G -space, where $G = C_2$ is a cyclic group with generator γ such that $\gamma^2 = e$. The group C_2 has two irreducible real representations, namely the trivial

representation denoted by 1 (or \mathbb{R}) and the sign representation denoted by σ (or \mathbb{R}_-). The regular representation is isomorphic to $\rho_{C_2} = 1 + \sigma$ (it is denoted by ρ if there is no confusion). Thus the representation ring $RO(C_2)$ is free abelian of rank 2, so every representation V can be expressed as $V = r\sigma + s$.

Definition 1. A G -**universe** is a countably infinite-dimensional G -representation which contains the trivial G -representation and which contains infinitely many copies of each of its finite-dimensional subrepresentations. Also, a **complete G -universe** is just a G -universe that contains infinitely many copies of every irreducible G -representation.

Definition 2. A G -spectrum E on a G -universe \mathcal{U} is a collection E_V of based G -spaces together with basepoint-preserving G -maps

$$\sigma_{V,W} : \Sigma^{W-V} E_V \longrightarrow E_W$$

whenever $V \subset W \subset \mathcal{U}$, where $W - V$ denotes the orthogonal complement of V in W . It is required that $\sigma_{V,V}$ is identity, and the commutativity of the diagram

$$\begin{array}{ccc} \Sigma^{W-V} \Sigma^{V-U} E_U & \xrightarrow{\Sigma_{W-V} \sigma_{U,V}} & \Sigma^{W-V} E_V \\ \searrow \sigma_{U,W} & & \swarrow \sigma_{V,W} \\ & E_W & \end{array}$$

for $U \subset V \subset W \subset \mathcal{U}$.

Definition 3. If the adjoint structure maps

$$\tilde{\sigma}_{V,W} : E_V \longrightarrow \Omega^{W-V} E_W$$

are weak homotopy equivalences for $V \subset W \subset \mathcal{U}$, then a G -spectrum is called $G - \Omega$ -spectrum.

A G -spectrum indexed on a complete(trivial) G -universe is called genuine(naive).

For an actual representation V of G and a G -space X , the V -th homotopy group of X is the Mackey functor $\underline{\pi}_V(X)$ determined by

$$\underline{\pi}_V(X)(G/H) = [S^V, X]^H$$

for every $H < G$.

For a virtual representation $V \in RO(G)$ and a G -spectrum E , the V -th homotopy group of E is the Mackey functor $\underline{\pi}_V(E)$ determined by

$$\underline{\pi}_V(E)(G/H) = \operatorname{colim}_n \pi_0(\Omega^{V+W_n} E_{W_n})^H$$

where $\{W_n | n \in \mathbb{N}\}$ is an increasing sequence of representations

$$\cdots \subset W_n \subset W_{n+1} \subset \cdots$$

such that any finite dimensional representation V of G admits an equivariant embedding in some W_n .

Lewis, May and McClure [8] defined an ordinary cohomology theory $H_G^*(-; \underline{M})$ on G -spaces with Mackey functor \underline{M} coefficients and the graded by real orthogonal representations.

Throughout this paper, the Mackey functor will typically be the constant Mackey functor $\underline{M} = \underline{Z/2}$, which can be given the following diagram in Lewis notation.

$$(2.1) \quad \begin{array}{c} \mathbb{Z}/2 \\ \text{Id} \left(\begin{array}{c} \downarrow \\ \uparrow \end{array} \right)_0 \\ \mathbb{Z}/2 \\ \text{Id} \end{array}$$

The ordinary equivariant cohomology $M_2^{C_2}$ of a point with this coefficient is given in the Figure 1 below. Every \bullet in the figure represents a copy of $Z/2$.

As you see in the Figure 1 below, there are two elements of interest. The inclusion map of the fixed point set (the north and south poles) $a : S^0 \rightarrow S^\sigma$ defines an element in $\pi_{-\sigma}^{C_2}(S^{-0})$, and we will use the same symbol for its mod 2 Hurewicz image. It is called an Euler class. One can show that

$$H_1^{C_2}(S^\sigma; Z/2) = H_{1-\sigma}^{C_2}(S^{-0}; Z/2) = Z/2$$

and we denote its generator by u . Dually, we have $a \in H_{C_2}^\sigma(S^{-0}; Z/2)$ and $u \in H_{C_2}^{\sigma-1}(S^{-0}; Z/2)$. These are the analog of elements ρ and τ in real motivic homotopy theory, respectively.

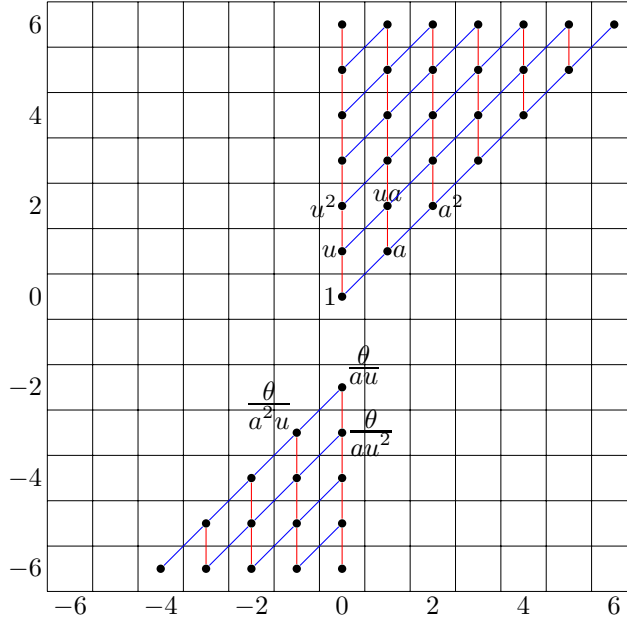


FIGURE 1. The equivariant cohomology $M_2^{C_2}$ of a point

The coordinate (x, y) represents degree $(x - y) + \sigma y$, which is convenient with the motivic bidegree. Red and blue lines represent multiplication by u and a , respectively.

Now, we will give the definition of equivariant connectivity of G -spaces.

Definition 4. [10]

- (i) A function ν^* from the set of conjugacy classes of subgroups of G to the integers is called a **dimension function**. The value of ν^* on the conjugacy class of $K \subset G$ is denoted by ν^K . Let ν^* and μ^* be two dimension functions. If $\nu^K \geq \mu^K$ for every subgroup K , then $\nu^* \geq \mu^*$. Associated to any G -representation V is the dimension function $|V^*|$ whose value at K is the real dimension of the K -fixed subspace V^K of V . The dimension function with constant integer value n is denoted n^* for any integer n .
- (ii) Let ν^* be a non-negative dimension function. If for each subgroup K of G , the fixed point space Y^K is ν^K -connected, then a G -space Y is called G - ν^* -**connected**. If a G -space Y is G - 0^* -connected, then it is called G -**connected**. Also, if it is G - 1^* -connected, it is called **simply G -connected**. A G -space Y is **homologically G - ν^* -connected** if, for every subgroup K of G and every integer m with $0 \leq m \leq \nu^K$, the homology group $H_m^K(Y)$ is zero.
- (iii) Let ν^* be a non-negative dimension function and let $f : Y \rightarrow Z$ be a G -map. If, for every subgroup K of G ,

$$(f^K)_* : \pi_m(Y^K) \rightarrow \pi_m(Z^K)$$

is an isomorphism for every integer m with $0 \leq m < \nu^K$ and an epimorphism for $m = \nu^K$, then f is called G - ν^* -**equivalence**. A G -pair (Y, B) is said to be G - ν^* -**connected** if the inclusion of B into Y is a G - ν^* -equivalence. The notions of **homology G - ν^* -equivalence** and of **homology G - ν^* -connectedness** for pairs are defined similarly, but with homotopy groups replaced by homology groups.

- (iv) Let V be a G -representation. For each subgroup K of G , let $V(K)$ be the orthogonal complement of V^K ; then $V(K)$ is a K -representation. If $\pi_{V(K)+m}^K(Y)$ is zero for each subgroup K of G and each integer m with $0 \leq m \leq |V^K|$, the G -space Y is called G - V -**connected**. Similarly, if $H_{V(K)+m}^G(Y)$ is zero for each subgroup K of G and each integer m with $0 \leq m \leq |V^K|$, then the G -space Y is called **homologically G - V -connected**.
- (v) Let V be a G -representation. A G - 0^* -equivalence $f : Y \rightarrow Z$ is said to be a G - V -**equivalence** if, for every subgroup K of G , the map

$$f_* : \pi_{V(K)+m}^K(Y) \rightarrow \pi_{V(K)+m}^K(Z)$$

is an isomorphism for every integer m with $0 \leq m < |V^K|$ and an epimorphism for $m = |V^K|$. A **homology G - V -equivalence** is defined similarly. A G -pair (Y, B) is called G - V -**connected** (respectively, **homologically G - V -connected**) if the inclusion of B into Y is a G - V -equivalence (respectively, homology G - V -equivalence).

3. C_2 -EQUIVARIANT STEENROD ALGEBRA

The analog of the mod 2 Steenrod algebra is defined by Voevodsky [19] in the motivic case, and Po Hu and Igor Kriz [5] in the equivariant case. The two descriptions are essentially the same.

One has squaring operations $Sq_{C_2}^k$ for $k \geq 0$, whose degrees

$$|Sq_{C_2}^k| = \begin{cases} i(1 + \sigma) & \text{for } k = 2i \\ i(1 + \sigma) + 1 & \text{for } k = 2i + 1. \end{cases}$$

$Sq_{C_2}^0 = 1$ as in the classical case. The C_2 -equivariant Steenrod algebra acts on the coefficient ring $M_{C_2}^2$ by

$$(3.1) \quad Sq_{C_2}^k(u) = \begin{cases} u & \text{for } k = 0 \\ a & \text{for } k = 1 \\ 0 & \text{else.} \end{cases}$$

$$(3.2) \quad Sq_{C_2}^{2m+\delta}(u^{2l+\epsilon}) = \binom{2l+\epsilon}{2m+\delta} u^{2l+\epsilon-m-\delta} a^{2m+\delta}$$

The difficulty in deriving the formula 3.2 is the C_2 -equivariant Cartan formula 3.7, 3.8. Since

$$|Sq_{C_2}^{2m+\delta}| = m(1 + \sigma) + \delta \quad \text{for } 0 \leq \delta \leq 1,$$

we have

$$(3.3) \quad \begin{cases} \Delta(Sq_{C_2}^{2m+1}) = \sum_{0 \leq i \leq 2m+1} Sq_{C_2}^i \otimes Sq_{C_2}^{2m+1-i} \\ \Delta(Sq_{C_2}^{2m}) = \sum_{0 \leq j \leq m} Sq_{C_2}^{2i} \otimes Sq_{C_2}^{2m-2j} + u \sum_{1 \leq j \leq m} Sq_{C_2}^{2j-1} \otimes Sq_{C_2}^{2m-2j+1}. \end{cases}$$

The terms divisible by u make things difficult. Here we are using cohomological degree, so $|u| = \sigma - 1$. Note that

$$|u^{-m} Sq_{C_2}^{2m+\delta}| = m(1 - \sigma) + m(1 + \sigma) + \delta = 2m + \delta$$

and define

$$\mathcal{S}q_{C_2}^{2m+\delta} := u^{-m} Sq_{C_2}^{2m+\delta}.$$

We will see that these operations satisfy the classical Cartan formula. We have

$$\begin{aligned} \Delta(\mathcal{S}q_{C_2}^{2m+1}) &= u^{-m} \Delta(Sq_{C_2}^{2m+1}) \\ &= u^{-m} \sum_{0 \leq i \leq 2m+1} Sq_{C_2}^i \otimes Sq_{C_2}^{2m+1-i} \\ &= \sum_{0 \leq i \leq 2m+1} u^{-[i/2]} Sq_{C_2}^i \otimes u^{-[(2m+1-i)/2]} Sq_{C_2}^{2m+1-i} \\ &= \sum_{0 \leq i \leq 2m+1} \mathcal{S}q_{C_2}^i \otimes \mathcal{S}q_{C_2}^{2m+1-i} \end{aligned}$$

since $[i/2] + [(2m+1-i)/2] = m$. And also,

$$\Delta(\mathcal{S}q_{C_2}^{2m}) = u^{-m} \Delta(Sq_{C_2}^{2m})$$

$$\begin{aligned}
&= u^{-m} \sum_{0 \leq j \leq m} Sq_{C_2}^{2j} \otimes Sq_{C_2}^{2m-2j} + u^{1-m} \sum_{1 \leq j \leq m} Sq_{C_2}^{2j-1} \otimes Sq_{C_2}^{2m-2j+1} \\
&= \sum_{0 \leq j \leq m} u^{-j} Sq_{C_2}^{2j} \otimes u^{j-m} Sq_{C_2}^{2m-2j} + \sum_{1 \leq j \leq m} u^{1-j} Sq_{C_2}^{2j-1} \otimes u^{j-m} Sq_{C_2}^{2m-2j+1} \\
&= \sum_{0 \leq j \leq m} Sq_{C_2}^{2j} \otimes Sq_{C_2}^{2m-2j} + \sum_{1 \leq j \leq m} Sq_{C_2}^{2j-1} \otimes Sq_{C_2}^{2m-2j+1} \\
&= \sum_{0 \leq i \leq 2m} Sq_{C_2}^i \otimes Sq_{C_2}^{2m-i}.
\end{aligned}$$

Now, if we use homological degree, then

$$|Sq^m| = -m, |a| = -\sigma, \text{ and } |u| = 1 - \sigma.$$

We know that

$$(3.4) \quad Sq_{C_2}^m(u) = \begin{cases} u & \text{for } m = 0 \\ a & \text{for } m = 1 \\ 0 & \text{else.} \end{cases}$$

Consider the total Steenrod operation

$$(3.5) \quad Sq_t = \sum_{i \geq 0} t^i Sq^i,$$

where t is a dummy variable. Although this sum is infinite, it yields a finite sum when applied to any monomial in a and u . The classical Cartan formula satisfied by operations Sq^i implies that it is a ring homomorphism, meaning that

$$Sq_t(xy) = Sq_t(x)Sq_t(y).$$

Then 3.4 implies that

$$\begin{aligned}
Sq_t(u) &= u + ta \\
Sq_t(u^l) &= (u + ta)^l \\
&= \sum_{0 \leq m \leq l} \binom{l}{m} t^m u^{l-m} a^m \\
&= \sum_{0 \leq m \leq l} t^m Sq^m(u^l).
\end{aligned}$$

Hence, $Sq^m(u^l)$ is the coefficient of t^m in the first sum above.

It follows that

$$\begin{aligned}
Sq^{2m+\delta}(u^{2l+\epsilon}) &= \binom{2l+\epsilon}{2m+\delta} u^{2l+\epsilon-2m-\delta} a^{2m+\delta} \\
Sq^{2m+\delta}(u^{2l+\epsilon}) &= u^m Sq^{2m+\delta}(u^{2l+\epsilon}) \\
&= \binom{2l+\epsilon}{2m+\delta} u^{2l+\epsilon-m-\delta} a^{2m+\delta}.
\end{aligned}$$

As a result, we have the following:

Lemma 5.

$$Sq_{C_2}^{2m+\delta}(u^{2l+\epsilon}) = \binom{2l+\epsilon}{2m+\delta} u^{2l+\epsilon-m-\delta} a^{2m+\delta}.$$

The natural action of the Steenrod algebra in homology is on the right, not on the left. Classically, the mod p cohomology of a space or a spectrum X is a left module over the Steenrod algebra \mathcal{A} , so there is a map

$$c_X : \mathcal{A} \otimes H^*X \rightarrow H^*X.$$

The Steenrod algebra has a multiplication

$$\phi^* : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$$

(the symbol ϕ^* and its dual ϕ_* are taken from Milnor's paper [14]) and the following diagram commutes

$$(3.6) \quad \begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} \otimes H^*X & \xrightarrow{\phi^* \otimes H^*X} & \mathcal{A} \otimes H^*X \\ \downarrow \mathcal{A} \otimes c_X & & \downarrow c_X \\ \mathcal{A} \otimes H^*X & \xrightarrow{c_X} & H^*X. \end{array}$$

Milnor defines a right action of \mathcal{A} on H_*X by the rule

$$\langle xa, y \rangle = \langle x, ay \rangle \in \mathbb{F}_p$$

for $x \in H_*X$, $a \in \mathcal{A}$ and $y \in H^*X$, where the brackets denotes the evaluation of the cohomology class on the right on the homology class on the left. Milnor denotes by λ^* the resulting map

$$H_*X \otimes \mathcal{A} \rightarrow H_*X.$$

The same thing happens in the C_2 -equivariant case. For example, we have

$$(u^2)Sq_{C_2}^3 = (u^2)Sq_{C_2}^1 Sq_{C_2}^2 = 0$$

because $(u^2)Sq_{C_2}^1 = 0$. And,

$$(u^2)\chi(Sq_{C_2}^3) = (u^2)Sq_{C_2}^2 Sq_{C_2}^1 = (ua^2)Sq_{C_2}^1 = a^3,$$

where $\chi(-)$ means the conjugate Steenrod operations. Hence, 3.2 should really read as

$$(u^{2l+\epsilon})Sq_{C_2}^{2m+\delta} = \binom{2l+\epsilon}{2m+\delta} u^{2l+\epsilon-m-\delta} a^{2m+\delta}.$$

For example,

$$Sq_{C_2}^l(u^{-1}) = \binom{-1}{l} a^l u^{-1-l} = \begin{cases} \binom{-1}{0} u^{-1} = u^{-1} & \text{for } l = 0 \\ \binom{-1}{1} a u^{-2} = a u^{-2} & \text{for } l = 1 \\ \binom{-1}{2} a^2 u^{-3} = a^2 u^{-3} & \text{for } l = 2 \\ 0 & \text{for } l \geq 3 \end{cases}$$

Action on the other elements is determined by the Cartan formula (iv) given below. We now give axioms for the squares $Sq_{C_2}^k$. For the motivic case, you can check Voevodsky paper [19]. But, the Adem relation is fixed by Joël Riou in [17].

- (i) $Sq_{C_2}^0 = 1$ and $Sq_{C_2}^1 = \beta_{C_2}$, Bockstein homomorphism.
- (ii) $\beta Sq_{C_2}^{2k} = Sq_{C_2}^{2k+1}$.

(iii) $\beta Sq_{C_2}^{2k+1} = 0.$

(iv) (Cartan formula)

(3.7)
$$Sq_{C_2}^{2k}(xy) = \sum_{r=0}^k Sq_{C_2}^{2r}(x)Sq_{C_2}^{2k-2r}(y) + u \sum_{s=0}^{k-1} Sq_{C_2}^{2s+1}(x)Sq_{C_2}^{2k-2s-1}(y)$$

(3.8)
$$Sq_{C_2}^{2k+1}(xy) = \sum_{r=0}^{2k+1} Sq_{C_2}^r(x)Sq_{C_2}^{2k+1-r}(y) + a \sum_{s=0}^{k-1} Sq_{C_2}^{2s+1}(x)Sq_{C_2}^{2k-2s-1}(y)$$

(v) (Adem relation) If $0 < i < 2j$, then when $i + j$ is even

$$Sq_{C_2}^i Sq_{C_2}^j = \sum_{k=0}^{[i/2]} \binom{b-1-k}{i-2k} u^\epsilon Sq_{C_2}^{i+j-k} Sq_{C_2}^k$$

where

$$\epsilon = \begin{cases} 1 & \text{for } k \text{ is odd and } i \text{ and } j \text{ are even} \\ 0 & \text{else} \end{cases}$$

when $i + j$ is odd

$$Sq_{C_2}^i Sq_{C_2}^j = \sum_{k=0}^{[i/2]} \binom{j-1-k}{i-2k} Sq_{C_2}^{i+j-k} Sq_{C_2}^k + a \sum_{k=\text{odd}} \epsilon Sq_{C_2}^{i+j-k} Sq_{C_2}^k$$

where

$$\epsilon = \begin{cases} \binom{j-1-k}{i-2k} & \text{for } i \text{ is odd} \\ \binom{j-1-k}{i-2k-1} & \text{for } j \text{ is odd} \end{cases}$$

(vi) If x has a degree $k\sigma + k$, then $Sq_{C_2}^{2k}(x) = x^2$.(vii) (instability) If x has a degree V , $V < k\sigma + k$ then $Sq_{C_2}^{2k}(x) = 0$, where $V < V'$ if and only if $V' = V + W$ for some actual representations W with positive degree.Note that setting $u = 1$ and $a = 0$ reduces the Cartan formula (iv) to the classical Cartan formula, and Adem relation (v) to the classical Adem relation.**Examples 6.** We have

$$Sq_{C_2}^1 Sq_{C_2}^n = \begin{cases} Sq_{C_2}^{n+1} & \text{for } n \text{ is even} \\ 0 & \text{for } n \text{ is odd} \end{cases}$$

$$Sq_{C_2}^2 Sq_{C_2}^n = \begin{cases} Sq_{C_2}^{n+2} + u Sq_{C_2}^{n+1} Sq_{C_2}^1 & \text{for } n \equiv 0 \pmod{4} \\ Sq_{C_2}^{n+1} Sq_{C_2}^1 & \text{for } n \equiv 1 \pmod{4} \\ u Sq_{C_2}^{n+1} Sq_{C_2}^1 & \text{for } n \equiv 2 \pmod{4} \\ Sq_{C_2}^{n+2} + Sq_{C_2}^{n+1} Sq_{C_2}^1 & \text{for } n \equiv 3 \pmod{4} \end{cases}$$

and

$$Sq_{C_2}^3 Sq_{C_2}^n = \begin{cases} Sq_{C_2}^{n+3} + a Sq_{C_2}^{n+1} Sq_{C_2}^1 & \text{for } n \equiv 0 \pmod{4} \\ Sq_{C_2}^{n+2} Sq_{C_2}^1 & \text{for } n \equiv 1 \pmod{4} \\ a Sq_{C_2}^{n+1} Sq_{C_2}^1 & \text{for } n \equiv 2 \pmod{4} \\ Sq_{C_2}^{n+2} Sq_{C_2}^1 & \text{for } n \equiv 3 \pmod{4} \end{cases}$$

Now, let $Sq_{C_2}^I$ denote $Sq_{C_2}^{i_1} Sq_{C_2}^{i_2} \cdots Sq_{C_2}^{i_n}$ for a sequence of integers $I = (i_1, \cdots, i_n)$. The sequence I is said to be **admissible** if $i_s \geq 2i_{s+1}$ for all $s \geq 1$, where $i_{s+1} = 0$.

The operations $Sq_{C_2}^I$ with admissible I are called admissible monomials. We also call $Sq_{C_2}^0$ admissible, where $Sq_{C_2}^0 = Sq_{C_2}^I$ for empty I .

Lemma 7. *The admissible monomials form a basis for the C_2 -equivariant Steenrod algebra \mathcal{A}_{C_2} as a $H^*(pt)$ -module.*

Proof. The proof follows from the Adem relations and the Cartan formula as in the classical case. \square

For the graded \mathcal{A}_{C_2} -module structure and Hopf algebra structure of equivariant Steenrod algebra, one can look [16]. We will now give unstable module structure of it.

Definition 8. An \mathcal{A}_{C_2} -module is unstable if it satisfies the preceding instability condition (vii).

We define the **excess** of $Sq_{C_2}^k$ to be the degree of $Sq_{C_2}^k$

$$e(Sq_{C_2}^k) = \begin{cases} i\rho & \text{for } k = 2i \\ i\rho + 1 & \text{for } k = 2i + 1. \end{cases}$$

So, $e(Sq_{C_2}^k) = |Sq_{C_2}^k|$. Then the **excess** of $Sq_{C_2}^I = Sq_{C_2}^{i_1} Sq_{C_2}^{i_2} \cdots Sq_{C_2}^{i_k}$ to be

$$e(Sq_{C_2}^I) = \sum_j e(Sq_{C_2}^{i_j}) - \rho e(Sq_{C_2}^{i_{j+1}})$$

where $\rho(r\sigma + s) = (r + s)\rho$.

Examples 9.

- The monomial with $e(Sq_{C_2}^I) = 0$ is $Sq_{C_2}^0$.
- The monomials with $e(Sq_{C_2}^I) = 1$ are $Sq_{C_2}^1, Sq_{C_2}^2 Sq_{C_2}^1, Sq_{C_2}^4 Sq_{C_2}^2 Sq_{C_2}^1, \cdots$
- There is no monomial with $e(Sq_{C_2}^I) = \sigma$.
- The monomials with $e(Sq_{C_2}^I) = 2$ are $Sq_{C_2}^3 Sq_{C_2}^1, Sq_{C_2}^6 Sq_{C_2}^3 Sq_{C_2}^1, Sq_{C_2}^{12} Sq_{C_2}^6 Sq_{C_2}^3 Sq_{C_2}^1, \cdots$
- The monomials with $e(Sq_{C_2}^I) = \rho$ are $Sq_{C_2}^2, Sq_{C_2}^4 Sq_{C_2}^2, Sq_{C_2}^8 Sq_{C_2}^4 Sq_{C_2}^2, \cdots$
- There is no monomial with $e(Sq_{C_2}^I) = 2\sigma$,
- The monomials with $e(Sq_{C_2}^I) = 3$ are $Sq_{C_2}^7 Sq_{C_2}^3 Sq_{C_2}^1, Sq_{C_2}^{11} Sq_{C_2}^5 Sq_{C_2}^2 Sq_{C_2}^1, \cdots$
- The monomials with $e(Sq_{C_2}^I) = 2 + \sigma$ are $Sq_{C_2}^3, Sq_{C_2}^4 Sq_{C_2}^1, Sq_{C_2}^5 Sq_{C_2}^2, Sq_{C_2}^6 Sq_{C_2}^3, Sq_{C_2}^6 Sq_{C_2}^2 Sq_{C_2}^1, Sq_{C_2}^8 Sq_{C_2}^4 Sq_{C_2}^1, \cdots$
- There is no monomial with $e(Sq_{C_2}^I) = 1 + 2\sigma$.

Remark 10. There is no monomial with $e(Sq_{C_2}^I) = r\sigma + s$ if $r > s$.

Let $\mathbf{t}_{j,k} = Sq_{C_2}^{j2^{k-1}} \cdots Sq_{C_2}^j$. Then the set of elements with total excess 1 is

$$\{\mathbf{t}_{1,k_1} | k_1 > 0\}.$$

The set of elements with total excess 2 is

$$\{\mathbf{t}_{1+2^{k_1}, k_2+1} \mathbf{t}_{1,k_1} | k_1, k_2 \geq 0\}.$$

The set of elements with total excess 3 is

$$\{\mathbf{t}_{1+2k_2+2k_1+k_2, k_3+1} \mathbf{t}_{1+2k_1, k_2} \mathbf{t}_{1, k_1} | k_1, k_2, k_3 \geq 0\}.$$

The C_2 -equivariant mod 2 dual Steenrod algebra (one can check [16], or [5] for details) is

$$\mathcal{A}^{C_2} = M_2^{C_2}[\tau_i, \xi_i]/(\tau_i^2 + a\tau_{i+1}\eta_R(u)\xi_{i+1})$$

such that

$$\eta_R(u) = u + a\tau_0$$

$$\eta_R(a) = a$$

$$|\xi_i| = (2^i - 1)\rho$$

$$|\tau_i| = 1 + |\xi_i|$$

$$\Delta(\xi_i) = \sum_{j=0}^i \xi_{i-j}^{2^j} \otimes \xi_j, \text{ where } \xi_0 = 1$$

$$\Delta(\tau_i) = \tau_i \otimes 1 + \sum_{j=0}^i \xi_{i-j}^{2^j} \otimes \tau_j.$$

4. EQUIVARIANT EILENBERG-MAC LANE SPACES

For each Mackey functor \underline{M} , there is an Eilenberg-Mac Lane G -spectrum $H\underline{M}$ which has the property as Mackey functors

$$\pi_n^G(H\underline{M}) = \begin{cases} \underline{M} & n = 0 \\ 0 & n \in \mathbb{Z}, n \neq 0 \end{cases}$$

One can check [13, Chapter XIII, page 162] for the proof of the existence.

Let \underline{M} be a Mackey functor, the V th space in the Ω -spectrum for $H\underline{M}$ is called an equivariant Eilenberg-Mac Lane space of type $K(\underline{M}, V)$, which is a classifying space for the functor $H_G^V(-; \underline{M})$. That is, given any real orthogonal representations V, W , there is a G -homotopy equivalence $K(\underline{M}; V) \simeq \Omega^W K(\underline{M}, V+W)$ satisfying various compatibility properties. Such spaces are constructed in [9], or one can look [3] for a construction with a different method. Here, I will give the definition of them for consistency.

Definition 11. [9] Let V be a real orthogonal representation with $|V^G| \geq 1$ and \underline{M} be a Mackey functor. An equivariant Eilenberg-Mac Lane space $K(\underline{M}, V)$ is a based, $(|V^*| - 1)$ -connected G -space with the G -homotopy type of a G -CW complex such that $\pi_V^G(K(\underline{M}, V)) = \underline{M}$, and for $\pi_{V+k}^G(K(\underline{M}, V)) = 0$ $k \neq 0$.

Remark 12. One can ask what $\pi_{V+n\sigma}^G(K(\underline{M}, V))$ is for $n > 0$. Our main interest is $K(\underline{Z}/2, V)$. Then,

$$\pi_{V+n\sigma}^{C_2}(K(\underline{Z}/2, V))(C_2/e) = \pi_{V+n\sigma}^e(K(\underline{Z}/2, V)) = 0$$

and

$$\begin{aligned} \pi_{V+n\sigma}^{C_2}(K(\underline{Z}/2, V))(C_2/C_2) &= \pi_{V+n\sigma}^{C_2}(K(\underline{Z}/2, V)) \\ &\cong \tilde{H}_{V+n\sigma}^{C_2}(S^V; \underline{Z}/2) \end{aligned}$$

$$\begin{aligned} &\cong \tilde{H}_{n\sigma}^{C_2}(S^{0,0}; \underline{Z/2}) \\ &\cong H_{n\sigma}^{C_2}(*; \underline{Z/2}) \end{aligned}$$

So, as a Mackey functor, the homotopy $\pi_{V+n\sigma}^G(K(\underline{M}, V))$ is one of the

$$\begin{array}{ccc} \begin{array}{c} Z/2 \\ \left(\uparrow \downarrow \right) \\ 0 \\ \left(\uparrow \downarrow \right) \\ Id \end{array} & \text{or} & \begin{array}{c} 0 \\ \left(\uparrow \downarrow \right) \\ 0 \\ \left(\uparrow \downarrow \right) \\ Id \end{array} \end{array}$$

depending on the dimension of the representation V and n .

As mentioned before, one can check [9] for existence and some properties of these spaces.

Another approach to construct equivariant Eilenberg-Mac Lane spaces is Dos Santos [3] approach. As we know in the classical case, the free abelian group on the n -sphere is a model for the Eilenberg-Mac Lane space $K(\mathbb{Z}, n)$, and the free \mathbb{F}_2 -vector space on the n -sphere is a model for the Eilenberg-Mac Lane space $K(\mathbb{F}_2, n)$. Dos Santos constructed a topological abelian group $M \otimes X$ in [3, Definition 2.1.], which is the equivariant generalization of previous sentence for a Mackey functor M , and proved an $RO(G)$ -graded version of equivariant Dold-Thom theorem proved by Lima-Filho for \mathbb{Z} -graded case in [11].

Let M be a $\mathbb{Z}[G]$ -module, \underline{M} be the Mackey functor associated to M : the value of \underline{M} on G/H is M^H and the value on the projection $G/K \rightarrow G/H$, for $K < H < G$, is the inclusion of $M^H \hookrightarrow M^K$. We define $M \otimes X$ as the $\mathbb{Z}[G]$ -module with a topology as follows([3, Definition 2.1.]): Let $(X, *)$ be a based G -set, $M \otimes X$ denote the $\mathbb{Z}[G]$ -module $\bigoplus_{x \in X - \{*\}} M$. The action of $g \in G$ is given by $(g.m)_x = g.m_{g^{-1}.x}$, where m_x denotes the x th coordinate of $m \in \bigoplus_{x \in X - \{*\}} M$. Given $(X, *)$ a based G -space, $M \otimes X$ can be equivalently defined as the quotient

$$M \otimes X = \coprod_{n \geq 0} M^n \times X^n / \sim,$$

where \sim is the equivalence relation generated by:

- (i) $(r, \phi^*x) \sim (\phi_*r, x)$, for each based map $\phi : \{0, \dots, n\} \rightarrow \{0, \dots, m\}$, $n, m \in \mathbb{N}$, where $\phi^*x = x \circ \phi$, and $(\phi_*r)_i = \sum_{k \in \phi^{-1}(i)} r_k$.
- (ii) $((r, r'), (x, *)) \sim (r, x)$, for each $r \in M^n$, $r' \in M$, $x \in X$.

We give the discrete topology to M and endow $M \otimes X$ with the quotient topology corresponding to the relation \sim .

We can define Eilenberg-Mac Lane spaces as $K_V = M \otimes S^V$. In our case,

$$K_{m+n\sigma} = \mathbb{Z}/2 \otimes S^{m+n\sigma}.$$

Theorem 13. [3] *Let X be a based G -CW-complex and let V be a finite dimensional G -representation, then $M \otimes X$ is an equivariant infinite loop space and there is a natural equivalence*

$$\pi_V^G(M \otimes X) \cong \tilde{H}_V^G(X; \underline{M})$$

As a corollary to this theorem we have that $M \otimes S^V$ is a $K(\underline{M}, V)$ space (as Definition 11). Thus we have a simple model for the equivariant Eilenberg-Mac Lane spectrum $H\underline{M}$.

Examples 14.

- (i) $K(\underline{\mathbb{Z}/2}, 1)$ is RP^∞ , with trivial action.
- (ii) Recall that $RP_{tw}^\infty = \mathbb{P}(\mathcal{U})$ is the space of lines in the complete universe (Definition 1)

$$\mathcal{U} = (\mathbb{R}^\rho)^\infty$$

[13]. The cohomology of RP_{tw}^∞ is calculated by Kronholm in [?]. The space RP_{tw}^∞ is equivalent to $K(\underline{\mathbb{Z}/2}, \sigma)$, since it is equivalent to $\mathbb{Z}/2 \otimes S^\sigma$.

Theorem 15. [6] $H^*(RP_{tw}^\infty) \cong H^*(pt)[c, d]/(c^2 = ac + ud)$, where $\deg(c) = \sigma$, and $\deg(d) = \rho$.

Now, we will give a structure of fixed points of equivariant Eilenberg-Mac Lane spaces, which is useful to calculate the cohomology of them.

Theorem 16. [2, Corollary 10]

- (i) $(K(\underline{\mathbb{Z}/2}, r\sigma + s))^e \simeq K(\mathbb{Z}/2, r + s)$.
- (ii) $(K(\underline{\mathbb{Z}/2}, r\sigma + s))^{C_2} \simeq K(\mathbb{Z}/2, s) \times \cdots \times K(\mathbb{Z}/2, r + s)$.

5. COHOMOLOGY OF EILENBERG-MAC LANE SPACES

In classical case the cohomology of Eilenberg-Mac Lane spaces K_n with $\mathbb{Z}/2$ -coefficients, which is given by Serre in [18] is a polynomial ring

$$H^*(K_n; \mathbb{Z}/2) = P(Sq^I(\iota_n) | e(I) < n)$$

where I are admissible sequences, ι_n is the fundamental class, and $e(Sq^I) = \sum_j (i_j - 2i_{j+1})$. We thought that we can give similar description for $RO(C_2)$ -graded C_2 -equivariant cohomology of C_2 -equivariant Eilenberg-Mac Lane spaces, but these are more complicated than we expect.

Let $\mathbf{s}_{V,l}$ is the operation that sends x to x^{2^l} for $x \in H^V$. It is possible to express $\mathbf{s}_{V,l}$ as a linear combination of Steenrod operations.

$$\mathbf{s}_{V,0} = 1$$

If $x \in H^{a+b\sigma}$, and $b = r_1 + \lfloor \frac{a+b}{2} \rfloor$, then $(u^{-r_1}x)^2 = Sq_{C_2}^{a+b}(u^{-r_1}x)$, so

$$x^2 = u^{2r_1} Sq_{C_2}^{a+b}(u^{-r_1}x)$$

By using C_2 -equivariant Cartan formula and the formula 3.2

$$Sq_{C_2}^{2m+\delta}(u^{2l+\epsilon}) = \binom{2l+\epsilon}{2m+\delta} u^{2l+\epsilon-m-\delta} a^{2m+\delta}$$

one has general formula for $Sq_{C_2}^{a+b}(u^{-r_1}x)$. By iterating this method one can find a formula for every x^{2^l} , so $\mathbf{s}_{V,l}$ exist. For example, if $x \in H^{3+\sigma}$, then

$$\begin{aligned} (ux)^2 &= Sq_{C_2}^4(ux) = \sum_{r=0}^2 Sq_{C_2}^{2r}(u) Sq_{C_2}^{4-2r}(x) + \sum_{s=0}^1 Sq_{C_2}^{2s+1}(u) Sq_{C_2}^{3-2s}(x) \\ &= u Sq_{C_2}^4(x) + ua Sq_{C_2}^3(x) \end{aligned}$$

Thus

$$x^2 = u^{-1} Sq_{C_2}^4(x) + u^{-1} a Sq_{C_2}^3(x).$$

The set of elements x_i whose finite distinct products form a basis for a graded ring A is called a **simple system of generators**. For example, a polynomial algebra $k[x]$ has a simple system of generators $\{x^{2^i} \mid i \geq 0\}$.

Theorem 17. (Borel) *Let $F \rightarrow E \rightarrow B$ be a C_2 -fibration with E contractible. Suppose that $H^*(F)$ has a simple system $\{x_i\}$ of transgressive generators. Then $H^*(B)$ is a polynomial ring in the $\{\Sigma(x_i)\}$.*

E_2 -page of $RO(G)$ -graded Serre spectral sequence of Kronholm [7] depends only on the total degree of representations, not the dimension of twisted part. The proof of the theorem is completely same as the classical case. See, for example, [15, Page 88, Theorem 1].

A simple system of generators for $H^*(K_\sigma) \cong H^*(pt)[c, d]/(c^2 = ac + ud)$ is

$$\{c, d^{2^l} \mid l \geq 0\} = \{c, \mathbf{s}_{1+\sigma, l}(d) \mid l \geq 0\}$$

By applying the Borel Theorem to the path space fibration

$$K_\sigma \rightarrow P(K_\rho) \rightarrow K_\rho$$

we have

$$H^*(K_\rho) = P(x_\rho, \mathbf{s}_{\rho, l}(x_{1+\rho}) \mid l \geq 0).$$

A simple system of generator for $H^*(K_\rho)$ is

$$\{x_\rho^{2^j}, (\mathbf{s}_{\rho, l}(x_{1+\rho}))^{2^j} \mid j, l \geq 0\} = \{\mathbf{s}_{\rho, j}(x_\rho), \mathbf{s}_{2^l \rho + 1, j} \mathbf{s}_{\rho, l}(x_{1+\rho}) \mid j, l \geq 0\}$$

where $|\mathbf{s}_{\rho, l}(x_{1+\rho})| = 2^l \rho + 1$. Then,

$$H^*(K_{1+\rho}) = P(\mathbf{s}_{\rho, j}(x_{1+\rho}), \mathbf{s}_{2^l \rho + 1, j} \mathbf{s}_{\rho, l}(x_{2+\rho}) \mid j, l \geq 0).$$

A simple system of generators for $H^*(K_{1+\rho})$ is

$$\begin{aligned} & \{(\mathbf{s}_{\rho, j} x_{1+\rho})^{2^k}, (\mathbf{s}_{2^j \rho + 1, j} \mathbf{s}_{\rho, l} x_{2+\rho})^{2^k} \mid j, l, k \geq 0\} \\ & = \{\mathbf{s}_{2^j \rho + 1, k} \mathbf{s}_{\rho, j} x_{2+\sigma}, \mathbf{s}_{2^j(2^l \rho + 1) + 1, k} \mathbf{s}_{2^l \rho + 1, j} \mathbf{s}_{\rho, l} x_{2+\rho} \mid j, l, k \geq 0\} \end{aligned}$$

where $|\mathbf{s}_{2^j \rho + 1, j} \mathbf{s}_{\rho, l} x_{2+\rho}| = 2^j(2^l \rho + 1) + 1$. Then,

$$H^*(K_{2+\rho}) = P(\mathbf{s}_{2^j \rho + 1, k} \mathbf{s}_{\rho, j} x_{2+\rho}, \mathbf{s}_{2^j(2^l \rho + 1) + 1, k} \mathbf{s}_{2^l \rho + 1, j} \mathbf{s}_{\rho, l} x_{3+\rho} \mid j, l, k \geq 0).$$

Thus, by iterating this process, one can find the $RO(C_2)$ -graded cohomology of C_2 -equivariant Eilenberg-Mac Lane spaces $K_{n+\sigma}$

$$H^*(K_{n+\sigma})$$

for $n \geq 0$.

Conjecture 18. *We know that*

$$\begin{aligned} H^*(K_1) &= P(x_1) \\ H^*(K_\sigma) &= P(x_\sigma, x_{1+\sigma}) / (x_\sigma^2 + ax_\sigma + ux_{1+\sigma}). \end{aligned}$$

If we knew $H^(K_{1+k(\sigma-1)})$ for all $k \geq 0$, then we could use the Borel theorem to find $H^*(K_{1+m+k(\sigma-1)})$ for $m \geq 0$.*

We conjectured that $H^(K_{1+k(\sigma-1)})$ is a polynomial algebra on $k+1$ generators with k relations, saying that the square of each of the first k generators is a linear combination of the other generators. The dimensions of the first k generators of*

the $H^*(K_{1+k(\sigma-1)})$ are obtained by adding $\sigma - 1$ to those of the generators of the $H^*(K_{1+(k-1)(\sigma-1)})$, and the dimension of the last generator is $k\sigma + 2^k - k$.

Example 19. Let $k = 3$. Then

$$H^*(K_{3\sigma-2}) = P(x_{3\sigma-2}, x_{3\sigma-1}, x_{3\sigma+1}, x_{3\sigma+5}) / (x_{3\sigma-2}^2 + \cdots, x_{3\sigma-1}^2 + \cdots, x_{3\sigma+1}^2 + \cdots)$$

where the other terms in the relations are linear. The resulting simple system of generators is

$$\{x_{3\sigma-2}, x_{3\sigma-1}, x_{3\sigma+1}\} \cup \{x_{3\sigma+5}^i \mid i \geq 0\}.$$

Now, we give another useful lemma for computations, which is the cohomology analogous of the Lemma 2.7. in [1].

There is a forgetful map

$$\Phi^e : H_{C_2}^V(X; \underline{Z/2}) \longrightarrow H^{|V|}(X^e; Z/2)$$

from the equivariant cohomology to the non-equivariant cohomology with $Z/2$ -coefficients. And also, we have a fixed point map

$$\Phi^{C_2} : H_{C_2}^V(X; \underline{Z/2}) \longrightarrow H^{|V^{C_2}|}(X^{\Phi C_2}; Z/2)$$

where $X^{\Phi C_2}$ is a geometric fixed point of a G -space X . Now, I will state the lemma, whose proof is the analog of Lemma 2.7. in [1].

Lemma 20. *Let X be a genuine C_2 -spectrum, and suppose that $\{b_i\}$ is a set of elements of $H^*(X)$ such that*

- (i) $\{\Phi^e(b_i)\}$ is a basis of $H^*(X^e)$, and
- (ii) $\{\Phi^{C_2}(b_i)\}$ is a basis of $H^*(X^{\Phi C_2})$

Then $H^(X)$ is free over $H^*(pt)$ with the basis $\{b_i\}$.*

One project is to finish calculations of the $RO(C_2)$ -graded C_2 -equivariant cohomology of C_2 -equivariant Eilenberg-Mac Lane spaces by using Caruso theorem 16 and lemma 20, and then Eilenberg-Moore spectral sequences of Michael A. Hill [4, Chapter 5].

Conjecture 21. *$H^*(K_{r\sigma+s})$ is a polynomial algebra on certain C_2 -equivariant Steenrod operations $Sq_{C_2}^I(\iota_{r\sigma+s})$ divided by certain powers of u , where $e(I) < r\sigma + s$, and $\iota_{r\sigma+s}$ is the fundamental class, and $V < V'$ if and only if $V' = V + W$ for some actual representations W with positive degree.*

Example 22. $H^*(K(\underline{Z/2}, 1 + \sigma))$ is the polynomial algebra generated by elements $Sq^I(\iota_{1+\sigma})$, where I is admissible and $e(I) < 1 + \sigma$. So, it is a polynomial algebra

$$P(Sq^0(\iota_{1+\sigma}), Sq^2 Sq^1(\iota_{1+\sigma}), Sq^4 Sq^2 Sq^1(\iota_{1+\sigma}), \cdots).$$

Then, it is shortly

$$P(x_\rho, x_{1+2\rho}, x_{1+4\rho}, x_{1+8\rho}, \cdots).$$

By now, we have calculated the cohomology of $K(\underline{Z/2}, n + \sigma)$ for $n \geq 0$. To calculate other cases, if knew $H^*(K_{n\sigma})$ for $n \geq 2$, we could use the Eilenberg-Moore spectral sequence [4, Chapter 5] and the Borel theorem for the path-space fibration

$$\Omega K(\underline{Z/2}, V) \longrightarrow P(K(\underline{Z/2}, V)) \longrightarrow K(\underline{Z/2}, V).$$

For example, for the path-space fibration

$$K(\underline{\mathbb{Z}/2}, \sigma) \longrightarrow P(K(\underline{\mathbb{Z}/2}, 1 + \sigma)) \longrightarrow K(\underline{\mathbb{Z}/2}, 1 + \sigma).$$

E_∞ -term of the Eilenberg-Moore spectral sequence is

$$E_\infty = E(x_\sigma, x_\rho, x_{2\rho}, x_{4\rho}, \dots)$$

with the relations

$$\begin{aligned} x_\sigma^2 &= ax_\sigma + ux_{1+\sigma} \\ x_{1+\sigma}^2 &= x_{2+2\sigma} \\ x_{2+2\sigma}^2 &= x_{4+4\sigma} \\ &\vdots \\ x_{2^i\rho}^2 &= x_{2^{i+1}\rho} \\ &\vdots \end{aligned}$$

for $i \geq 0$. As a result, $H^*(K(\underline{\mathbb{Z}/2}, \sigma), \underline{\mathbb{Z}/2})$ is a quadratic extension of a polynomial algebra, as it has already known.

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ISTANBUL MEDENIYET UNIVERSITY, ISTANBUL, TURKEY

Email address: `ugur.yigit@medeniyet.edu.tr`

Current address: Department of Mathematics, Istanbul Medeniyet University, H1-20, Istanbul, TURKEY 34700