On the extension problems for the 33-stem homotopy groups of the 6-, 7- and 8-spheres

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Abstract This paper tackles the extension problems for the homotopy groups $\pi_{39}(S^6)$, $\pi_{40}(S^7)$, and $\pi_{41}(S^8)$ localized at 2, the puzzles having remained unsolved for forty-five years. We introduce a tool for the theory of determinations of unstable homotopy groups, namely, the rectangular Toda bracket, by which we are able to solve the extension problems with respect to these three homotopy groups.

Key Words and Phrases unstable homotopy group, rectangular Toda bracket, *EHP* sequence

1 Introduction

The extension problems from the 1979 literature by N. Oda ([7, pp. 145]), which have gone uncracked for the past 45 years, are as follows:

$$\pi_{39}(S^6) \approx (\mathbb{Z}/2)^8 \text{ or } (\mathbb{Z}/2)^6 \oplus \mathbb{Z}/4,$$

$$\pi_{40}(S^7) \approx (\mathbb{Z}/2)^6 \text{ or } (\mathbb{Z}/2)^4 \oplus \mathbb{Z}/4$$

$$\text{and } \ \pi_{41}(S^8) \approx \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^8 \text{ or } \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^6 \oplus \mathbb{Z}/4 \ \text{ localized at 2.}$$

The key aspect of addressing these problems lies in determining the order of the element $\kappa'_6 \in \pi_{39}(S^6_{(2)})$ with nontrivial *Hopf* invariant $\bar{\nu}_{11}\bar{\kappa}_{19}$. The puzzles have proven insurmountable by conventional strategies. In this paper, we introduce the rectangular Toda bracket, by which we are able to determine the group extensions for these three homotopy groups. It is noteworthy that our capacity to employ the rectangular Toda bracket in resolving Oda's extension problems arises from our establishment of the H-formula (see Theorem 2), which cannot be straightforwardly or congruently derived from Toda's H-formula ([1, Proposition 2.6]), but rather deduced through the theory of groups $[(C\Sigma X, X), (Y, B)]$.

Homotopy groups occupy a central and foundational position in homotopy theory, encapsulating the essence of building spaces. These groups serve as pivotal invariants, providing profound insights into the inherent geometric and algebraic characteristics of spaces. Among the myriad of homotopy groups, those pertaining to spheres hold a particularly preeminent and influential status. Through meticulous analysis of these groups, researchers have devised sophisticated techniques and unearthed unexpected results, significantly impacting areas such as algebraic topology, differential geometry, and theoretical physics.

Given $k \in \mathbb{Z}_+$, the groups $\pi_{n+k}(S^n)\,(n \geq 2)$ are called the k-stem homotopy groups of spheres. The initial systematic and effective computation of homotopy groups of spheres is presented in [1] (the *1* to *19*-stems) by H. Toda in 1962, followed subsequently by [3] (the *20*-stem), [4] (the *21,22*-stems localized at 2) and [5] (the *23,24*-stems localized at 2).

In 1979, N. Oda ([7]) studies the homomotopy groups of spheres $\pi_{n+k}(S^k)$ $(25 \leq n \leq 31)$ for all $k \geq 2$ and $\pi_{n+k}(S^k)\,(n = 32, 33)$ for $2 \leq k \leq 8$ localized at 2. This work epitomizes an exceptional blend of rigorous logic and artistic flair. Oda employs an array of sophisticated techniques to compute these homotopy groups, manifesting an extraordinary level of mathematical prowess. The employed methodologies encompass the classical Toda bracket method, the *4*-fold Toda bracket method, the Adams spectral sequence and Im(J)-theory. However, the *33*-stem homotopy groups $\pi_{39}(S^6)$, $\pi_{40}(S^7)$ and $\pi_{41}(S^8)$ localized at 2 are incompletely determined.

In 2017, the determinations of the *32*-stem homotopy groups localized at 2, namely, the groups $\pi_{32+k}(S^k_{(2)})$ for all $k \geq 2$, are comprehensively concluded by T. Miyauchi and J. Mukai ([11]). The authors provide a new tool for determinations of unstable homotopy groups, that is, the matrix Toda bracket indexed by n, which makes better use of the desuspension property of homotopy classes if

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$n \geq 1$ and is a key ingredient to compute $\pi_{46}(S^{14}_{(2)})$. Notably, [11] gives a comprehensive computation of the homotopy group $\pi_{42}(S^9_{(2)})$, a *33*-stem homotopy group localized at 2. While the three groups $\pi_{33+n}(S^n_{(2)})$ $(n = 6, 7, 8)$ are still unresolved within [11].

Herein lies our first main theorem, which constitutes the primary objective of this manuscript and addresses the extension problems posed by Oda ([7, pp. 145]), (see Proposition 4.2.1).

Theorem 1 *Localized at 2,*

$$\pi_{39}(S^6) \approx (\mathbb{Z}/2)^8, \ \ \mathrm{ord}(\kappa'_6) = 2; \ \pi_{40}(S^7) \approx (\mathbb{Z}/2)^6; \ \pi_{41}(S^8) \approx \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^8.$$

To prove Theorem 1, we define rectangular Toda brackets and study their properties in the spirit of [12]. In particular, we propose a theorem that explicates the interaction of the homomorphism H with P in the process of using the rectangular Toda bracket, which theorem constitutes the most pivotal technique in our conquest of Oda's extension problems.

Our second main theorem which generalizes Toda's H-formula given by [1, Proposition 2.6, pp. 22-23], is stated as follows, (see Proposition 3.1.4).

Theorem 2

$$\begin{tikzcd}
 & \Sigma X_1 \arrow[dl, "\Sigma\alpha_1"'] & \Sigma Y_1 \arrow[l, "\Sigma\beta_1"'] & \\
S^{m+1} & & & \Sigma Z \arrow[ul, "\Sigma\gamma_1"'] \arrow[dl, "\Sigma\gamma_2"] \\
 & \Sigma X_2 \arrow[ul, "\Sigma\alpha_2"] & \Sigma Y_2 \arrow[l, "\Sigma\beta_2"] &
\end{tikzcd}$$

Localized at 2, given the above diagram of spaces and homotopy classes with Y_1, Y_2 and Z suspensions such that $\Sigma\alpha_1 \circ \Sigma\beta_1 = \Sigma\alpha_2 \circ \Sigma\beta_2 = \beta_1\gamma_1 + \beta_2\gamma_2 = 0$, we have

$$H \begin{bmatrix} \Sigma\alpha_1 & \Sigma\beta_1 & \Sigma\gamma_1 \\ \Sigma\alpha_2 & \Sigma\beta_2 & \Sigma\gamma_2 \end{bmatrix}_1 = -P^{-1}(\alpha_1\beta_1) \circ \Sigma^2\gamma_1 - P^{-1}(\alpha_2\beta_2) \circ \Sigma^2\gamma_2.$$

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2 Preliminaries

2.1 Notations

In this paper, all spaces, maps, homotopy classes are pointed. By a space we mean a path-connected CW complex. If we take the 2-localization, we always use the original symbols of the spaces, maps and homotopy classes to denote them after localization at 2. Basepoints and constant maps are denoted by $*$, homotopy classes of constant maps are denoted by 0. To indicate the domain and codomain, the trivial element in $\pi_{n+k}(S^n)$ is also denoted by $0^{(k)}_n$. In our discourse on the sphere S^m, we consistently presuppose $m \geq 1$. For a map or a homotopy class f, the notation C_f stands for the homotopy cofibre of f. We shall not differentiate between a map and its homotopy class when no confusion arises.

Let $\alpha \in \pi_n(X), \beta \in \pi_m(S^n)$ where $n \geq 2$ and let $k \in \mathbb{Z}$; usually and reasonably, $\alpha\beta$ is the the abbreviation of $\alpha \circ \beta$ and $k\alpha \circ \beta$ is the the abbreviation of $(k\alpha) \circ \beta$; in this article we only use the symbol $k\alpha\beta$ to denote $k(\alpha\beta)$. So it's necessary to point out that,

$$k\alpha\beta \neq k\alpha \circ \beta \quad \text{in general.}$$

Of course $k\alpha\beta = k\alpha \circ \beta$ always holds if β is a suspension or the codomain of α is S^7 or a group-like H-space ([6, p. 118]), especially a topological group.

For a non-negative integer m, let $\mathbb{Z}/m$ denote $\mathbb{Z}/m\mathbb{Z}$; let $\mathbb{Z}_{(2)}$ be the group or the ring of 2-local integers; for a $\mathbb{Z}_{(2)}$-module A of form $\mathbb{Z}_{(2)}$ or $\mathbb{Z}/2^k$, we use $G = A\{\mathtt{x}\}$ to denote a $\mathbb{Z}_{(2)}$-module G which is ismorphic to A and generated by $\mathtt{x}$. For example, $G = \mathbb{Z}/4\{\mathtt{x}\}$ stands for $G \approx \mathbb{Z}/4$ and G is generated by $\mathtt{x}$. $(\mathbb{Z}/m)^k$ denotes the direct sum of k-copies of $\mathbb{Z}/m$; we use $\oplus$ to denote both the internal direct sum and the external direct sum. And $\mathrm{ord}(\mathtt{x})$ denotes the order of the element $\mathtt{x}$ of a group.

Let G be an abelian group and A be a subgroup, $g, g' \in G$; following Oda ([7]), we write $g \equiv g' \bmod A$ if $g - g' \in A$. If A is generated by $\{a_\lambda\}_{\lambda \in \Lambda}$, then " $\bmod A$ " is also denoted by " $\bmod a_\lambda$, $(\lambda \in \Lambda)$ ".

2.2 Some fundamental facts

Given the following sequence of spaces and homotopy classes where Z is a suspension such that $\alpha\circ\Sigma^n\beta=\beta\gamma=0$,

$$W \xleftarrow{\ \alpha\ } \Sigma^n X \xleftarrow{\Sigma^n\beta} \Sigma^n Y \xleftarrow{\Sigma^n\gamma} \Sigma^n Z,$$

the (ordinary) Toda bracket which is denoted by $\{\alpha,\Sigma^n\beta,\Sigma^n\gamma\}_n$ is defined to be the set of all compositions of the form

$$(-1)^n\mathrm{ext}_{\Sigma^n\beta}(\alpha)\circ\Sigma^n\mathrm{coext}_\beta(\gamma),$$

where $\mathrm{ext}_{\Sigma^n\beta}(\alpha)\in[C_{\Sigma^n\beta},W]$ is an extension of α with respect to $\Sigma^n\beta$, and $\mathrm{coext}_\beta(\gamma)\in[\Sigma Z,C_\beta]$ is a coextsion of γ with respect to β, (in [1, p. 13], $\mathrm{ext}_{\Sigma^n\beta}(\alpha)$ and $\mathrm{coext}_\beta(\gamma)$ are denoted by $\overline{\alpha}$ and $\widetilde{\gamma}$ respectively $^\sharp$). Such a Toda bracket is a coset included in $[\Sigma^{n+1}Z,W]$, that is,

$$\{\alpha,\Sigma^n\beta,\Sigma^n\gamma\}_n\in[\Sigma^{n+1}Z,W]/A,$$

where $A=\alpha\circ\Sigma^n[\Sigma Z,X]+[\Sigma^{n+1}Y,W]\circ\Sigma^{n+1}\gamma$. Furthermore, as a group homorphism sends a coset to a coset, it is clear that $f\circ\{\alpha,\Sigma^n\beta,\Sigma^n\gamma\}_n$ is a coset of $f\circ A$, and $\{\alpha,\Sigma^n\beta,\Sigma^n\gamma\}_n\circ\Sigma g$ is a coset of $A\circ\Sigma g$, (f,g : maps). For a fixed n, $\{\alpha,\Sigma^n\beta,\Sigma^n\gamma\}_n$ depends only on (α,β,γ) but not $(\alpha,\Sigma^n\beta,\Sigma^n\gamma)$. It is necessary to point out that even if $\Sigma^n\beta'=\Sigma^n\beta,\Sigma^n\gamma'=\Sigma^n\gamma$ and $\beta'\gamma'=0$,

$$\{\alpha,\Sigma^n\beta',\Sigma^n\gamma'\}_n\neq\{\alpha,\Sigma^n\beta,\Sigma^n\gamma\}_n \text{ in general,}$$

(see [10, Remark 3.1]). For more basic properties of Toda brackets, see [1, pp. 10-12].

Next, let us reminisce upon the the generalized *EHP* sequence. The following is from [12, Proposition 4.2].

Proposition 2.2.1 *After localization at 2, let the following sequence (called the generalized EHP sequence) be given,*

$$\cdots\xrightarrow{H}[\Sigma^2Z,S^m]\xrightarrow{\Sigma}[\Sigma^3Z,S^{m+1}]\xrightarrow{H}[\Sigma^3Z,S^{2m+1}]\xrightarrow{P}[\Sigma Z,S^m]\xrightarrow{\Sigma}[\Sigma^2Z,S^{m+1}],$$

where each H is induced by H_2 up to the obvious isomorphism, and each P : $[\Sigma^{r+1}Z,S^{2m+1}]\to[\Sigma^{r-1}Z,S^m]$ is defined to be $P=\partial\circ(H_{2*})^{-1}\circ\Omega_1$ with $\Omega_1:[\Sigma-\,,S^{2m+1}]\xrightarrow{\approx}[-,\ JS^{2m}]$ the classical isomorphism and $\partial:[(C\Sigma^{r-1}Z,\Sigma^{r-1}Z),(JS^m,S^m)]\to[\Sigma^{r-1}Z,S^m]$ $(r\geq 2)$ the boundary homomorphism. Then this sequence is exact in the category of groups. Moreover,

$$\left\{[\Sigma^3Z,S^{2m+1}]\xrightarrow{P}[\Sigma Z,S^m]\mid Z\text{ is a 2-local space}\right\}$$

is a natural transformation from the cofounctor $[\Sigma-,\ S^m]$ to the cofounctor $[\Sigma^2-,\ S^{2m+1}]$. □

Remark 2.2.2 (1) *We can derive a long exact sequence with ease by applying the functor $[\Sigma Z,-]$ to the 2-local homotopy fibration sequence, $\cdots\longrightarrow\Omega^2S^{2m+1}\to S^m\hookrightarrow\Omega S^{m+1}\xrightarrow{H_2}\Omega S^{2m+1}$. Proposition 2.2.1 primarily highlights the* **specific constructions** *of the boundary homomorphisms P, in order to use [9, Proposition 2.4] to derive the H-formulas such as Theorem 2.*

(2) *Worth mentioning is the absence of the relative Whitehead theorem (also see [6, pp. 184-185]). That is, after localization at 2, the fact that $\pi_k(JS^m,S^m)\xrightarrow{H_{2*}}\pi_k(JS^{2m},*)$, $(k\geq 1)$ are isomorphisms for the path-connected CW pairs (JS^m,S^m) and $(JS^{2m},*)$, does not imply $(JS^m,S^m)\simeq(JS^{2m},*)$. $^{\sharp\sharp}$ The invocation of the theory of groups of relative homotoy classes $[(C\Sigma X,\Sigma X),(Y,B)]$ is imperative in confirming that the homomorphisms $H_{2*}:[(C\Sigma^{r-1}Z,\Sigma^{r-1}Z),\ (JS^m,S^m)]\xrightarrow{\approx}[(C\Sigma^{r-1}Z,\Sigma^{r-1}Z),(JS^{2m},*)]$ $(r\geq 1)$ are isomorphisms localized at 2. See [12] for more details.*

The ensuing result from [12, Corrollary 4.2] is particularly advantageous in the proof of the generalized H-formula (Proposition 3.1.4).

Lemma 2.2.1 *After localization at 2, suppose L_1,L_2 are suspensions; then there exists a commutative diagram where $p_1:L_1\vee L_2\to L_1$ is the pinch map and $i_2:L_2\to L_1\vee L_2$ is the inclusion map. Moreover, the vertical arrows are short exact sequences which split into direct products and the split homomorphisms are induced by the obvious pinch and inclusion maps.*

$^\sharp$ For more properties of extensions and coextensions, see [10, Lemma 2.2 and Lemma 2.3]

$^{\sharp\sharp}$ $(JS^m,S^m)\not\simeq(JS^{2m},*)$ localized at 2, by checking the $\mathbb{Z}/2$-homology.

$$\begin{tikzcd}
{[\Sigma^2 L_1, S^{2m+1}]} \arrow[r, "P"] \arrow[d, "(\Sigma^2 p_1)^*"'] & {[L_1, S^m]} \arrow[d, "p_1^*"] \\
{[\Sigma^2(L_1 \vee L_2), S^{2m+1}]} \arrow[r, "P"] \arrow[d, "(\Sigma^2 i_2)^*"'] & {[L_1 \vee L_2, S^m]} \arrow[d, "i_2^*"] \\
{[\Sigma^2 L_2, S^{2m+1}]} \arrow[r, "P"] & {[L_2, S^m]}
\end{tikzcd}$$

□

3 Rectangular Toda brackets

In the spirit of [12], we define the rectangular Toda bracket indexed by n, (Definition 3.1), which is essentially a special case of the *3*-fold Toda bracket. Recall that in the seminal works [1], [3],[4],[5],[7] and [11] on computing homotopy groups of spheres, to determine $\pi_*(S^n)$, in particular, to define elements with nontrivial H-images, all instances of *3*-fold Toda brackets are brackets of three elements of homotopy groups of spheres. Within this purview, our rectangular Toda bracket elaborates on the use of the *3*-fold Toda bracket. It is worthy mentioning that the *3*-fold, matrix and rectangular Toda brackets indexed by n ($n \geq 1$) make better use of the desuspension property of homotopy classes, and play an important role in unstable homotopy theory.

Our ability to address Oda's extension problems is largely attributed to the utilization of a rectangular Toda bracket indexed by *2*, see Lemma 4.2.1.

Additionally, while the rectangular Toda bracket is a particular case of the *3*-fold one, not all properties of the *3*-fold one can be preserved in a "rectangular pattern" for the rectangular Toda bracket, (see Proposition 3.1.1, Lemma 3.1.1 (4), (5)). Hence, a specific investigation into its properties is warranted.

Suppose $f_i : U_i \to W$ ($i = 1, 2$) are maps or homotopy classes. By abuse of notation, the following composition is still denoted by $f_1 \vee f_2$,

$$W \xleftarrow[\text{folding}]{\nabla} W \vee W \xleftarrow{f_1 \vee f_2} U_1 \vee U_2.$$

Definition 3.1

$$\begin{tikzcd}
 & \Sigma^n X_1 \arrow[dl, "\alpha_1"'] & \Sigma^n Y_1 \arrow[l, "\Sigma^n \beta_1"'] & \\
W & & & \Sigma^n Z \arrow[ul, "\Sigma^n \gamma_1"'] \arrow[dl, "\Sigma^n \gamma_2"] \\
 & \Sigma^n X_2 \arrow[ul, "\alpha_2"] & \Sigma^n Y_2 \arrow[l, "\Sigma^n \beta_2"] &
\end{tikzcd}$$

*For the above diagram of spaces and homotopy classes with Z a suspension such that $\alpha_1 \circ \Sigma^n \beta_1 = \alpha_2 \circ \Sigma^n \beta_2 = \beta_1\gamma_1 + \beta_2\gamma_2 = 0$, let $Y_k \overset{j_k}{\hookrightarrow} Y_1 \vee Y_2$ be the inclusions ($k = 1, 2$); the **rectangular Toda bracket** indexed by n*

$$\begin{bmatrix} \alpha_1 & \Sigma^n\beta_1 & \Sigma^n\gamma_1 \\ \alpha_2 & \Sigma^n\beta_2 & \Sigma^n\gamma_2 \end{bmatrix}_n$$

is defined to be the ordinary Toda bracket

$$\{\alpha_1 \vee \alpha_2,\ \Sigma^n(\beta_1 \vee \beta_2),\ \Sigma^n(j_1\gamma_1 + j_2\gamma_2)\}_n$$

with respect to the following sequence

$$W \xleftarrow{\alpha_1 \vee \alpha_2} \Sigma^n(X_1 \vee X_2) \xleftarrow{\Sigma^n(\beta_1 \vee \beta_2)} \Sigma^n(Y_1 \vee Y_2) \xleftarrow{\Sigma^n(j_1\gamma_1 + j_2\gamma_2)} \Sigma^n Z.$$

Proposition 3.1.1 *Under the condition of Definition 3.1, further suppose $n \geq 1$, X_1 and X_2 are suspensions. Then,*

$$\begin{bmatrix} \alpha_1 & \Sigma^n\beta_1 & \Sigma^n\gamma_1 \\ \alpha_2 & \Sigma^n\beta_2 & \Sigma^n\gamma_2 \end{bmatrix}_n \in [\Sigma^{n+1}Z, W]/A, \quad \textit{where}$$

$$A = \alpha_1 \circ \Sigma^n[\Sigma Z, X_1] + \alpha_2 \circ \Sigma^n[\Sigma Z, X_2] + [\Sigma^{n+1}Y_1, W] \circ \Sigma^{n+1}\gamma_1.$$

Proof. By the assumption, we see that $n \geq 1$, X_1 and X_2 are suspensions. Then, according to the proof of [12, Proposition 3.2], we derive

$$[\alpha_1 \vee \alpha_2] \circ \Sigma^n[\Sigma Z, X_1 \vee X_2] = \alpha_1 \circ \Sigma^n[\Sigma Z, X_1] + \alpha_2 \circ \Sigma^n[\Sigma Z, X_2]. \tag{3.1}$$

Therefore, the proposition follows from Definition 3.1 and [1, Lemma 1.1, pp. 9]. $\square$

Remark 3.1.2 *In Proposition 3.1.1, the condition "$n \geq 1$" is necessary. Since Formula (3.1) is not true in general if $n = 0$ is allowed. See [12, Example 3.3].*

Proposition 3.1.3 *Under the condition of Definition 3.1, further suppose $n \geq 1$, X_1, X_2 are suspensions, and $\beta_1\gamma_1 = \beta_2\gamma_2 = 0$. Then,*

$$\begin{bmatrix} \alpha_1 & \Sigma^n\beta_1 & \Sigma^n\gamma_1 \\ \alpha_2 & \Sigma^n\beta_2 & \Sigma^n\gamma_2 \end{bmatrix}_n \subseteq \{\alpha_1, \Sigma^n\beta_1, \Sigma^n\gamma_1\}_n + \{\alpha_2, \Sigma^n\beta_2, \Sigma^n\gamma_2\}_n.$$

Proof. Recall from Definition 3.1 that the maps $Y_k \overset{j_k}{\hookrightarrow} Y_1 \vee Y_2$ are the inclusions $(k = 1, 2)$. Let $X_k \overset{i_k}{\hookrightarrow} X_1 \vee X_2$ be the inclusions $(k = 1, 2)$. Then, $(\beta_1 \vee \beta_2) \circ j_1 = i_1\beta_1$, $(\beta_1 \vee \beta_2) \circ j_2 = i_2\beta_2$. And so,

$$\begin{aligned}
\begin{bmatrix} \alpha_1 & \Sigma^n\beta_1 & \Sigma^n\gamma_1 \\ \alpha_2 & \Sigma^n\beta_2 & \Sigma^n\gamma_2 \end{bmatrix}_n &= \{\alpha_1 \vee \alpha_2,\ \Sigma^n(\beta_1 \vee \beta_2),\ \Sigma^n(j_1\gamma_1 + j_2\gamma_2)\}_n \\
&\subseteq \{\alpha_1 \vee \alpha_2,\ \Sigma^n(\beta_1 \vee \beta_2),\ \Sigma^n(j_1\gamma_1)\}_n + \{\alpha_1 \vee \alpha_2,\ \Sigma^n(\beta_1 \vee \beta_2),\ \Sigma^n(j_2\gamma_2)\}_n \\
&\subseteq \{\alpha_1 \vee \alpha_2,\ \Sigma^n((\beta_1 \vee \beta_2) \circ j_1),\ \Sigma^n\gamma_2\}_n + \{\alpha_1 \vee \alpha_2,\ \Sigma^n((\beta_1 \vee \beta_2) \circ j_2),\ \Sigma^n\gamma_2\}_n \\
&= \{\alpha_1 \vee \alpha_2,\ \Sigma^n(i_1\beta_1),\ \Sigma^n\gamma_2\}_n + \{\alpha_1 \vee \alpha_2,\ \Sigma^n(i_2\beta_2),\ \Sigma^n\gamma_2\}_n \qquad (3.2) \\
&\supseteq \{(\alpha_1 \vee \alpha_2) \circ \Sigma^n i_1,\ \Sigma^n\beta_1,\ \Sigma^n\gamma_2\}_n + \{(\alpha_1 \vee \alpha_2) \circ \Sigma^n i_2,\ \Sigma^n\beta_2,\ \Sigma^n\gamma_2\}_n \qquad (3.3) \\
&= \{\alpha_1,\ \Sigma^n\beta_1,\ \Sigma^n\gamma_2\}_n + \{\alpha_2,\ \Sigma^n\beta_2,\ \Sigma^n\gamma_2\}_n. \qquad (3.4)
\end{aligned}$$

It suffices to show that the relation "$\supseteq$" in Equation (3.3) is in fact an equality. By Formula (3.1), we infer the sum in Equation (3.2) is a coset of

$$M = \alpha_1 \circ \Sigma^n[\Sigma Z, X_1] + \alpha_2 \circ \Sigma^n[\Sigma Z, X_2] + [\Sigma^{n+1}Y_1, W] \circ \Sigma^{n+1}\gamma_1 + [\Sigma^{n+1}Y_2, W] \circ \Sigma^{n+1}\gamma_2.$$

Through straightforward verification, we can easily deduce that the sum in Equation (3.4) is also a coset of M. Therefore, the sum in Equation (3.4) is equal to the sum in Equation (3.2). $\square$

Proposition 3.1.4

$$\begin{tikzcd}
 & \Sigma X_1 \arrow[dl, "\Sigma\alpha_1"'] & \Sigma Y_1 \arrow[l, "\Sigma\beta_1"'] & \\
S^{m+1} & & & \Sigma Z \arrow[ul, "\Sigma\gamma_1"'] \arrow[dl, "\Sigma\gamma_2"] \\
 & \Sigma X_2 \arrow[ul, "\Sigma\alpha_2"] & \Sigma Y_2 \arrow[l, "\Sigma\beta_2"] &
\end{tikzcd}$$

Localized at 2, given the above diagram of spaces and homotopy classes with Y_1, Y_2 and Z suspensions such that $\Sigma\alpha_1 \circ \Sigma\beta_1 = \Sigma\alpha_2 \circ \Sigma\beta_2 = \beta_1\gamma_1 + \beta_2\gamma_2 = 0$, we have

$$H \begin{bmatrix} \Sigma\alpha_1 & \Sigma\beta_1 & \Sigma\gamma_1 \\ \Sigma\alpha_2 & \Sigma\beta_2 & \Sigma\gamma_2 \end{bmatrix}_1 = -P^{-1}(\alpha_1\beta_1) \circ \Sigma^2\gamma_1 - P^{-1}(\alpha_2\beta_2) \circ \Sigma^2\gamma_2.$$

Proof. By Proposition 2.2.1 and Lemma 2.2.1, we infer

$$P^{-1}(\alpha_1\beta_1 + \alpha_2\beta_2) = (\Sigma^2 p_1)^*(P^{-1}(\alpha_1\beta_1)) + (\Sigma^2 p_2)^*(P^{-1}(\alpha_2\beta_2))$$

where $p_i : Y_1 \vee Y_2 \to Y_i\,(i = 1, 2)$ are the pinch maps. Then the proposition follows from Definition 3.1 and [9, Proposition 2.4]. (Notice that the proof is similar to [12, Proof of Theorem 2 and 3]). $\square$

Suppose $k \in \mathbb{Z}$; following Fred. R. Cohen ([8, pp. 16]), we use $[k]$ to denote the self-map of degree k whose domain is a suspension. Notice that $[k] \circ \alpha = \alpha \circ [k] = k\alpha$ if $\Sigma X \xrightarrow{\alpha} \Sigma Y$ is a suspension.

The lemma presented below can be directly concluded from Definition 3.1 and the well-known properties of Toda brackets given by [1, pp. 10-12].

Lemma 3.1.1 *Under the condition of Definition 3.1 and setting $Z = \Sigma Z'$, we have the following.*

(1)
$$\begin{bmatrix}\alpha_1 & \Sigma^{n-1}(\Sigma\beta_1) & \Sigma^{n-1}(\Sigma\gamma_1)\\ \alpha_2 & \Sigma^{n-1}(\Sigma\beta_2) & \Sigma^{n-1}(\Sigma\gamma_2)\end{bmatrix}_{n-1} \subseteq \begin{bmatrix}\alpha_1 & \Sigma^n\beta_1 & \Sigma^n\gamma_1\\ \alpha_2 & \Sigma^n\beta_2 & \Sigma^n\gamma_2\end{bmatrix}_n, \ (n\geq 1).$$

(2)
$$\Sigma\begin{bmatrix}\alpha_1 & \Sigma^n\beta_1 & \Sigma^n\gamma_1\\ \alpha_2 & \Sigma^n\beta_2 & \Sigma^n\gamma_2\end{bmatrix}_n \subseteq -\begin{bmatrix}\Sigma\alpha_1 & \Sigma^{n+1}\beta_1 & \Sigma^{n+1}\gamma_1\\ \Sigma\alpha_2 & \Sigma^{n+1}\beta_2 & \Sigma^{n+1}\gamma_2\end{bmatrix}_{n+1}.$$

(3)
$$\begin{bmatrix}\alpha_1 & \Sigma^n\beta_1 & \Sigma^n\gamma_1\\ \alpha_2 & \Sigma^n\beta_2 & \Sigma^n\gamma_2\end{bmatrix}_n \circ \Sigma^{n+1}\delta \subseteq \begin{bmatrix}\alpha_1 & \Sigma^n\beta_1 & \Sigma^n(\gamma_1\circ\Sigma\delta)\\ \alpha_2 & \Sigma^n\beta_2 & \Sigma^n(\gamma_2\circ\Sigma\delta)\end{bmatrix}_n, \quad (\delta\in[K,Z'],\ K\colon \textit{a space});$$

$$\xi\circ\begin{bmatrix}\alpha_1 & \Sigma^n\beta_1 & \Sigma^n\gamma_1\\ \alpha_2 & \Sigma^n\beta_2 & \Sigma^n\gamma_2\end{bmatrix}_n \subseteq \begin{bmatrix}\xi\alpha_1 & \Sigma^n\beta_1 & \Sigma^n\gamma_1\\ \xi\alpha_2 & \Sigma^n\beta_2 & \Sigma^n\gamma_2\end{bmatrix}_n, \quad (\xi\in[W,L],\ L\colon \textit{a space}).$$

(4)
$$\begin{bmatrix}\alpha_1 & \Sigma^n\beta_1 & \Sigma^n([k]\circ \mathbb{y}_1)\\ \alpha_2 & \Sigma^n\beta_2 & \Sigma^n([k]\circ \mathbb{y}_2)\end{bmatrix}_n \subseteq \begin{bmatrix}\alpha_1 & \Sigma^n(k\beta_1) & \Sigma^n\mathbb{y}_1\\ \alpha_2 & \Sigma^n(k\beta_2) & \Sigma^n\mathbb{y}_2\end{bmatrix}_n, (k\in\mathbb{Z}),$$

if the spaces Y_s are suspensions and $\gamma_s = [k]\circ\mathbb{y}_s$, $(s=1,2)$.

(5)
$$\begin{bmatrix}k\alpha_1 & \Sigma^n\mathbb{b}_1 & \Sigma^n\gamma_1\\ k\alpha_2 & \Sigma^n\mathbb{b}_1 & \Sigma^n\gamma_2\end{bmatrix}_n \subseteq \begin{bmatrix}\alpha_1 & \Sigma^n(k\mathbb{b}_1) & \Sigma^n\gamma_1\\ \alpha_2 & \Sigma^n(k\mathbb{b}_1) & \Sigma^n\gamma_2\end{bmatrix}_n, (k\in\mathbb{Z}),$$

if the elements α_s are suspensions and $\beta_s = k\mathbb{b}_s$ with $\mathbb{b}_s$ suspensions, $(s=1,2)$. □

4 On the extension problems for π^6_{39}, π^7_{40} and π^8_{41}

In this section, all spaces and homotopy classes are *localized at 2* unless otherwise stated. The 2-local homotopy group $\pi_m(S^n)$ is also denoted by π^n_m. We follow Toda's notations in [1] of the generators of π^n_*, whose notations and naming convention are also adopted by [3],[4],[5],[7],[11] and so on. Recall from [1] that a set containing a single element is identified with its element. We denote Toda's E by Σ, the suspension functor, and we denote Toda's Δ by P, the boundary homomorphism of the *EHP* sequence. For the convenience of readers, in next paragraph we shall give a brief introduction of Toda's naming convention for the generators of π^n_* to help readers to read some of our lemmas and their proofs. Since Toda's 2-primary component method in [1] naturally corresponds to the 2-localization method. In this article we use the language of the 2-localization to state the facts on the 2-primary components.

Roughly speaking, for $\mathbb{x}$ which represents a Greek letter, $\mathbb{x}_n$ denotes one of the generators of π^n_{n+r} for some r, the subscript n indicates the codomain of $\mathbb{x}_n$. Moreover, $\mathbb{x}_{n+k} := \Sigma^k\mathbb{x}_n$, $\mathbb{x} := \Sigma^\infty\mathbb{x}_n$, and $\mathbb{x}^\ell_n$ is the abbreviation of $\mathbb{x}_n\circ\mathbb{x}_{n+r}\circ\cdots\circ\mathbb{x}_{n+(\ell-1)r}$, ($\ell$ factors). The usages of $\overline{\mathbb{x}}_n$ and $\mathbb{x}^*_n$ are similar to above. In π^j_{j+r} (not a stable homotopy group), if a generator is written without a subscript, then this generator does not survive in the stable homotopy group $\pi^S_r(S^0)$ or its Σ^∞- image is divisible by 2. For example, for $\theta\in\pi^{12}_{24}$, $\sigma'''\in\pi^5_{12}$, their Σ^∞- images satisfy $\Sigma^\infty\theta = 0$, $\Sigma^\infty\sigma''' = 8\sigma$ of order 2. There is an advantage of this naming convention, that is, we can examine the commutativity of the unstable composition conveniently,

$$\mathbb{x}_n\circ\mathbb{y}_i = \pm\mathbb{y}_n\circ\mathbb{x}_j \quad \text{for some } i,j \quad \text{if } \ n\geq a+b,$$

where $\{\mathbb{x}_k\}$ was born in π^a_* and $\{\mathbb{y}_\ell\}$ was born in π^b_*, (see [1, Proposition 3.1]). For instance, for the elements in [1],

$$\sigma_n\in\pi^n_{7+n}\,(n\geq 8) \text{ and } \mu_n\in\pi^n_{9+n}\,(n\geq 3),$$

we have $\sigma_{8+3}\mu_i = \pm\mu_{8+3}\sigma_j$, successively, $\sigma_{11}\mu_{18} = \mu_{11}\sigma_{20}$, ($\pm$ is not necessary, since μ_3 is of order 2). But $\sigma_{10}\mu_{17}\neq\mu_{10}\sigma_{19}$, (see [1, p. 156]). Some common generators are summarized in [1, p. 189] and [2, (1.1), p. 66].

4.1 Some relations on the generators of π_*^n

The following lemma is intended to lay the groundwork for the subsequent subsection.

Lemma 4.1.1 (1) $\eta_7\bar{\nu}_8\bar{\kappa}_{16} = \nu_7^3\bar{\kappa}_{16} = \nu_7\bar{\kappa}_{10}\nu_{30}^2$.

(2) $\sigma_{11}\rho_{18}\nu_{33}^2 = \varepsilon_{11}\kappa_{19}\nu_{33}^2 = \nu_{11}\bar{\sigma}_{14}\nu_{33}^2 = 0_{11}^{(26)}$.

(3) $P^{-1}(0_{11}^{(26)}) \circ \nu_{33}^2 = 0$.

(4) $P(\iota_{11}) = \nu_5\eta_8$.

(5) $\nu_6 \circ \nu_9\sigma_{12}\bar{\kappa}_{19} = \nu_6 \circ \phi_9\sigma_{32} = 0$.

(6) $\nu_6 \circ \Sigma^2\pi_{37}^7 = 0$.

Proof.

(1) It follows from [7, Formula (7.5), pp. 130] and $\eta_5\bar{\nu}_6 = \nu_5^3$,([1]).

(2) By [2, Table, pp. 105], we have $\rho_{18}\nu_{33} = 4E^5\lambda = 8\nu_{18}^* = 0$. Therefore, $\sigma_{11}\rho_{18}\nu_{33}^2 = 0$. By [2, Table, pp. 104], we have $\nu_7\varepsilon_{10} = 0$. Hence, $\varepsilon_{11}\kappa_{19}\nu_{33}^2 = \nu_{11}\varepsilon_{14}\nu_{22}\kappa_{25} = 0$. By [7, Proposition 2.3 (3), pp. 86], we see that $\nu_6^3\bar{\sigma}_{15} = 0$. Then, $\nu_{11}\bar{\sigma}_{14}\nu_{33}^2 = 0$.

(3) π_{22+11}^{11} is given by [4]; then (2) of our lemma shows that $\pi_{22+11}^{11} \circ \nu_{33}^2 = 0$. Notice that $\pi_{22+11}^{11} \supseteq P^{-1}(0_{11}^{(26)})$. Hence the result holds.

(4) See [1, Formula (5.10), pp. 44].

(5) By [1, Formula (7.19), pp. 71], we know $\nu_9\sigma_{12} = 2a\sigma_9\nu_{16}$ for some odd a. Notice $2\Sigma\pi_{38}^5 = 0$, ([7, Formula (9.25), pp. 146]). So, $\nu_6 \circ \nu_9\sigma_{12}\bar{\kappa}_{19} = 2a\nu_6\sigma_9\nu_{16}\bar{\kappa}_{19} \in 2\Sigma\pi_{38}^5 = 0$. By [7, Proposition 3.5 (5), pp. 60], we see that $2\bar{\sigma}_6\sigma_{25} = \nu_6\phi_9$. So, $\nu_6 \circ \phi_9\sigma_{32} = 2\bar{\sigma}_6\sigma_{25}\sigma_{32} = \bar{\sigma}_6 \circ 2\sigma_{25}\sigma_{32} = 0$.

(6) Notice that π_{37}^7 is given by [7, pp. 105] and $\Sigma^2\sigma' = 2\sigma_9$. Then the result follows from (5) of our lemma.

□

4.2 The element $\kappa_6' \in \pi_{39}^6$ and its order

Leveraging the groundwork laid out above, we can devise a more refined construction for the element $\kappa_6' \in \pi_{39}^6$, allowing for the precise determination of its order. And we are therefore able to solve Oda's extension problems for the homotopy groups π_{39}^6, π_{40}^7 and π_{41}^8.

Recall from [7, Formula (9.16), pp. 144] that the element $\bar{\nu}_{11}\bar{\kappa}_{19} \in \pi_{39}^{11}$ is of order 2.

Lemma 4.2.1 *The following rectangular Toda bracket*

$$T = \begin{bmatrix} \nu_6 & \eta_9 & \bar{\nu}_{10}\bar{\kappa}_{18} \\ 2\nu_6 & \nu_9\bar{\kappa}_{12} & \nu_{32}^2 \end{bmatrix}_2 \subseteq \pi_{39}^6$$

is well-defined. And $H(T) = \bar{\nu}_{11}\bar{\kappa}_{19} \neq 0$.

Proof. By Lemma 4.1.1 (1), we know $\eta_7\bar{\nu}_8\bar{\kappa}_{16} = \nu_7\bar{\kappa}_{10}\nu_{30}^2$; equivalently saying, $\eta_7\bar{\nu}_8\bar{\kappa}_{16} + \nu_7\bar{\kappa}_{10}\nu_{30}^2 = 0$. It is well-known that $\nu_6\eta_9 = 2\nu_6 \circ \nu_9 = 0$, ([1]). Then, by Definition 3.1, we know T is well-defined. Recall Proposition 3.1.4. By Lemma 4.1.1 (3) and (4), we have

$$\begin{aligned} H(T) &\subseteq H\begin{bmatrix} \nu_6 & \eta_9 & \bar{\nu}_{10}\bar{\kappa}_{18} \\ 2\nu_6 & \nu_9\bar{\kappa}_{12} & \nu_{32}^2 \end{bmatrix}_1 \\ &= -P^{-1}(\nu_5\eta_8) \circ \bar{\nu}_{11}\bar{\kappa}_{19} - P^{-1}(2\nu_5 \circ \nu_8\bar{\kappa}_{11}) \circ \nu_{33}^2 \\ &= \bar{\nu}_{11}\bar{\kappa}_{19} + P^{-1}(0_5^{(26)}) \circ \nu_{33}^2 \\ &= \bar{\nu}_{11}\bar{\kappa}_{19} \neq 0. \end{aligned}$$

□

Lemma 4.2.2 *Choosing $\kappa'_6 \in T$, we have $H(\kappa'_6) = \bar{\nu}_{11}\bar{\kappa}_{19}$, $2\kappa'_6 = 0$. Successively, κ'_6 is of order 2. All properties of the original κ'_6 defined in [7, pp. 145] satisfies, this new choice of κ'_6 still satisfies.*

Proof. By Lemma 4.2.1, we see that $H(\kappa'_6) = \bar{\nu}_{11}\bar{\kappa}_{19} \neq 0$ for $\kappa'_6 \in T$. Recall from [1] that $2\bar{\nu}_8 = 2\nu_{30}^2 = 0$. Then, by Proposition 3.1.3 and Lemma 4.1.1 (6), we derive

$$\begin{aligned}
2\kappa'_6 &\in \begin{bmatrix} \nu_6 & \eta_9 & \bar{\nu}_{10}\bar{\kappa}_{18} \\ 2\nu_6 & \nu_9\bar{\kappa}_{12} & \nu_{32}^2 \end{bmatrix}_2 \circ 2\iota_{39} \\
&\subseteq \begin{bmatrix} \nu_6 & \eta_9 & 2\bar{\nu}_{10}\bar{\kappa}_{18} \\ 2\nu_6 & \nu_9\bar{\kappa}_{12} & 2\nu_{32}^2 \end{bmatrix}_2 \\
&= \begin{bmatrix} \nu_6 & \eta_9 & 0_{10}^{(28)} \\ 2\nu_6 & \nu_9\bar{\kappa}_{12} & 0_{32}^{(29)} \end{bmatrix}_2 \\
&\subseteq \{\nu_6, \eta_9, 0_{10}^{(28)}\}_2 + \{2\nu_6, \nu_9\bar{\kappa}_{12}, 0_{32}^{(29)}\}_2. \\
&= \nu_6 \circ \Sigma^2\pi_{37}^7 \\
&= 0.
\end{aligned}$$

Thus, κ'_6 is of order 2. Notice the proof of [7, Formula (9.27), pp. 145-146]. For any element $\mathbb{x} \in \pi_{39}^6$ satisfying $H(\mathbb{x}) = \bar{\nu}_{11}\bar{\kappa}_{19}$, the original κ'_6 in [7] can be taken as $\kappa'_6 = \mathbb{x}$. Hence the result holds. □

Following [7], $\kappa'_{6+i} := \Sigma^i\kappa'_6$.

Recall the following facts on the determination of π_{39}^6 from [7]. The *EHP* sequence

$$\pi_{40}^{11} \xrightarrow{P} \pi_{38}^5 \xrightarrow{\Sigma} \pi_{39}^6 \xrightarrow{H} \pi_{39}^{11} \xrightarrow{P} \pi_{37}^5$$

induces a short exact sequence

$$0 \longrightarrow \Sigma\pi_{38}^5 \longrightarrow \pi_{39}^6 \xrightarrow{H} \mathrm{Im}(H) \longrightarrow 0,$$

where $$\Sigma\pi_{38}^5 = \mathrm{span}\{\nu_6\sigma_9\nu_{16}\bar{\kappa}_{19},\ \mu_{4,6},\ \eta_6\mu_{3,7}\sigma_{32}\} \approx (\mathbb{Z}/2)^3$$

and $$\mathrm{Im}(H) = \mathrm{span}\{C_1\omega_{23},\ \Sigma F_1^{(1)},\ \sigma_{11}^4,\ C_1^{(2)},\ \bar{\nu}_{11}\bar{\kappa}_{19}\} \approx (\mathbb{Z}/2)^5.$$

Moreover, $$H(P(\Sigma A_1 \circ \omega_{25})) = C_1\omega_{23},\ H(P(\Sigma A_1 \circ \sigma_{25}\mu_{32})) = \Sigma F_1^{(1)},$$

$$H(\bar{\sigma}_6\sigma_{25}^2) = \sigma_{11}^4 \text{ and } H(P(\Sigma A_1^{(2)})) \equiv C_1^{(2)} \bmod C_1\omega_{23},\ \Sigma F_1^{(1)},\ \sigma_{11}^4,\ \bar{\nu}_{11}\bar{\kappa}_{19}.$$

The four liftings $P(\Sigma A_1 \circ \omega_{25}), P(\Sigma A_1 \circ \sigma_{25}\mu_{32})$, $\bar{\sigma}_6\sigma_{25}^2$ and $P(\Sigma A_1^{(2)})$ are all of order 2.

By these facts, Lemma 4.2.2 together with [7, Proposition 9.20, pp. 144-145], we infer the following proposition, which solves the group extension problems left by Oda ([7]).

Proposition 4.2.1

$$\begin{aligned}
\pi_{39}^6 &= \mathrm{span}\{\kappa'_6,\ P(\Sigma A_1 \circ \omega_{25}),\ P(\Sigma A_1 \circ \sigma_{25}\mu_{32}),\ \bar{\sigma}_6\sigma_{25}^2, \\
&\qquad P(\Sigma A_1^{(2)}),\ \nu_6\sigma_9\nu_{16}\bar{\kappa}_{19},\ \mu_{4,6},\ \eta_6\mu_{3,7}\sigma_{32}\} \\
&\approx (\mathbb{Z}/2)^8, \\
\pi_{40}^7 &= \mathrm{span}\{\sigma'\eta_{14}\mu_{3,15},\ \kappa'_7,\ \bar{\sigma}_7\sigma_{26}^2,\ \nu_7\sigma_{10}\nu_{17}\bar{\kappa}_{20},\ \mu_{4,7},\ \eta_7\mu_{3,8}\sigma_{33}\} \\
&\approx (\mathbb{Z}/2)^6, \\
\pi_{41}^8 &= \mathrm{span}\{\sigma_8 \circ \Sigma^3\tau^{\mathrm{IV}},\ \sigma_8\eta_{15}\mu_{3,16},\ \sigma_8\nu_{15}^2\bar{\kappa}_{21},\ \Sigma\sigma' \circ \eta_{15}\mu_{3,16},\ \kappa'_8,\ \bar{\sigma}_8\sigma_{27}^2, \\
&\qquad \nu_8\sigma_{11}\nu_{18}\bar{\kappa}_{21},\ \mu_{4,8},\ \eta_8\mu_{3,9}\sigma_{34}\} \\
&\approx \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^8.
\end{aligned}$$

□

4.3 A problem

Suppose p is a prime and X (without taking the 2-localization) is simply-connected CW complex of finite type. After localization at p, let $r_n(X) = \dim(\pi_n(X) \otimes \mathbb{Z}/p)$, i.e., the dimension of $\pi_n(X) \otimes \mathbb{Z}/p$ as a $\mathbb{Z}/p$-vector space. It is clear that $r_n(X)$ is an invariant dependent solely on n and the homotopy type of X. And it is the count of non-trivial direct summands of $\pi_n(X)$. $^\sharp$

This invariant will afford formidable aid in addressing extension problems for homotopy groups, also assists in resolving the group extension problems left by spectral sequences of homotopy groups such as Adams spectral sequences. Notice that Oda's first extension problem can state as $r_{39}(S^6) = 7$ *or* 8 in the case $p = 2$.

Problem: How can we analyze the invariant $r_n(X)$ from some perspectives, whether geometric or algebraic? Alternatively, how might we construct a new theoretical framework to examine this invariant, thereby assisting in our exploration of extension problems for homotopy groups?

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$^\sharp$ Of course, the term " the count of nontrivial direct summands" refers to the count of nontrivial cyclic $\mathbb{Z}_{(p)}$-modules such that $\pi_n(X)$ is decomposed into a direct sum of these modules. For example, $M = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is regarded as having 2 but not 3 direct summands, despite $\langle (1,0) \rangle$, $\langle (0,1) \rangle$, and $\langle (1,1) \rangle$ each being its direct summands.