

On the Thom Spectra over  $\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))$  and Mahowald's  $X_k$  Spectra

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ON THE THOM SPECTRA OVER  $\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))$   
AND MAHOWALD'S  $X_k$  SPECTRA

DUNG YUNG YAN

(Communicated by Frederick R. Cohen)

ABSTRACT. The Thom spectra  $M(n)$  ( $2^k \leq n \leq 2^{k+1} - 1$ ) induced from  $\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)) \rightarrow BO$  is a wedge of suspensions of Mahowald's  $X_k$  spectra that is induced from  $\Omega J_{2^k-1} S^2 \rightarrow \Omega^2 S^3 \rightarrow BO$ , where  $J_i$  is the  $i$ th stage of the James construction.

Given a connected  $H$ -space  $L$  and an  $H$ -map  $f: L \rightarrow BO$ , the resulting Thom spectrum  $T(f)$  is a ring spectrum with a two-sided unit [1, 5]. If  $L$  and  $BO$  admit associating homotopies compatible under  $f$ , then  $T(f)$  is an associative ring spectrum, and if  $L$  has higher multiplicative structure compatible with  $BO$  under  $f$ , then  $T(f)$  has analogous structure in the multiplication of  $T(f)$ .

$T(f)$  is  $(-1)$ -connected and  $\pi_0(T(f))$  is either  $\mathbb{Z}$  or  $\mathbb{Z}/2$ . If  $f$  is nonorientable, i.e.,  $f^*(w_1) \neq 0$ , then  $\pi_0(T(f)) = \mathbb{Z}/2$ . Otherwise  $\pi_0(T(f)) = \mathbb{Z}$ .

For our purpose a ring spectrum is a spectrum with a multiplication that is associative and with a two-sided unit, but not necessarily commutative. Furthermore "a ring map" will mean "a map between two ring spectra that is multiplicative and carries the unit," otherwise "a map" even between two ring spectra is not necessarily multiplicative.

Let  $\eta: S^1 \rightarrow BO$  represent the generator of  $\pi_1(BO) = \mathbb{Z}/2$ . Since  $BO$  is a double loop space there is an induced map  $\gamma: \Omega^2 S^3 \rightarrow BO$ . Then one takes the composite map  $\Omega J_{2^k-1} S^2 \rightarrow \Omega^2 S^3 \rightarrow BO$ , where  $J_i$  is the  $i$ th stage of the James construction. These maps result in Thom spectra which will be denoted by  $X_k$  due to Mahowald [1, 3, 4, 5]. Also let

$$\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)) \rightarrow \Omega(\mathrm{SU}/\mathrm{SO}) = BO$$

be the usual inclusion map. These maps yield the Thom spectra  $M(n)$ . In the present note, we intend to show  $M(n)$  is a wedge of suspensions of  $X_k$  for  $2^k \leq n \leq 2^{k+1} - 1$ , which reflects a well-known splitting in the complex case originally due to Ravenel, elucidated by Hopkins [2]. An application of this paper will appear in [8].

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**Theorem 1.**  $M(n)$  is a wedge of suspensions of  $X_k$  for  $2^k \leq n \leq 2^{k+1} - 1$ .

From now on all unstated coefficient groups are  $\mathbb{Z}/2$ .  $A$  is the mod 2 dual Steenrod algebra, and let  $\xi_i \in A_{2^i-1}$  be the generator defined by Milnor. Then as an algebra  $A = \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots]$ , and the coproduct is determined by  $\Delta \xi_k = \sum_{i=0}^k \xi_{k-i}^{2^i} \otimes \xi_i$ . The Thom spectra  $M(n)$  and  $X_k$  are ring spectra, so one will know that  $H_*(M(n)), H_*(X_k)$  are (left)  $A$  comodule algebras in the following two propositions.

**Proposition 2** [1, 5]. As a subcomodule algebra of  $A$ , it follows that  $H_*(X_k) \cong \mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_k]$ .

There is a well-known inclusion  $\mathbb{R}P^{n-1} \rightarrow \Omega(\text{SU}(n)/\text{SO}(n))$  such that

$$H_*(\Omega(\text{SU}(n)/\text{SO}(n))) \cong \mathbb{Z}/2[c_1, c_2, \dots, c_{n-1}], \quad |c_i| = i,$$

$c_i$  is induced from the  $\mathbb{R}P^i$ ,  $1 \leq i \leq n - 1$ , and

$$H_*(\Omega(\text{SU}(n)/\text{SO}(n))) \rightarrow H_*(BO)$$

is injective. The Thom isomorphism yields

$$H_*(M(n)) \cong \mathbb{Z}/2[\bar{c}_1, \bar{c}_2, \dots, \bar{c}_{n-1}].$$

Since  $H_*(\Omega(\text{SU}(n)/\text{SO}(n)))$  injects in  $H_*(BO)$ ,  $H_*(M(n)) \rightarrow H_*(MO)$  is an injection. It is an isomorphism for  $* \leq n - 1$ . To get the mod two homology splitting of  $M(n)$  over the dual Steenrod algebra, recall the splitting of  $H_*(MO)$  over the dual Steenrod algebra

$$H_*(MO) \cong \mathbb{Z}/2[\xi_1, \xi_2, \dots] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\bar{\beta}_i / i \neq 2^l - 1], \quad |\bar{\beta}_i| = i,$$

and  $\bar{\beta}_i$  has the trivial coaction over the dual Steenrod algebra.

*Note.* We actually choose the above splitting from the following isomorphism (as comodule algebras)

$$\pi_*(MO) \otimes_{\mathbb{Z}/2} H_*(M(\gamma)) \xrightarrow{h_* \otimes \gamma_*} H_*(MO) \otimes_{\mathbb{Z}/2} H_*(MO) \xrightarrow{m} H_*(MO),$$

where  $h_*: \pi_*(MO) \rightarrow H_*(MO)$  is the mod 2 Hurewicz map,  $M(\gamma)$  is the Thom spectrum from  $\gamma: \Omega^2 S^3 \rightarrow BO$ ,  $\gamma_*: H_*(M(\gamma)) \rightarrow H_*(MO)$  is the map induced from  $\gamma: \Omega^2 S^3 \rightarrow BO$ , and  $m$  is the multiplication map.

**Proposition 3.**  $H_*(M(n)) \cong \mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_p] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\bar{y}_i / i \neq 2^l - 1]$ ,  $2^p \leq n$ ,  $i \leq n - 1$ ,  $|\bar{y}_i| = i$ , and  $\bar{y}_i$  has the trivial coaction over  $A$ .

*Proof.* Since  $H_*(M(n)) \rightarrow H_*(MO)$  is injective and isomorphic for  $* \leq n - 1$ , it follows that  $H_*(M(n))$  is a left  $\mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_p]$  comodule algebra from the splitting of  $H_*(MO)$ . Moreover it maps onto  $\mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_p]$ . Hence, by a theorem of Milnor and Moore [6],

$$H_*(M(n)) \cong \mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_p] \otimes_{\mathbb{Z}/2} \mathbf{C}$$

as left  $\mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_p]$  comodule algebras, where

$$\mathbf{C} = \mathbb{Z}/2 \square_{\mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_p]} H_*(M(n)).$$

An easy counting argument shows that  $\mathbf{C}$  must have the indicated form.  $\square$

Before going to the Thom splitting results, we would like to prove the following property.

**Proposition 4.**  $X_k$  and  $M(n)$  are noncommutative ring spectra for  $k \geq 1$  and  $n \geq 2$  respectively.

The proof uses the following result.

**Theorem 5** [2, 2.2.1]. *Let  $E$  be a commutative ring spectrum. If  $\pi_*(E)$  contains an invertible element of order two then  $E$  is weakly equivalent to a wedge of suspensions of Eilenberg-Mac Lane spectra.*

Also recall that  $X_k$  is taken to be the Thom spectrum induced from

$$\Omega J_{2^k-1} S^2 \rightarrow \Omega^2 S^3 \rightarrow BO,$$

and we know that  $\Omega^2 S^3 \rightarrow BO$  induces the mod 2 Eilenberg-Mac Lane spectrum  $H(\mathbb{Z}/2)$  [1, 4, 5], hence one has the ring map  $X_k \rightarrow H(\mathbb{Z}/2)$  that induces the natural embedding of mod 2 homology.

*Proof of Proposition 4.* Since the pull back of the first Stiefel-Whitney class is nonzero for  $\Omega J_{2^k-1} S^2 \rightarrow BO$  ( $k \geq 1$ ), then  $\pi_0(X_k) = \mathbb{Z}/2$ , i.e., the unit is the invertible element of order two. If  $X_k$  were a commutative ring spectrum, then  $X_k$  would be a wedge of suspensions of Eilenberg-Mac Lane spectra. So if the cohomology of  $X_k$  was a free module over the Steenrod algebra, then one would have the ring map  $X_k \rightarrow H(\mathbb{Z}/2)$ , which would induce cohomology injective, a contradiction. An analogous argument holds for  $M(n)$ .  $\square$

**Theorem 6.**  $M(2^{k+1} - 1)$  is a wedge of suspensions of  $X_k$ .

**Lemma 7.** *The usual inclusion  $\mathrm{SU}(n)/\mathrm{SO}(n) \rightarrow \mathrm{SU}(n+1)/\mathrm{SO}(n+1)$  induces an isomorphism on  $\pi_*$  for  $* \leq n-1$ , an epimorphism for  $* = n$  if  $n$  is odd and  $n \geq 3$ , and an isomorphism on  $\pi_*$  for  $* \leq n-2$ , an epimorphism for  $* = n-1$  if  $n \geq 3$ .*

To prove the lemma, we quote the classical Zeeman's comparison theorem [9]. A homomorphism between two spectral sequences (both are 1st quadrant spectral sequences). That is to say,

$$\begin{aligned} f_{p,q}^r &: E_{p,q}^r \rightarrow \bar{E}_{p,q}^r & (r = 2, 3, \dots, \infty \text{ and } -\infty < p, q < \infty), \\ f_n^A &: A_n \rightarrow \bar{A}_n & (n = 0, 1, \dots), \\ f_p^B &: B_p \rightarrow \bar{B}_p & (p = 0, 1, \dots), \\ f_q^C &: C_q \rightarrow \bar{C}_q & (q = 0, 1, \dots), \end{aligned}$$

where  $A_n = \bigoplus_{p+q=n} E_{p,q}^\infty$ ,  $B_p = E_{p,0}^2$ ,  $C_q = E_{0,q}^2$ , and the same definitions as for  $\bar{A}_n, \bar{B}_p, \bar{C}_q$ . Furthermore the spectral sequences satisfy the following commutative diagram with rows exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_{p,0}^2 \otimes E_{0,q}^2 & \longrightarrow & E_{p,q}^2 & \longrightarrow & \mathrm{Tor}(E_{p-1,0}^2, E_{0,q}^2) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bar{E}_{p,0}^2 \otimes \bar{E}_{0,q}^2 & \longrightarrow & \bar{E}_{p,q}^2 & \longrightarrow & \mathrm{Tor}(\bar{E}_{p-1,0}^2, \bar{E}_{0,q}^2) \longrightarrow 0 \end{array}$$

We say that  $f^A$  is an isomorphism up to dimension  $N$  if  $f_n^A$  is an isomorphism for each  $n$  ( $0 \leq n \leq N$ ). Then

**Theorem 8** [9]. *If  $f^A, f^C$  are isomorphisms up to dimensions  $N$  and  $Q$  respectively, then  $f^B$  is an isomorphism up to dimension  $P = \min(N, Q)$ . If further,  $f_{P+1}^A$  is onto, then  $f_{P+1}^B$  is onto.*

*Proof of Lemma 7.* Consider the following diagram with each row and each column a fibration:

$$\begin{array}{ccccc}
 \mathrm{SO}(n) & \longrightarrow & \mathrm{SO}(n+1) & \longrightarrow & \mathcal{S}^n \\
 \downarrow & & \downarrow & & \\
 \mathrm{SU}(n) & \longrightarrow & \mathrm{SU}(n+1) & \longrightarrow & \mathcal{S}^{2n+1} \\
 \downarrow & & \downarrow & & \\
 \mathrm{SU}(n)/\mathrm{SO}(n) & \longrightarrow & \mathrm{SU}(n+1)/\mathrm{SO}(n+1) & & 
 \end{array}$$

It follows from the fibrations that

$$H_*(\mathrm{SO}(n), \mathbb{Z}) \rightarrow H_*(\mathrm{SO}(n+1), \mathbb{Z})$$

is an isomorphism for  $* \leq n - 2$ , a surjection for  $* = n - 1$ , and

$$H_*(\mathrm{SU}(n), \mathbb{Z}) \rightarrow H_*(\mathrm{SU}(n+1), \mathbb{Z})$$

is an isomorphism for  $* \leq 2n - 1$ . In order to let Zeeman’s comparison work, we use the  $\mathbb{Z}/p$  coefficient for each prime  $p$  in the Serre spectral sequences. Then to prove the second statement is easy. Since  $n - 2 < n - 1 < 2n - 1$  for  $n \geq 3$ , from Theorem 8 the map  $\mathrm{SU}(n)/\mathrm{SO}(n) \rightarrow \mathrm{SU}(n+1)/\mathrm{SO}(n+1)$  induces an isomorphism on  $H_*(\ , \mathbb{Z}/p)$  for  $* \leq n - 2$  and a surjection on  $H_{n-1}(\ , \mathbb{Z}/p)$  for each prime  $p$ . Since the integral homology of  $\mathrm{SU}(n)/\mathrm{SO}(n)$  and  $\mathrm{SU}(n+1)/\mathrm{SO}(n+1)$  are of finite type, a trivial application of the mapping cylinder of

$$\mathrm{SU}(n)/\mathrm{SO}(n) \rightarrow \mathrm{SU}(n+1)/\mathrm{SO}(n+1),$$

one can show  $\mathrm{SU}(n)/\mathrm{SO}(n) \rightarrow \mathrm{SU}(n+1)/\mathrm{SO}(n+1)$  induces an isomorphism on  $H_*(\ , \mathbb{Z})$  for  $* \leq n - 2$  and a surjection for  $* = n - 1$ . Hence by Hurewicz’s theorem the map induces an isomorphism on  $\pi_*$  for  $* \leq n - 2$  and a surjection for  $* = n - 1$ . This proves the second statement. In the first statement, if one can show  $H_{n-1}(\mathrm{SO}(n), \mathbb{Z}) \rightarrow H_{n-1}(\mathrm{SO}(n+1), \mathbb{Z})$  is an isomorphism when  $n$  is odd and  $n \geq 3$ , i.e.,  $H_*(\mathrm{SO}(n), \mathbb{Z}) \rightarrow H_*(\mathrm{SO}(n+1), \mathbb{Z})$  is an isomorphism for  $* \leq n - 1$ , then by Theorem 8 again the map  $\mathrm{SU}(n)/\mathrm{SO}(n) \rightarrow \mathrm{SU}(n+1)/\mathrm{SO}(n+1)$  induces an isomorphism on  $H_*(\ , \mathbb{Z}/p)$  for  $* \leq n - 1$  and a surjection on  $H_n(\ , \mathbb{Z}/p)$  for each prime  $p$  since  $n - 1 < n < 2n - 1$ . Therefore, with an argument analogous to the one above we can prove the first statement. So it remains to prove that  $H_{n-1}(\mathrm{SO}(n), \mathbb{Z}) \rightarrow H_{n-1}(\mathrm{SO}(n+1), \mathbb{Z})$  is an isomorphism when  $n$  is odd. In the following paragraph we will use the integral coefficients.

By the Whitehead theorem and the long exact sequence of the pair  $(\mathrm{SO}(n+1), \mathrm{SO}(n))$  one has

$$H_n(\mathrm{SO}(n+1), \mathrm{SO}(n)) \xrightarrow{\partial} H_{n-1}(\mathrm{SO}(n)) \rightarrow H_{n-1}(\mathrm{SO}(n+1)) \rightarrow 0.$$

Let  $\mathrm{SO}^n(n+1) = K^n \cup \mathrm{SO}(n)$  where  $K^n$  is the  $n$ -skeleton of  $\mathrm{SO}(n+1)$ . Then for the triple  $(\mathrm{SO}(n+1), \mathrm{SO}^n(n+1), \mathrm{SO}(n))$  we also have a long exact sequence

$$\begin{aligned} \rightarrow H_n(\mathrm{SO}^n(n+1), \mathrm{SO}(n)) &\rightarrow H_n(\mathrm{SO}(n+1), \mathrm{SO}(n)) \\ &\rightarrow H_n(\mathrm{SO}(n+1), \mathrm{SO}^n(n+1)) \rightarrow . \end{aligned}$$

From the cellular homology,  $H_n(\mathrm{SO}(n+1), \mathrm{SO}^n(n+1)) = 0$ . So the first map is surjective. Again from the cellular homology, the well-known CW-decomposition of  $\mathrm{SO}(n+1) = \mathrm{O}(n+1)/\mathrm{O}(1)$  [7], and the fact that  $n$  is odd ( $n \geq 3$ ), one has

$$\begin{array}{ccc} \mathbb{Z} \cong H_n(\mathbb{R}P_+^n, \mathbb{R}P_+^{n-1}) & \xrightarrow{\cong} & H_n(\mathrm{SO}^n(n+1), \mathrm{SO}(n)) \cong \mathbb{Z} \\ & \searrow & \downarrow \text{onto} \\ & & H_n(\mathrm{SO}(n+1), \mathrm{SO}(n)) \end{array}$$

Hence

$$\begin{array}{ccc} H_n(\mathbb{R}P_+^n, \mathbb{R}P_+^{n-1}) & \xrightarrow{\text{onto}} & H_n(\mathrm{SO}(n+1), \mathrm{SO}(n)) \\ \downarrow & & \downarrow \partial \\ H_{n-1}(\mathbb{R}P_+^{n-1}) & \longrightarrow & H_{n-1}(\mathrm{SO}(n)) \end{array}$$

But  $H_{n-1}(\mathbb{R}P_+^{n-1}) = 0$  since  $n$  is odd and  $n \geq 3$ . So  $\partial = 0$ , therefore

$$H_{n-1}(\mathrm{SO}(n)) \xrightarrow{\cong} H_{n-1}(\mathrm{SO}(n+1)).$$

This completes the proof.  $\square$

**Lemma 9.** *There is a ring map of ring spectra  $f: X_k \rightarrow M(2^{k+1} - 1)$  such that  $H_*(X_k)$  is isomorphic onto  $\mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_k]$  of*

$$H_*(M(2^{k+1} - 1)) \cong \mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_k] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\bar{y}_i \mid i \neq 2^l - 1]$$

under  $f_*$ , that is,  $f_*$  projects the mod 2 homology of  $X_k$  onto a wedge summand of the mod 2 homology of  $M(2^{k+1} - 1)$ .

*Proof.* Using Lemma 7, it is easy to see  $\mathrm{SU}(2^{k+1} - 1)/\mathrm{SO}(2^{k+1} - 1) \rightarrow \mathrm{SU}/\mathrm{SO}$  induces an isomorphism on  $\pi_*$  for  $* \leq 2^{k+1} - 2$  and an epimorphism for  $* = 2^{k+1} - 1$  when  $k \geq 1$ . It is known that the CW-complex  $J_{2^k-1}S^2$  has dimension  $2^{k+1} - 2$ , hence we have the following lifting (up to homotopy).

$$\begin{array}{ccc} \mathrm{SU}(2^{k+1} - 1)/\mathrm{SO}(2^{k+1} - 1) & & \\ \nearrow f & & \downarrow \\ J_{2^k-1}S^2 & \longrightarrow & \mathrm{SU}/\mathrm{SO} \end{array}$$

Looping the diagram and then Thomifying the resulting diagram, one has

$$\begin{array}{ccc} M(2^{k+1} - 1) & & \\ \nearrow f & & \downarrow \\ X_k & \longrightarrow & MO \end{array}$$

This ring map  $f: X_k \rightarrow M(2^{k+1} - 1)$  is the desired one. Since each map in the above diagram is a ring map and  $H_*(X_k) \rightarrow H_*(MO)$ ,  $H_*(M(2^{k+1} - 1)) \rightarrow H_*(MO)$  are injective, it follows that  $f_*$  maps  $H_*(X_k)$  onto  $\mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_k]$ . This completes the proof.  $\square$

*Proof of Theorem 6.* Since

$$\pi_0(M(2^{k+1} - 1)) = \pi_0(MO) = \mathbb{Z}/2,$$

not only are  $M(2^{k+1} - 1)$  and  $MO$  2-local but also  $\pi_*(M(2^{k+1} - 1))$ ,  $\pi_*(MO)$  are of characteristic 2. And the natural ring map  $M(2^{k+1} - 1) \rightarrow MO$  induces an isomorphism on  $H_*(\ , \mathbb{Z}/2)$  for  $* \leq 2^{k+1} - 2$ . So  $\pi_*(M(2^{k+1} - 1)) \rightarrow \pi_*(MO)$  is an epimorphism for  $* \leq 2^{k+1} - 2$ . Therefore, in the splitting

$$H_*(M(2^{k+1} - 1)) \cong \mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_k] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\bar{y}_2, \bar{y}_4, \dots, \bar{y}_{2^{k+1}-2}],$$

each polynomial generator in  $\mathbb{Z}/2[\bar{y}_2, \bar{y}_4, \dots, \bar{y}_{2^{k+1}-2}]$  is a mod 2 Hurewicz image. Finally, construct ring spectra  $L_{2^{k+1}-1}$  out of suitable wedges of spheres, satisfying

$$H_*(L_{2^{k+1}-1}) \cong \mathbb{Z}/2[\bar{y}_2, \bar{y}_4, \dots, \bar{y}_{2^{k+1}-2}].$$

Using Lemma 9, one has the map

$$X_k \wedge L_{2^{k+1}-1} \rightarrow M(2^{k+1} - 1) \wedge M(2^{k+1} - 1) \rightarrow M(2^{k+1} - 1),$$

where the last map is the multiplication map. By the construction, it follows that the above map induces isomorphism on mod 2 homology, hence, by the Hurewicz theorem, the map is an equivalence since  $X_k \wedge L_{2^{k+1}-1}$ ,  $M(2^{k+1} - 1)$  are 2-local. This completes the proof.  $\square$

*Proof of Theorem 1.* Again, construct a ring spectra  $L_n$  out of suitable wedges of spheres, satisfying

$$H_*(L_n) \cong \mathbb{Z}/2[\bar{y}_i \mid i \neq 2^l - 1, 1 \leq i \leq n - 1], \quad 2^k \leq n \leq 2^{k+1} - 1.$$

Obviously, we have a map  $g_n: L_{2^{k+1}-1} \rightarrow L_n$ . Then using Theorem 6, one yields a map

$$M(n) \rightarrow M(2^{k+1} - 1) \rightarrow X_k \wedge L_{2^{k+1}-1} \xrightarrow{1 \wedge g_n} X_k \wedge L_n,$$

the first map is from the natural inclusion

$$\Omega(\text{SU}(n)/\text{SO}(n)) \rightarrow \Omega(\text{SU}(2^{k+1} - 1)/\text{SO}(2^{k+1} - 1)).$$

It is trivial to check the mod 2 homology is an isomorphism, hence an equivalence.  $\square$

We finish the paper by pointing out a remark.

*Remark 10.* Under the splitting of Theorem 6, i.e.,  $M(2^{k+1} - 1)$  is a wedge of suspensions of  $X_k$ , one has  $X_k \xrightarrow{f} M(2^{k+1} - 1) \xrightarrow{P} X_k$  and  $Pf \simeq 1$ , where  $f$  is the ring map in Lemma 9. Then

(a)  $P$  carries the unit.

(b) The following diagram commutes

$$\begin{array}{ccccc} X_k & \xrightarrow{f} & M(2^{k+1} - 1) & \xrightarrow{P} & X_k \\ \downarrow & & \downarrow & & \downarrow \\ H(\mathbb{Z}/2) & \longrightarrow & MO & \longrightarrow & H(\mathbb{Z}/2) \end{array}$$

where  $MO \rightarrow H(\mathbb{Z}/2)$  is the Thom class.

(c)  $P_*: H_*(M(2^{k+1} - 1)) \rightarrow H_*(X_k)$  is a ring map.

*Proof.* (a) is obvious. For (b), the first commutative diagram is induced from

$$\begin{array}{ccc} \Omega J_{2^k-1} S^2 & \longrightarrow & \Omega(\mathrm{SU}(2^{k+1} - 1)/\mathrm{SO}(2^{k+1} - 1)) \\ \downarrow & & \downarrow \\ \Omega^2 S^3 & \longrightarrow & BO \end{array}$$

This diagram is actually induced from the diagram in the proof of Lemma 9. The second commutative diagram is due to

$$\begin{aligned} H^o(M(2^{k+1} - 1)) &\cong [M(2^{k+1} - 1), H(\mathbb{Z}/2)]_0 \\ &\cong \mathrm{Hom}_{\mathbb{Z}/2}^o(H_*(M(2^{k+1} - 1)), \mathbb{Z}/2), \mathbb{Z}/2 \cong \mathbb{Z}/2, \end{aligned}$$

and both maps from  $M(2^{k+1} - 1)$  to  $H(\mathbb{Z}/2)$  are essential maps. To prove (c), from (b), one knows the map

$$H_*(M(2^{k+1} - 1)) \xrightarrow{P_*} H_*(X_k) \rightarrow H_*(H(\mathbb{Z}/2))$$

is a ring map. But the second map is the natural embedding. So  $P_*$  is a ring map.  $\square$

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