A LOCALIZATION THEOREM FOR EQUIVARIANT SPECTRA

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In this note, we generalize the localization theorem (Theorem C) in [Kri]. Her original theorem states that for a finite group G, a faithful orientable representation V, and any Eilenberg-MacLane spectrum $H\underline{M}$, the function spectrum $F(EG_+, H\underline{M}) \simeq H\underline{M}[u_V^{-1}]$, where $u_V \in H\underline{\mathbb{A}}^G_{|V|-V}$ is an orientation class, and we are using the $H\underline{\mathbb{A}}$-modules structure on $H\underline{M}$.

We observe that the condition on the faithfulness on the representation V in her theorem can be removed if we allow function spectrum out of the universal space of some general family beyond the family $\{e\}$. Also the Eilenberg-MacLane spectra can be replaced by bounded-above spectra. The author believes Kriz knows this generalizaiton, and we will also use a key observation of hers. We will provide a spectrum-level proof of the generalization in this paper that the author feels is conceptually clearer, rather than mimicking the chain-level argument that was used in her paper. Another motivation for the theorem is from the author's computation of the $RO(C_{2^n})$-graded homotopy groups of $H\underline{\mathbb{Z}}$ using generalized Tate squares.

The generalized localization theorem has potential applications in the slice spectral sequences of $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}, N_{C_2}^{C_{2^n}} BP_{\mathbb{R}}$ and variants of them. For example, it provides a comparison between the E_2-terms of the slice spectral sequence (SSS) and the $\mathscr{F}$-completed SSS, as well as a comparision between the E_2-terms of the $\mathscr{F}$-localized SSS and the $\mathscr{F}$-Tate construction of the SSS. Of course, it applies more generally to the $\mathscr{F}$-Tate square of any tower of spectra, as long as we can find a suitable notion of orientation classes there.

Let $\tau_{\leqslant n}E$ be the Postnikov section of a G-spectrum E, and let $X^{(m)}$ be the m-skeleton of X when X is a G-CW complex or G-CW spectrum.

Lemma 1. *Let $X = \tau_{\leqslant n}X$ be a bounded-above G-spectrum, and Y a finite G-CW spectrum with top cells in dimension m, then $X \wedge Y = \tau_{\leqslant n+m}(X \wedge Y)$ is bounded-above.*

Proof. The cellular filtration of Y gives a filtration of $X \wedge Y$, with filtration quotients $X \wedge Y^{(i)}/Y^{(i-1)} = X \wedge (\vee_{I_i} S^i), i \leqslant m$. Now we have a strongly convergent spectral sequence

$$E^1_{s,t} = \pi_t(X \wedge Y^{(s)}/Y^{(s-1)}) \Rightarrow \pi_t(X \wedge Y)$$

with E^1-term vanishing for $t > m + n$, which proves the lemma. □

Theorem 2. *Let G be a finite group, V a real G-representation, E a bounded-above G-spectrum, say $E = \tau_{\leqslant b}E$. Then the map $E \simeq F(S^0, E) \to F(E\mathscr{F}_{V+}, E)$ induces isomorphisms*

$$\pi_{n|V|-nV+b+j}E \xrightarrow{\cong} \pi_{n|V|-nV+b+j}F(E\mathscr{F}_{V+}, E)$$

for $j \geqslant 2-n$. Here $n \geqslant 0$, $|V| = \dim_{\mathbb{R}} V$, and $\mathscr{F}_V$ is the family $\{H \subset G | i_H^*(V) = |V|\}$.

Proof. Given a real representation V of G, let

$$\mathscr{F}_V = \{H \subset G | i_H^*(V) = |V|\}.$$

where $|V|$ is regared as the trivial representation with dimension $|V|$. It is easily checked that $\mathscr{F}_V$ is a family of subgroups. By [GM95], we have the following generalized Tate square for $\mathscr{F}_V$, which is a homotopy pullback of commutative ring spectra if E is.

$$\begin{array}{ccc} E & \longrightarrow & \widetilde{E\mathscr{F}_V} \wedge E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ F(E\mathscr{F}_{V+}, E) & \longrightarrow & \widetilde{E\mathscr{F}_V} \wedge F(E\mathscr{F}_{V+}, E) \end{array} \quad . \tag{1}$$

Let F be the fiber of the left vertical map $f : E \to F(E\mathscr{F}_{V+}, E)$, then F is also the fiber of the right vertical map since it is a homotopy pullback square. The key properties of F are that

(1) F is bounded-above, more precisely, $F = \tau_{\leqslant b} F$;
(2) F is $\widetilde{E\mathscr{F}_V}$-local, in the sense that the canonical map $F \to \widetilde{E\mathscr{F}_V} \wedge F$ is an equivalence.

(1) is an easy consequence of the long exact sequence of the homotopy groups of the fiber sequence

$$F \to E \to F(E\mathscr{F}_{V+}, E),$$

once we know $F(E\mathscr{F}_{V+}, E) = \tau_{\leqslant b} F(E\mathscr{F}_{V+}, E)$. This is the case since we have an Atiyah-Hirzebruch spectral sequence

$$H^s(E\mathscr{F}_V; \underline{\pi_t} E) \Rightarrow \pi_{t-s} F(E\mathscr{F}_{V+}, E) \tag{2}$$

and as a space, $E\mathscr{F}_V$ does not have negative cohomology. (2) is a consequence of F being the fiber of two $\widetilde{E\mathscr{F}_V}$-local spectra, using the fact that for any G-spectrum X, $\widetilde{E\mathscr{F}_V} \wedge X$ is $\widetilde{E\mathscr{F}_V}$-local by the equivalence of G-spaces

$$\widetilde{E\mathscr{F}_V} \wedge \widetilde{E\mathscr{F}_V} \simeq \widetilde{E\mathscr{F}_V}.$$

Now the choice of $\mathscr{F}_V$ is that any $K \in \mathscr{F}_V$ acts trivally on V. Now let $H \notin \mathscr{F}_V$, then $V^H \subsetneq V$, and $S^{V^H} \subsetneq S^V$ is a subcomplex of a lower dimension. Thus the top cells ($|V|$-cells) of S^V can only have orbit type G/H for $H \in \mathscr{F}_V$. Now we use the product cell structure on S^{nV} and we can see that the cells of dimension i for $n(|V|-1) < i \leqslant n|V|$ all have orbit type G/H for $H \in \mathscr{F}_V$.

Now the cellular filtration makes sure the inclusion of skeleton

$$(S^{nV})^{(n|V|-n)} \xhookrightarrow{i_n} S^{nV}$$

can be factored into n maps with subquotients $\vee G/H_{i_+} \wedge S^j, H_i \in \mathscr{F}_V$. Since F is $\widetilde{E\mathscr{F}_V}$-local, $G/H_{i_+} \wedge S^j \wedge F \simeq *$. Thus the map i_n induces an equivalence

$$(S^{nV})^{(n|V|-n)} \wedge F \xrightarrow{i_n \wedge 1} S^{nV} \wedge F. \tag{3}$$

By Lemma 1, since $(S^{nV})^{(n|V|-n)}$ is a finite G-CW complex of dimension $n|V|-n$, we know $(S^{nV})^{(n|V|-n)} \wedge F$ is bounded-above, with homotopy groups π_m vanishing for $m > n|V|-n+b$. Combining with (3), we get

$$\pi_m S^{nV} \wedge F = 0 \quad \text{when} \quad m \geqslant n|V|-n+b+1.$$

Using the cofiber sequence

$$S^{nV} \wedge F \to S^{nV} \wedge E \xrightarrow{1\wedge f} S^{nV} \wedge F(E\mathscr{F}_{V+}, E),$$

we know that the map $1 \wedge f$ induce isomorphisms on homotopy groups π_m for $m \geqslant n|V|-n+b+2$. □

Example 1. *(1) All Eilenberg-MacLane spectra $H\underline{M}$, or more generally, bounded-above graded Eilenberg-MacLane spectra.*

(2) The slice sections P^nX and slices P^n_nX for any $n \in \mathbb{Z}$ and any G-spectrum X, see [HHR16, Thm 4.42].
(3) The Postnikov sections $\tau_{\leqslant n}X$ for any G-spectrum X.

Remark 1. *The above argument actually works for all families $\mathscr{F}$ such that $\mathscr{F} \supset \mathscr{F}_V$.*

Corollary 3. *If a G-ring spectrum F has an orientation class $u_V \in \pi_{|V|-V}F$, and E is a F-module, then $F(E\mathscr{F}_{V+}, E)$ is u_V-local. If further E is bounded-above, then $f : E \to F(E\mathscr{F}_{V+}, E)$ in (1) induces an equivalence*

$$F(E\mathscr{F}_{V+}, E) \simeq E[u_V^{-1}].$$

Moreover, the map $g : \widetilde{E\mathscr{F}_V} \wedge E \to \widetilde{E\mathscr{F}_V} \wedge F(E\mathscr{F}_{V+}, E)$ in (1) is also inverting u_V.

Proof. The spectrum $F(E\mathscr{F}_{V+}, E)$ is u_V-local, since $E\mathscr{F}_{V+} = hocolim_m\, E\mathscr{F}_{V}{}_+^{(m)}$ with filtration quotients $\vee G/H_+ \wedge S^n, H \in \mathscr{F}_V$, thus

$$F(E\mathscr{F}_{V+}, E) = holim_m\, F(E\mathscr{F}_{V}{}_+^{(m)}, E)$$

with filtration quotients $F(\vee G/H_+ \wedge S^n, E) \simeq \vee G/H_+ \wedge S^{-n} \wedge E, H \in \mathscr{F}_V$, by the self-duality of orbits and the fact that $E\mathscr{F}_{V}{}_+^{(m)}$ can be constructed as a finite G-CW complex. By induction, we only need to prove that multiplication by u_V induces equivalences on the filtration quotients. But this is indeed true by the shearing isomorphism

$$G/H_+ \wedge S^{-n} \wedge E \xrightarrow{\simeq} S^{V-|V|} \wedge G/H_+ \wedge S^{-n} \wedge E.$$

Now Theorem 2 implies that

$$\pi_k E[u_V^{-1}] = colim_n \pi_{k+n|V|-nV} E \cong colim_n \pi_{k+n|V|-nV} F(E\mathscr{F}_{V+}, E) = \pi_k F(E\mathscr{F}_{V+}, E)$$

where the colimit is over multiplications by powers of u_V. Since this is also true when we restrict to subgroups $K \subset G$, we actually get an equivalence of homotopy Mackey functors of both sides, and the equivalence

$$F(E\mathscr{F}_{V+}, E) \simeq E[u_V^{-1}]$$

is proved.

Now for for the last statement, we have the following commutative square with the vertical maps in the middle an equivalence.

$$\begin{array}{ccccc}
(E\mathscr{F}_{V+}\wedge E)[u_V^{-1}] & \longrightarrow & E[u_V^{-1}] & \longrightarrow & (\widetilde{E\mathscr{F}_V}\wedge E)[u_V^{-1}] \\
\simeq\downarrow & & \simeq\downarrow & & \downarrow \\
E\mathscr{F}_{V+}\wedge E & \longrightarrow & F(E\mathscr{F}_{V+},E) & \longrightarrow & \widetilde{E\mathscr{F}_V}\wedge F(E\mathscr{F}_{V+},E)
\end{array}.$$

The left vertical map is an equivalence since $E\mathscr{F}_{V+}\wedge E$ is already u_V-local. We deduce the rightmost vertical map is an equivalence. □

Remark 2. *For a bounded-above spectrum E, the function spectrum $F(E\mathscr{F}_+,E)$ is again bounded-above by the same bound. So the localization process can be iterated. If the G-ring spectrum F has two orientation classes u_V,u_W, then we will get*

$$F(E(\mathscr{F}_V\cap\mathscr{F}_W)_+,E)\simeq F(E\mathscr{F}_{V+}\wedge E\mathscr{F}_{W+},E)\simeq F(E\mathscr{F}_{V+},F(E\mathscr{F}_{W+},E))\simeq E[u_V^{-1},u_W^{-1}].$$

Example 2. *(1) $G=C_n$ a cyclic group, $E=H\underline{\mathbb{A}}$. All irreducibles V are rotations of the plane. If G_V is the stablizer of V, then $u_V=res^G_{G_V}:\pi_0H\underline{\mathbb{A}}\to\pi_{2-V}H\underline{\mathbb{A}}=\mathbb{A}(G/G_V)$, and u_V can be taken to be the image of 1. This can be generalized to all finite groups, since we still have $H\mathbb{A}_{|V|-V}=\mathbb{A}(G/G_V)$.*

(2) G any finite group, E is the postnikov section of any complex orientable cohomolgy theory, like $\tau_{\leqslant n}MU_G,\tau_{\leqslant n}KU_G,\tau_{\leqslant n}kU_G$.

(3) More concretely, take $F=KU_G$ and $E=\tau_{\leqslant n}KU_G$. For a complex G-representation V, we have the orientation class u_V, and thus $F(E\mathscr{F}_{V+},\tau_{\leqslant n}KU_G)\simeq\tau_{\leqslant n}KU_G[u_V^{-1}]$. Taking inverse limit with respect to n, we get

$$F(E\mathscr{F}_{V+},KU_G)\simeq holim_n(\tau_{\leqslant n}KU_G[u_V^{-1}]).$$

This formula also holds for other complex-oriented G-spectra.

Remark 3. *The theorem cannot be extended to a general G-spectrum. Take $G=C_2,V=2\sigma$ with σ the real sign representation, ,$E=KU_{C_2}$, then $E\mathscr{F}_V=EC_2$, and we know $KU_{C_2}[u_{2\sigma}^{-1}]\simeq KU_{C_2}$ is not equivalent to $F(EC_{2+},KU_{C_2})$.*

References

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