THE STRONG KERVAIRE INVARIANT PROBLEM IN DIMENSION 62

ZHOULI XU

Abstract. Using a Toda bracket computation $\langle\theta_4, 2, \sigma^2\rangle$ due to Daniel C. Isaksen [12], we investigate the 45-stem more thoroughly. We prove that $\theta_4^2 = 0$ using a 4-fold Toda bracket. By [2], this implies that θ_5 exists and there exists a θ_5 such that $2\theta_5 = 0$. Based on $\theta_4^2 = 0$, we simplify significantly the 9-cell complex construction in [1] to a 4-cell complex, which leads to another proof that θ_5 exists.

1. Introduction and main results

The Kervaire invariant problem is one of the most interesting problems that relates geometric topology and stable homotopy theory. One way of formulating it, due to Browder [5], is in terms of the classical Adams spectral sequence (ASS) at the prime 2:

For each n, the element $h_n^2 \in Ext^{2,2^{n+1}-2}$ survives in the ASS.

If h_n^2 survives, we denote the corresponding detecting elements in homotopy by $\theta_n \in \pi_{2^{n+1}-2}S^0$ and we say that θ_n exists. The strong Kervaire invariant problem for n is the following.

θ_n exists, and there exists a θ_n such that $2\theta_n = 0$.

It is well-known that the first three Kervaire invariant elements θ_1, θ_2 and θ_3 can be chosen to be η^2, ν^2 and σ^2. And they all have order 2. Mahowald and Tangora [17] showed that θ_4 exists and $2\theta_4 = 0$ by an ASS computation. In [1], Barratt, Jones and Mahowald showed that θ_5 exists by constructing a 9-cell complex and using the Peterson-Stein formula. Recently, using equivariant homotopy technology, Hill, Hopkins and Ravenel [10] in their marvelous paper showed that θ_n does not exist for all $n \geq 7$, which left the existence of θ_6 as the only open case.

In [2], Barratt, Jones and Mahowald gave the following inductive approach to the strong Kervaire invariant problem:

Theorem 1.1. *Suppose that there exists an element θ_n such that $2\theta_n = 0$ and $\theta_n^2 = 0$. Then there exists an element θ_{n+1} with $2\theta_{n+1} = 0$.*

In this paper, we prove the following:

Theorem 1.2. $\theta_4^2 = 0$.

Since θ_4 is unique and $2\theta_4 = 0$, we have the following corollary:

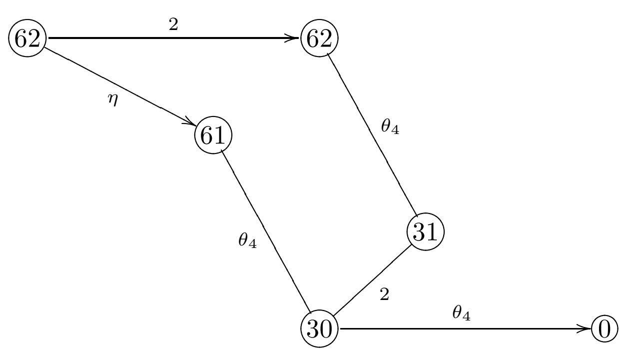
Corollary 1.3. *θ_5 exists and there exists a θ_5 such that $2\theta_5 = 0$.*

Remark 1.4. In [19], R. J. Milgram claims to show that under the same condition as in Theorem 1.1, one has θ_{n+2} exists. If this were true, then we would have that θ_6 exists. However, Milgram's argument fails because of a computational mistake [8].

Remark 1.5. Note that if one can further prove that the same θ_5 has the property $\theta_5^2 = 0$, then Theorem 1.1 will imply the open case θ_6 exists and that there exists a θ_6 such that $2\theta_6 = 0$.

For the case θ_5, Lin [16] shows that there exists a θ_5 such that $2\theta_5 = 0$ based on a computation of the Toda bracket $\langle \theta_4, 2, \sigma^2 \rangle$. Based on the same Toda bracket but a different computational result, Kochman [13] also shows that $\theta_4^2 = 0$ and hence that there exists a θ_5 such that $2\theta_5 = 0$. Recently, Isaksen [12] computed this Toda bracket using more straightforward arguments. His result contradicts the results of both Lin and Kochman. For more details about where Lin and Kochman's arguments fails, see Remark 3.4. Our proof uses Isaksen's computation. Since Isaksen's computation of $\langle \theta_4, 2, \sigma^2 \rangle$ gives a more complicated answer than the earlier claims, we must study several other Toda brackets to prove $\theta_4^2 = 0$.

Knowing $\theta_4^2 = 0$, we give a second proof of the existence of θ_5. In [1], Barratt, Jones and Mahowald constructed a 9-cell complex X', and maps $f' : S^{62} \to X'$, $g' : X' \to S^0$, such that the composite $g' \circ f' : S^{62} \to S^0$ realizes a θ_5. We simplify this 9-cell complex X' into a 4-cell complex X, and construct maps $f : S^{62} \to X$, $g : X \to S^0$ as indicated in the following cell diagram. We follow Barratt, Jones and Mahowald's notation of cell diagrams.



Here each circle represents a cell. The number in each circle represents the dimension of that cell. The middle 4 cells represent the cell structure of X, where the three lines without arrow heads represent attaching maps of X. The map g is an extension of θ_4, and the map f is a co-extension of $\eta \vee 2$. In other words, if we restrict the map g on the bottom cell of X: $g|_{S^{30}} : S^{30} \to S^0$, we have θ_4. If we pinch down the 31-skeleton of X: $p : X \to S^{61} \vee S^{62}$, then the composite $p \circ f : S^{62} \to S^{61} \vee S^{62}$ is $\eta \vee 2$. For more details about cell diagrams, see [1].

Theorem 1.6. *The composite of maps $g \circ f : S^{62} \to S^0$ realizes a θ_5.*

Proof. We first show that we can form this cell diagram. For primary obstructions, we have $2\theta_4 = 0$ and $\theta_4^2 = 0$. For secondary obstructions, we have $\eta\theta_4 \in \langle 2, \theta_4, 2 \rangle$

and $0 \in \langle \theta_4, 2, \theta_4 \rangle$. The latter is shown in [1]. It is straightforward to check that the following two facts are true: for $i \leq 4$ the functional cohomology operations

$$Sq_g^{2^i} : H^0 S^0 \longrightarrow H^{2^i - 1} X$$

are all zero, while $Sq_g^{32} : H^0 S^0 \to H^{31} X$ is nonzero; the functional cohomology operation Sq_f^{32} is nonzero on $Sq_g^{32} H^0 S^0 = H^{31} X$. Note that all cohomology is understood to have mod 2 coefficients. As used in [1], it follows from the Peterson-Stein formula ([20],[22]) that the composite $g \circ f$ is detected by the secondary cohomology operation $\phi_{5,5}$. Therefore $g \circ f$ realizes a θ_5. □

We present the proof of Theorem 1.2 in Section 2. The proof uses several theorems and lemmas whose proofs we postpone. We include Isaksen's computation of $\langle \theta_4, 2, \sigma^2 \rangle$ in Section 3 for completeness. In Section 4, we discuss two more Toda brackets in the 45-stem, namely $\langle \theta_4, 2, \kappa \rangle$ and $\langle \theta_4, 2, \sigma^2 + \kappa \rangle$. The proof of the main theorem depends on the computation of the latter bracket. We give a modified 4-fold Toda bracket for θ_4 in Section 5. We complete our proof of the main theorem by proving several lemmas in Section 6.

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2. The proof of the main theorem

We will use the following Toda brackets to prove Theorem 1.2.

Theorem 2.1. *$\langle \theta_4, 2, \sigma^2 + \kappa \rangle$ contains 0 with indeterminacy $\{0, \rho_{15}\theta_4\}$.*

Theorem 2.2. *$\theta_4 = \langle 2, \sigma^2 + \kappa, 2\sigma, \sigma \rangle$ with zero indeterminacy.*

Lemma 2.3. *$\sigma \pi_{53} = 0$.*

Lemma 2.4. *$\langle \rho_{15}\theta_4, 2\sigma, \sigma \rangle = 0$ with zero indeterminacy.*

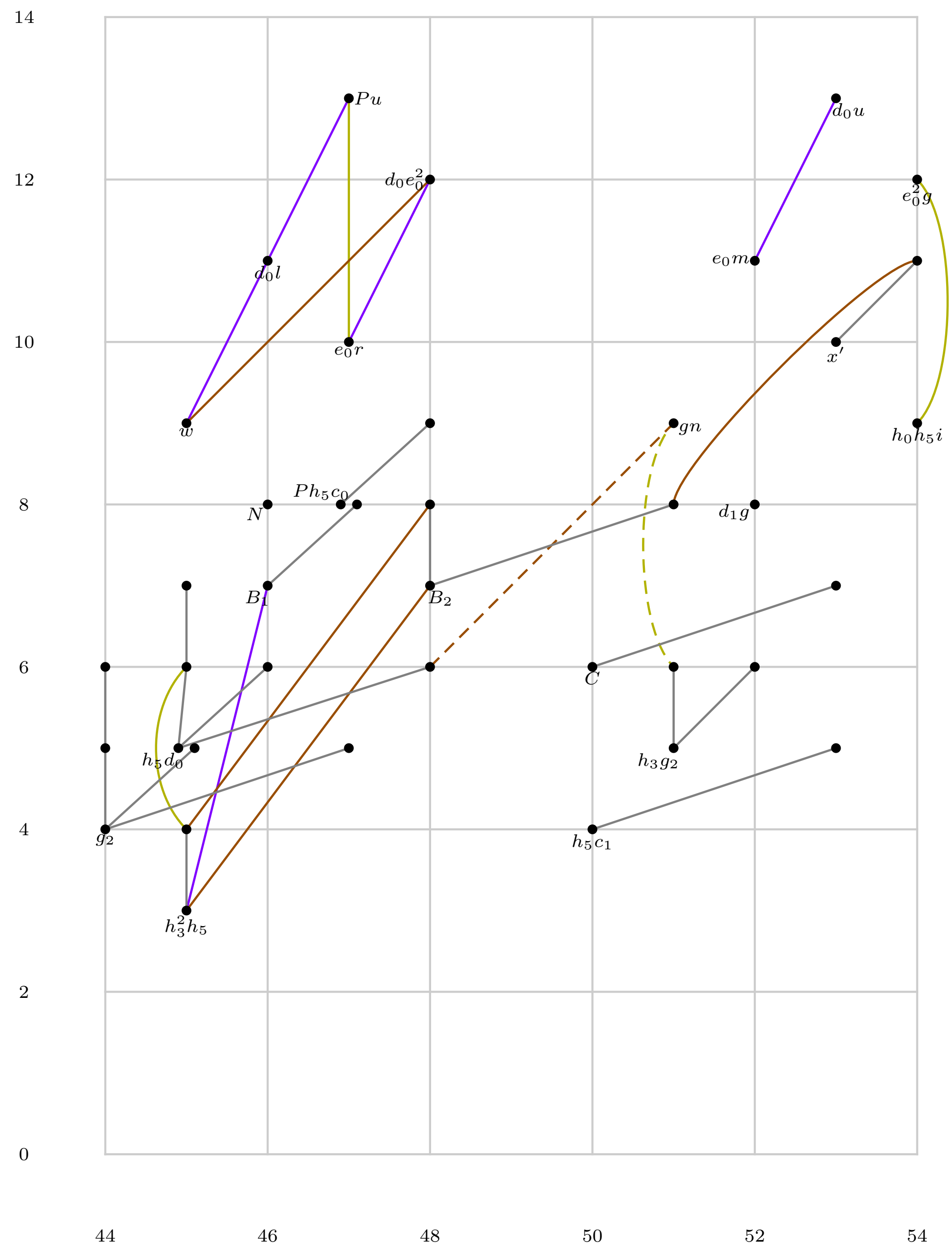
We postpone the proof of Theorem 2.1 to Section 4, the proof of Theorem 2.2 to Section 5 and the proofs of Lemma 2.3 and 2.4 to Section 6. Now we present the proof of Theorem 1.2.

Proof. Following Theorems 2.1 and 2.2, we have

$$\begin{aligned} \theta_4^2 &= \theta_4 \langle 2, \sigma^2 + \kappa, 2\sigma, \sigma \rangle \\ &\subseteq \langle \langle \theta_4, 2, \sigma^2 + \kappa \rangle, 2\sigma, \sigma \rangle \\ &= \text{the union of } \langle 0, 2\sigma, \sigma \rangle \text{ and } \langle \rho_{15}\theta_4, 2\sigma, \sigma \rangle \end{aligned}$$

By Lemma 2.3 and Lemma 2.4 above, both brackets contain a single element zero. Therefore, we have that $\theta_4^2 = 0$. □

If a is a surviving cycle in ASS, we use $\{a\}$ to denote the set of elements in the homotopy group that are detected by a. For elements in the E_∞-page of the ASS, we include part of Isaksen's chart [12].



We do not include elements in filtration higher than 14. Those elements are detected by the $K(1)$-local sphere, and are not relevant to our proof. Here we use colored lines to denote nontrivial extensions. For example, the line between Pu and e_0r indicates that $2\{e_0r\}$ is nontrivial and is detected by Pu. The 2, η and ν-extensions are completely known in this range except for a possible 2-extension from $h_0h_3g_2$ to gn and a possible ν-extension from $h_2h_5d_0$ to gn. We use dashed lines to denote them. In fact, Isaksen [11] showed that these two possible extensions either both occur or neither occur. But these extensions are irrelevant to our purpose.

3. A Toda bracket $\langle\theta_4, 2, \sigma^2\rangle$

The following theorem is due to Isaksen [11]. For completeness, we include the proof.

Theorem 3.1. *$\langle\theta_4, 2, \sigma^2\rangle$ contains an element of order 2 that can be detected by $h_0h_4^3$.*

Remark 3.2. Before presenting the proof, we mention that the indeterminacy of this Toda bracket is well-known. Namely, it is the set $\{0, \rho_{15}\theta_4\}$, where ρ_{15} is the generator of ImJ in π_{15}, and is detected by $h_0^3h_4$. Furthermore, $\rho_{15}\theta_4 \neq 0$ is detected by $h_0^2h_5d_0$. This is shown by Tangora in [23].

Proof. In the Adams E_3-page, we have $\langle h_4^2, h_0, h_3^2\rangle = h_4^2h_4 + h_5h_3^2 = 0$ in the Adams filtration 3. Therefore, by the Moss Theorem [21], there is an element in $\langle\theta_4, 2, \sigma^2\rangle$ that is detected by some element of filtration at least 4. Since the nontrivial element in the indeterminacy has filtration 7, any element in $\langle\theta_4, 2, \sigma^2\rangle$ has filtration at least 4. We have

$$2\langle\theta_4, 2, \sigma^2\rangle = \langle 2, \theta_4, 2\rangle\sigma^2 = \eta\theta_4\sigma^2 = 0.$$

Note that the indeterminacy of $\langle 2, \theta_4, 2\rangle\sigma^2$ is $2\sigma^2\pi_{31} = 0$. Therefore, any element in $\langle\theta_4, 2, \sigma^2\rangle$ has order 2.

Now consider the product $\nu_4\theta_4$.

$$\nu_4\theta_4 = \langle\sigma, \nu, \sigma\rangle\theta_4 \subseteq \langle\sigma, \nu, \sigma\theta_4\rangle \subseteq \langle\sigma, \nu, \{x\}\rangle.$$

Here, since $2\theta_4 = 0$, we can ignore the difference between ν_4, which is by definition $\langle\nu, \sigma, 2\sigma\rangle$, and $\langle\sigma, \nu, \sigma\rangle = 7\nu_4$. In the Adams E_2-page, we have $h_2h_5d_0 = \langle h_3, h_2, x\rangle$ with zero indeterminacy. In fact, this follows from

$$h_2\langle h_3, h_2, x\rangle = \langle h_2, h_3, h_2\rangle x = h_3^2x = h_2^2h_5d_0.$$

Therefore, $\nu_4\theta_4$ is contained in $\langle\sigma, \nu, \{x\}\rangle \subseteq \{h_2h_5d_0\}$.

On the other side, $\nu_4\theta_4$ is contained in $\theta_4\langle 2, \sigma^2, \nu\rangle = \langle\theta_4, 2, \sigma^2\rangle\nu$. For the indeterminacy, note that $\rho_{15}\theta_4\nu = 0$. Therefore, we actually have

$$\nu_4\theta_4 = \langle\theta_4, 2, \sigma^2\rangle\nu.$$

Combining this with the fact that $\nu_4\theta_4$ is also contained in $\{h_2h_5d_0\}$, we deduce that there exists an element in $\langle\theta_4, 2, \sigma^2\rangle$ such that ν times it is detected by $h_2h_5d_0$, which has filtration 6. Therefore, $\langle\theta_4, 2, \sigma^2\rangle$ contains an element with filtration at most 5. Furthermore, it cannot be detected by h_1g_2, which has filtration 5, since otherwise the ν multiple won't be detected by $h_2h_5d_0$. Therefore, the statement of the theorem is the only possibility left. $\square$

Remark 3.3. Another way to describe the statement of this theorem is the following:

$$\langle \theta_4, 2, \sigma^2 \rangle \text{ contains an order 2 element of the form } 2\alpha + \beta,$$

where α is detected by $h_3^2 h_5$ and β is detected by $h_5 d_0$. Note that the nontrivial 2-extension in the 45-stem means that there exist elements α and γ, which are detected by $h_3^2 h_5$ and $h_5 d_0$ respectively, such that $4\alpha = 2\gamma$. Since γ has order 8, one can choose β to be $-\gamma = 7\gamma$, so that $2\alpha + \beta$ has order 2.

Remark 3.4. In [16], Lin showed that this bracket contains 0. The step that rules out the element Isaksen got is invalid. In [13], Kochman showed that this bracket contains $\eta\{g_2\}$ or 0. His argument failed because essentially of the inconsistency of the $\nu-$extension on $\{h_2 h_5 d_0\}$ and the $\sigma-$extension on $\{h_0^2 g_2\}$, which allowed him to eliminate the right element. The inconsistency is discussed in [11].

4. More about the 45-stem

We first consider the Toda bracket $\langle \theta_4, 2, \kappa \rangle$ in π_{45}.

Lemma 4.1. *$\langle \theta_4, 2, \kappa \rangle$ contains an element of order 2 that can be detected by $h_0 h_4^3$.*

Proof. The Adams differential $d_3(h_0 h_4) = h_0 d_0$ implies that in the Adams E_4-page, $\langle h_4^2, h_0, d_0 \rangle = h_0 h_4^3$ in the Adams filtration 4. Then by the Moss convergence theorem [21], there is an element in $\langle \theta_4, 2, \kappa \rangle$ that is detected by $h_0 h_4^3$. From

$$2\langle \theta_4, 2, \kappa \rangle = \langle 2, \theta_4, 2 \rangle \kappa = \eta \theta_4 \kappa = 0,$$

we know that any element in $\langle \theta_4, 2, \kappa \rangle$ has order 2. The indeterminacy of $\langle 2, \theta_4, 2 \rangle \kappa$ is $2\kappa\pi_{31} = 0$. Here we also used that $\kappa\theta_4 = 0$, which is known for filtration reasons. In fact, since $d_0 h_4^2 = 0$ in Ext^6, $\kappa\theta_4$ must be detected by an element of filtration at least 7. However, in the 44-stem of the E_∞-page, there are no elements of filtration 7 or higher. Therefore $\langle \theta_4, 2, \kappa \rangle$ contains an element of order 2 that can be detected by $h_0 h_4^3$. □

Remark 4.2. The indeterminacy of this bracket is the same as that of $\langle \theta_4, 2, \sigma^2 \rangle$, i.e., $\{0, \rho_{15}\theta_4\}$. In fact, π_{31} is generated by $\eta\theta_4, \{n\}$ and ρ_{31}, where ρ_{31} is the generator of ImJ in π_{31}, and is detected by $h_0^{10} h_5$. Since $\kappa\theta_4 = 0$, $\eta\kappa\theta_4 = 0$. Again for filtration reasons, $\kappa\{n\} = 0$ and $\kappa\rho_{31} = 0$. Therefore $\kappa\pi_{31} = 0$. This shows that the indeterminacy of $\langle \theta_4, 2, \kappa \rangle$ is $\{0, \rho_{15}\theta_4\}$.

Although both $\langle \theta_4, 2, \kappa \rangle$ and $\langle \theta_4, 2, \sigma^2 \rangle$ contain an element of order 2 that can be detected by $h_0 h_4^3$, we do not necessarily know if they have an element in common. The following theorem confirms that they do.

Now we restate Theorem 2.1.

Theorem 4.3. *$\langle \theta_4, 2, \sigma^2 + \kappa \rangle$ contains 0 with indeterminacy $\{0, \rho_{15}\theta_4\}$.*

We need the following lemma to prove the theorem.

Lemma 4.4. *$\sigma^2 \pi_{33} = 0$.*

Proof. We know that π_{33} is generated by $\eta\eta_5$, $\nu\theta_4$, $\eta\{q\}$, $\eta^2\rho_{31}$ and $\{P^4 h_1\}$. Since $\eta\sigma^2 = 0$ and $\nu\sigma^2 = 0$, we only need to show that $\{P^4 h_1\}\sigma^2$=0. In fact, we have

$$\{P^4 h_1\}\sigma^2 = \eta\rho_{39}\sigma = 0$$

for filtration reasons. Here ρ_{39} is the generator of ImJ in π_{39}, and is detected by $P^2 h_0^2 i$. Therefore, $\sigma^2 \pi_{33} = 0$. □

Now we present the proof of Theorem 4.3.

Proof. The indeterminacy is straightforward, as in Remark 4.2.

Since all elements in $\langle \theta_4, 2, \kappa \rangle$ and $\langle \theta_4, 2, \sigma^2 \rangle$ have order 2 and can be detected by $h_0 h_4^3$ in the Adams filtration 4, elements in $\langle \theta_4, 2, \sigma^2 + \kappa \rangle$ must be detected by elements of filtration at least 5 and have order 2. To prove the theorem, we need to rule out both $\{w\}$ and $\eta\{g_2\}$.

For $\{w\}$, by Lemma 4.4, we have that

$$\eta^2 \langle \theta_4, 2, \sigma^2 \rangle = \langle \eta^2, \theta_4, 2 \rangle \sigma^2 \in \pi_{33} \sigma^2 = 0.$$

Next we have that

$$\eta^2 \langle \theta_4, 2, \kappa \rangle = \theta_4 \langle 2, \kappa, \eta^2 \rangle.$$

In the Adams E_4-page, we have that $\langle h_0, d_0, h_1^2 \rangle = h_0 h_4 h_1^2 = 0$ in the Adams filtration 4. Then the Moss Theorem tells us that $\langle 2, \kappa, \eta^2 \rangle$ might contain a non-trivial element of higher filtration, namely a combination of $\nu\kappa, \eta^2\rho_{15}$ and $\{P^2 h_1\}$. Note that we have that $\nu\kappa\theta_4 = 0$ and by Lemma 6.1 we have that $\eta^2 \rho_{15} \theta_4 = 0$. To show that $\{P^2 h_1\}\theta_4 = 0$, we first show that $\{Ph_1\}\theta_4 = 0$.

In fact, $\{Ph_1\}\theta_4 \in \langle \eta, 8\sigma, 2 \rangle \theta_4 = \eta \langle 8\sigma, 2, \theta_4 \rangle$, which contains 0. This holds since $\eta\langle 8\sigma, 2, \theta_4 \rangle$ intersects $\eta\{h_0^3 h_3 h_5\}$, which contains a single element zero. The indeterminacy is $\eta \pi_8 \theta_4 = 0$. This gives that $\{Ph_1\}\theta_4 = 0$. Then we have

$$\{P^2 h_1\}\theta_4 \in \theta_4 \langle \{Ph_1\}, 2, 8\sigma \rangle = \langle \theta_4, \{Ph_1\}, 2 \rangle 8\sigma \subseteq \pi_{40} 8\sigma = 0.$$

Therefore, no matter what $\langle 2, \kappa, \eta^2 \rangle$ equals, we always have that

$$\eta^2 \langle \theta_4, 2, \kappa \rangle = \langle 2, \kappa, \eta^2 \rangle \theta_4 \text{ contains } 0.$$

The indeterminacy of $\eta^2 \langle \theta_4, 2, \kappa \rangle$ is zero since $\eta^2 \theta_4 = 0$ and $\eta^2 \kappa = 0$. Then

$$\eta^2 \langle \theta_4, 2, \kappa \rangle = 0.$$

Therefore,

$$\eta^2 \langle \theta_4, 2, \sigma^2 + \kappa \rangle = 0.$$

Then the fact that $\eta^2\{w\} \neq 0$ rules out $\{w\}$, since otherwise we would have that $\eta^2 \langle \theta_4, 2, \sigma^2 + \kappa \rangle = \eta^2 \{w\} \neq 0$.

For $\eta\{g_2\}$, first note that $\sigma\eta\{g_2\} \neq 0$ is detected by $h_1 h_3 g_2$. We have that

$$\langle \theta_4, 2, \kappa \rangle \sigma = \theta_4 \langle 2, \kappa, \sigma \rangle \subseteq \theta_4 \pi_{22} = 0.$$

In fact, π_{22} is generated by $\nu\overline{\sigma}$ and $\eta^2\overline{\kappa}$. We have that $\eta^2 \overline{\kappa} \theta_4 = 0$ and $\nu\overline{\sigma}\theta_4 = 0$ for filtration reasons. As a remark, we can actually prove that $\langle 2, \kappa, \sigma \rangle = \nu\overline{\sigma}$ by studying the cofiber of 2, but we don't need this fact here.

On the other side, as explained in Remark 3.3, $\langle \theta_4, 2, \sigma^2 \rangle$ contains $2\alpha + \beta$. Therefore,

$$\langle \theta_4, 2, \sigma^2 \rangle \sigma \text{ contains } 2\alpha\sigma + \beta\sigma.$$

We have that $2\alpha\sigma \in 2\pi_{52} = 0$. In the Adams E_3-page, we compute directly that $\langle h_0, h_4^2, d_0 \rangle = h_5 d_0$. Then Moss's Theorem shows that $\langle 2, \theta_4, \kappa \rangle$ contains an element that equals to β plus possibly higher filtration terms. Note that $\sigma\{w\} = 0$ by using tmf. In fact, if $\sigma\{w\} \neq 0$, the only possibility is that $\sigma\{w\}$ is detected by $\{e_0 m\}$. This implies that $\eta\sigma\{w\} = \kappa\{u\}$ because of the two nontrivial η-extensions. Since both $\eta\{w\}$ and $\kappa\{u\}$ are detected by tmf and $\sigma = 0$ in $\pi_* tmf$, mapping this relation into tmf gives a contradiction. Besides, from tmf, we know that $\{d_0 l\}$ detects $\kappa\{q\}$, then the contradiction also follows from $\kappa\sigma = 0$. See [4],[9] for example.

Then we have that

$$\beta\sigma \in \langle 2, \theta_4, \kappa\rangle\sigma = 2\langle\theta_4, \kappa, \sigma\rangle \subseteq 2\pi_{52} = 0.$$

Therefore, $\langle\theta_4, 2, \sigma^2\rangle\sigma$ contains $2\alpha\sigma + \beta\sigma = 0$. Note that $\rho_{15}\theta_4\sigma \in \theta_4\pi_{22} = 0$, the indeterminacy is hence zero. Then we have that

$$\langle\theta_4, 2, \sigma^2\rangle\sigma = 0.$$

Therefore,

$$\langle\theta_4, 2, \sigma^2 + \kappa\rangle\sigma = 0.$$

Combined with the fact that $\eta\{g_2\}\sigma \neq 0$, this rules out $\eta\{g_2\}$.

This completes the proof. □

Remark 4.5. $\sigma^2 + \kappa$ is another element in π_{14} that deserves to be called θ_3.

Remark 4.6. We can actually show that the bracket $\langle 2, \theta_4, \eta^2\rangle$ contains $\eta\eta_5 + \nu\theta_4$ with indeterminacy $\{0, \eta^2\rho_{31}\}$.

5. A modified 4-fold Toda bracket for θ_4

We have the following well-known 4-fold Toda brackets for θ_4. See [3],[13],[14] for example.

$$\begin{aligned}\theta_4 &= \langle 2, \sigma^2, 2, \sigma^2\rangle \\ &= \langle 2, \sigma^2, \sigma^2, 2\rangle \\ &= \langle 2\sigma, \sigma, 2\sigma, \sigma\rangle \\ &= \langle 2, \sigma^2, 2\sigma, \sigma\rangle\end{aligned}$$

All of them have zero indeterminacy. This is partially discussed in [3],[13],[14]. For completeness, we include a proof here.

Lemma 5.1. *All four Toda brackets above have zero indeterminacy.*

Proof. In general, suppose a 4-fold Toda bracket $\langle\alpha_1, \alpha_2, \alpha_3, \alpha_4\rangle$ is defined, where $\alpha_i \in \pi_{n_i}$. Then its indeterminacy is contained in the union of three types of 3-fold Toda brackets:

$$\langle\alpha_1, \alpha_2, \pi_{n_3+n_4+1}\rangle, \langle\alpha_1, \pi_{n_2+n_3+1}, \alpha_4\rangle \text{ and } \langle\pi_{n_1+n_2+1}, \alpha_3, \alpha_4\rangle.$$

In our case, the indeterminacy for all of them is contained in the union of the following eight brackets:

$$\langle\pi_{15}, 2, \sigma^2\rangle, \langle 2, \pi_{15}, \sigma^2\rangle, \langle 2, \sigma^2, \pi_{15}\rangle, \langle 2, \pi_{29}, 2\rangle,$$

$$\langle\pi_{15}, 2\sigma, \sigma\rangle, \langle 2\sigma, \pi_{15}, \sigma\rangle, \langle 2\sigma, \sigma, \pi_{15}\rangle, \langle 2, \pi_{22}, \sigma\rangle.$$

We will show that they are all zero. Note that $\pi_{30} \cong \mathbb{Z}/2$ and is generated by θ_4, which is indecomposable. So for each of them, we only need to show that it does not contain θ_4. They all follow for filtration reasons.

For $\langle \pi_{15}, 2, \sigma^2\rangle$, $\langle 2, \sigma^2, \pi_{15}\rangle$, $\langle \pi_{15}, 2\sigma, \sigma\rangle$ and $\langle 2\sigma, \sigma, \pi_{15}\rangle$, the corresponding Massey products are all well-defined on the Adams E_3-page. Since π_{15} is generated by elements of filtration at least 4, the Massey products all take values in filtration at least 5. Therefore, by the Moss Theorem, all of them are all zero.

For $\langle 2, \pi_{15}, \sigma^2\rangle$ and $\langle 2\sigma, \pi_{15}, \sigma\rangle$, the corresponding Massey products are all well-defined on the Adams E_2-page. Since π_{15} is generated by elements of filtration at least 4, the Massey products all take values in filtration at least 6. Therefore, by the Moss Theorem, all of them are all zero.

For $\langle 2, \pi_{22}, \sigma\rangle$, there are essentially two Toda brackets to check: $\langle 2, \nu\overline{\sigma}, \sigma\rangle$ and $\langle 2, \eta^2\overline{\kappa}, \sigma\rangle$, where $\nu\overline{\sigma}$ is detected by h_2c_1. Both brackets have zero indeterminacy. We have that

$$\langle 2, \nu\overline{\sigma}, \sigma\rangle = \langle 2, \overline{\sigma}, \nu\sigma\rangle = \langle 2, \overline{\sigma}, 0\rangle = 0,$$

and that

$$\langle 2, \eta^2\overline{\kappa}, \sigma\rangle = \langle 2, \eta^2, \overline{\kappa}\sigma\rangle = \langle 2, \eta^2, 0\rangle = 0.$$

Here we used the fact that $2\overline{\sigma} = 0$ and $\overline{\kappa}\sigma = 0$.

At last, $\langle 2, \pi_{29}, 2\rangle = 0$, since $\pi_{29} = 0$. This completes the proof. $\square$

Now we prove a modified 4-fold Toda bracket based on the last one. Again, note that $\pi_{30} \cong \mathbb{Z}/2$ and is generated by θ_4.

Theorem 5.2. $\theta_4 = \langle 2, \sigma^2 + \kappa, 2\sigma, \sigma\rangle$ *with zero indeterminacy.*

Proof. We have $\langle \sigma^2 + \kappa, 2\sigma, \sigma\rangle \subseteq \pi_{29} = 0$. And

$$\langle 2, \sigma^2 + \kappa, 2\sigma\rangle \supseteq \langle 2, \sigma^2 + \kappa, 2\rangle\sigma \ni \eta(\sigma^2 + \kappa)\sigma = 0.$$

The indeterminacy of the bracket $\langle 2, \sigma^2 + \kappa, 2\sigma\rangle$ is $2\pi_{22} + 2\sigma\pi_{15} = 0$, and we have $\langle 2, \sigma^2 + \kappa, 2\sigma\rangle = 0$. Therefore, this 4-fold Toda bracket is strictly defined, and the indeterminacy is

$$\langle 2, \sigma^2 + \kappa, \pi_{15}\rangle + \langle 2, \pi_{22}, \sigma\rangle + \langle \pi_{15}, 2\sigma, \sigma\rangle.$$

Note that $\langle 2, \sigma^2 + \kappa, \pi_{15}\rangle = 0$ for filtration reasons as in the proof of Lemma 5.1. The other two parts of the indeterminacy follow from the indeterminacy of $\langle 2, \sigma^2, 2\sigma, \sigma\rangle$, which we know is zero. Then the theorem follows from the next lemma and the fact that $\theta_4 = \langle 2, \sigma^2, 2\sigma, \sigma\rangle$. $\square$

Lemma 5.3. $\langle 2, \kappa, 2\sigma, \sigma\rangle = 0$ *with zero indeterminacy.*

Proof. Again, $\langle \kappa, 2\sigma, \sigma\rangle \subseteq \pi_{29} = 0$. And

$$\langle 2, \kappa, 2\sigma\rangle \supseteq \langle 2, \kappa, 2\rangle\sigma \ni \eta\kappa\sigma = 0.$$

The indeterminacy of $\langle 2, \kappa, 2\sigma\rangle$ is zero. Therefore, this 4-fold Toda bracket is strictly defined. Again, $\langle 2, \kappa, \pi_{15}\rangle = 0$ for filtration reasons. And the other two parts of the indeterminacy are zero, which follows from the indeterminacy of $\langle 2, \sigma^2, 2\sigma, \sigma\rangle$.

To see this bracket contains zero, we multiply by ν.

$$\langle 2, \kappa, 2\sigma, \sigma\rangle\nu \subseteq \langle 2, \kappa, \langle 2\sigma, \sigma, \nu\rangle\rangle = \langle 2, \kappa, \nu_4\rangle.$$

Since in the Adams E_4-page $\langle h_0, d_0, h_2h_4\rangle = 0$ in the Adams filtration 4, there is an element in $\langle 2, \kappa, \nu_4\rangle$ that is detected by an element in filtration strictly higher than 4. The indeterminacy of this bracket is $2\pi_{33} + \nu_4\pi_{15} = \nu_4\pi_{15}$, which also contains elements in filtration strictly higher than 4. On the other side, $\nu\theta_4$ is detected by p in Ext^4. Therefore $\langle 2, \kappa, \nu_4\rangle$ does not contain $\nu\theta_4$. Then the lemma follows from the fact that $\pi_{30} \cong \mathbb{Z}/2$ and is generated by θ_4. $\square$

Remark 5.4. We can show directly that $\langle 2, \kappa, \nu_4\rangle = 0$ with zero indeterminacy.

6. A few proofs

We first prove Lemma 2.3 which states that $\sigma\pi_{53} = 0$.

Proof. As shown in [11], $\pi_{53} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. One set of generators can be chosen to be elements in $\nu\{h_5c_1\}, \nu\{C\}, \epsilon\{h_3^2h_5\}$ and $\kappa\{u\}$ respectively. Note that x' detects $\epsilon\{h_3^2h_5\}$. Then the lemma follows from $\nu\sigma = 0, \epsilon\sigma = 0$ and $\kappa\sigma = 0$. □

The following lemma is shown by Tangora in [23]. We first sketch his proof, then give a more direct proof.

Lemma 6.1. *$\rho_{15}\theta_4 = 2\sigma\{h_0^2h_3h_5\}$.*

Proof. Tangora first showed that $\rho_{15}\theta_4 \neq 0$ and is detected by $h_0^2h_5d_0$. We have

$$\rho_{15}\theta_4 = \rho_{15}\langle\sigma, 2\sigma, \sigma, 2\sigma\rangle = \langle\rho_{15}, \sigma, 2\sigma, \sigma\rangle 2\sigma.$$

Then the only possibility is that $\langle\rho_{15}, \sigma, 2\sigma, \sigma\rangle$ is detected by $h_0^2h_3h_5$.

We present another proof. In the Adams E_3-page, we have $\langle h_3, h_0h_3, h_0^3\rangle = h_0^3h_4$. Therefore, ρ_{15} is contained in $\langle\sigma, 2\sigma, 8\rangle$. Then we have

$$\begin{aligned}\rho_{15}\theta_4 &= \langle\sigma, 2\sigma, 8\rangle\theta_4\\ &= \sigma\langle 2\sigma, 8, \theta_4\rangle\\ &= \sigma\langle 8\sigma, 2, \theta_4\rangle\\ &= \sigma\{h_0^3h_3h_5\}\\ &= 2\sigma\{h_0^2h_3h_5\}.\end{aligned}$$

For the first equation, $\langle\sigma, 2\sigma, 8\rangle\theta_4$ has no indeterminacy, hence the equality. For the last equation, the difference between $\{h_0^3h_3h_5\}$ and $2\{h_0^2h_3h_5\}$ contains elements of higher filtration, namely $\eta\sigma\theta_4$ in this case. The equality holds since $\eta\sigma^2\theta_4 = 0$. □

Now we prove Lemma 2.4 which states that $\langle\rho_{15}\theta_4, 2\sigma, \sigma\rangle = 0$ with zero indeterminacy.

Proof. The indeterminacy is $\rho_{15}\theta_4\pi_{15} + \sigma\pi_{53} = \rho_{15}\theta_4\pi_{15}$. π_{15} is generated by $\eta\kappa$ and ρ_{15}. We have $\rho_{15}^2 = 0$ and $\kappa\theta_4 = 0$ both for filtration reasons. Therefore the indeterminacy is equal to $\rho_{15}\theta_4\pi_{15} = 0$.

By Lemma 6.1, $\langle\rho_{15}\theta_4, 2\sigma, \sigma\rangle = \langle 2\sigma\{h_0^2h_3h_5\}, 2\sigma, \sigma\rangle$ contains $\sigma\langle 2\{h_0^2h_3h_5\}, 2\sigma, \sigma\rangle$. Note that $\langle 2\{h_0^2h_3h_5\}, 2\sigma, \sigma\rangle \subseteq \pi_{53}$ and $\sigma\pi_{53} = 0$. This completes the proof. □

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Department of Mathematics, The University of Chicago, Chicago, IL 60637
E-mail address: `xu@math.uchicago.edu`