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A SPLITTING THEOREM FOR CERTAIN COHOMOLOGY THEORIES ASSOCIATED TO $BP^*(-)$

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Let $P(n)^*(-)$ be Brown-Peterson cohomology modulo I_n and put $B(n)^*(-) = P(n)^*(-)[1/v_n]$. In this note we construct a canonical multiplicative and idempotent operation Ω_n in a suitable completion $\overline{B}(n)^*(-)$ of $B(n)^*(-)$ which has the property that its image is canonically isomorphic to the n-th Morava K-theory $K(n)^*(-)$. In particular, the ring theory $\underline{K}(n)^*(-)$ is contained as a direct summand in the theory $\overline{B}(n)^*(-)$. A similar result is not true before completing. Because the completion map $B(n)^*(-) \to \overline{B}(n)^*(-)$ is injective, the above splitting theorem contains also information about $B(n)^*(-)$. The proof of the theorem depends on a result about the behaviour of formal groups of finite height over complete graded $\mathbb{F}_p$-algebras.

1. Introduction and results

Let BP denote the Brown-Peterson spectrum associated to the prime p (see [2][3][11]). Recall that $BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ where $|v_i| = 2(p^i-1)$. The v_i are always supposed to be Hazewinkel generators [4]. There is a sequence of associative (and commutative if $p > 2$) ring spectra (see [5][16][18]) $BP \to P(1) \to P(2) \to \cdots$ with the property that $P(n)_* \cong BP_*/I_n \cong \mathbb{F}_p[v_n, v_{n+1}, \ldots]$ where $I_n = (p, v_1, v_2, \ldots) \subset BP_*$ is the n-th invariant prime ideal of BP_* [6][5]. The

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$P(n)$'s may be viewed as BP-theory with coefficients in BP_*/I_n and provide a convenient way for describing the structure of $BP_*(-)$.

If we localize $P(n)_*(-)$ with respect to $T_n = \{1, v_n, v_n^2, \ldots\} \subset P(n)_*$ we get a new multiplicative homology theory $B(n)_*(-) = T_n^{-1}P(n)_*(-)$ which may be represented by the telescope spectrum $B(n) = \varinjlim(\Sigma^{2i(1-p^n)}P(n), \Theta_n)$ where Θ_n corresponds to multiplication by v_n. The $B(n)$'s record a great deal of the periodicity structure of BP. In this paper we are interested in the relation of the theories $B(n)^*(-)$ and suitable completions of them to the Morava K-theories $K(n)^*(-)$ (see [5][15] for a definition and some basic properties). $K(n)^*(-)$ is represented by a ring spectrum $K(n)$, there is a canonical morphism of ring spectra $\lambda_n: B(n) \to K(n)$ and $K(n)_* \cong \mathbb{F}_p[v_n, v_n^{-1}]$. The following theorem has been proved in [15].

<u>1.1.THEOREM</u>. <u>Suppose</u> $p > 2$. <u>There is a natural equivalence</u>

$$\omega_X: B(n)_*(X) \xrightarrow{\sim} B(n)_*(K(n)) \underset{K(n)_*K(n)}{\square} K(n)_*(X)$$

<u>of multiplicative homology theories with values in the abelian category of</u> $B(n)_*B(n)$-<u>comodules</u>.

<u>REMARKS</u>: (1) In 1.1., $K(n)_*(-)$ is viewed in the usual sense as a left $K(n)_*K(n)$-comodule. The right $K(n)_*K(n)$-coaction map of $B(n)_*K(n)$ is obtained by composing its left $B(n)_*(B(n))$-coaction map with $id \otimes \lambda_n$, see [15] for details. $\square$ denotes the cotensor product over $K(n)_*K(n)$.

(2) Theoretically, 1.1. contains a description of the $B(n)^*$-algebra $B(n)^*(X)$ in terms of the $K(n)^*$-algebra $K(n)^*(X)$.

We know from [15], lemma 3.14. (or see [9], 2.4.) that there is an isomorphism of $B(n)_*$-modules and right $K(n)_*(K(n))$-comodules

1.2. $$\Theta: B(n)_*K(n) \cong B(n)_* \underset{K(n)_*}{\otimes} K(n)_*K(n) .$$

Together with 1.1. this implies that there is an equivalence of $B(n)_*$- module valued homology theories

$$1.3. \qquad B(n)_*(X) \cong B(n)_* \underset{K(n)_*}{\otimes} K(n)_*(X)$$

and similarly for cohomology. This suggests the question if there exists a natural isomorphism of the form 1.3. of <u>multiplicative</u> theories. Unfortunately, the answer is <u>no</u>. The reason may be found in the theory of formal groups: Both $B(n)^*(-)$ and $K(n)^*(-)$ are canonically complex-oriented . So if there would exist a multiplicative transformation $\alpha: K(n)^*(-) \to B(n)^*(-)$, the formal groups $\alpha_* F_{K(n)}$ and $F_{B(n)}$ had to be isomorphic over $B(n)^*$. But this is not the case (see remark 2.12.). The aim of this paper is to show that the situation changes if $B(n)^*(-)$ is suitably completed. Before we can state what we have in mind we must describe a result on formal groups.

By a formal group over a commutative ring A we always mean a one-dimensional commutative formal group law $F(x,y) \in A[\![x,y]\!]$. In our context, A will be graded and $F(x,y)$ is assumed to be a <u>homogeneous</u> power series of degree -2 (homological grading) resp. 2 (cohomological grading) where x and y have degree -2 resp. 2. What grading we use will (hopefully) be clear from the context. Similarly, homomorphisms $f: F \to G$ of formal groups are homogeneous power series of degree -2 resp. 2. All isomorphisms are supposed to be <u>strict</u>. For details on formal groups we refer the reader to [4].

Let J_n be the (homogeneous) ideal $(v_{n+1}, v_{n+2}, \ldots)$ of $B(n)^* \cong \mathbb{F}_p[v_n, v_n^{-1}, v_{n+1}, \ldots]$. J_n is a graded maximal ideal of $B(n)^*$ in the sense that $B(n)^*/J_n \cong K(n)^*$ is a graded field (i.e. all non-zero elements are invertible). In fact, $B(n)^*$ is a graded local ring in an obvious sense. Put

$$1.4. \quad \bar{B}(n)^* = \varprojlim_r B(n)^*/J_n^r \cong \mathbb{F}_p[v_n, v_n^{-1}][\![v_{n+1}, v_{n+2}, \ldots]\!] .$$

Thus, $\bar{B}(n)^*$ is a complete Hausdorff graded local ring with (graded) residue field $K(n)^*$. The completion map $c_n: B(n)^* \to \bar{B}(n)^*$ is clearly injective. The formal groups $F_{B(n)}$ and

$F_{K(n)}$ both extend to $\overline{B}(n)^*$.

1.5. THEOREM. There exists one and only one isomorphism $\phi_n: F_{B(n)} \to F_{K(n)}$ over $\overline{B}(n)^*$ such that $\phi_n(x) \equiv x \bmod \overline{J}_n$.

It should be noted that whereas $F_{B(n)}$ is a very comlicated formal group, $F_{K(n)}$ is rather easy to describe [4][13][15] . Put

$$f_n(x) = \sum_{i\geq 0} \frac{1}{p^i} v_n^{a_i} \cdot x^{p^{in}} \in \mathbb{Q}[v_n, v_n^{-1}][[x]]$$

where $a_i = (p^{in}-1)/(p^n-1)$. Then $F_n'(x,y) = f_n^{-1}(f_n(x)+f_n(y))$ is a formal group over $\mathbb{Z}_{(p)}[v_n, v_n^{-1}]$ and $F_{K(n)} = F_n' \bmod p$. Thus $F_{K(n)}$ is just the reduction mod p of the graded version of a Lubin-Tate formal group over $\mathbb{Z}_{(p)}$. In particular,

$$1.6. \qquad [p]_{F_{K(n)}}(x) = v_n \cdot x^{p^n} .$$

Theorem 1.5. and a slight generalisation of it will be proved in section 2. The proof is inspired by Hazewinkel's proof [4] of a well known theorem of Lazard which states that over a separably closed field of positive characteristic, formal groups of equal height are isomorphic.

Let $\underline{W}$ (resp. $\underline{W}_f$) be the category of CW-complexes (resp. finite CW-complexes). Using 1.3. one sees that for any $B(n)_*$-module A, there is a natural equivalence

$$1.7. \qquad \mathrm{Hom}^*_{B(n)_*}(B(n)_*(X),A) \cong \mathrm{Hom}^*_{K(n)_*}(K(n)_*(X),A) .$$

Because $K(n)_*$ is a graded field any $K(n)_*$-module is free, so the right term of 1.7. is an additive cohomology theory over $\underline{W}$. It follows in particular that the functor

$$1.8. \qquad \underline{W} \ni X \mapsto \overline{B}(n)^*(X) := \mathrm{Hom}^*_{B(n)_*}(B(n)_*X, \overline{B}(n)^*)$$

is an additive and multiplicative cohomology theory over $\underline{W}$ and thus representable by a ring spectrum $\overline{B}(n)$. Note that for X a finite complex, 1.9. $$\overline{B}(n)^*(X) \cong B(n)^*(X) \underset{B(n)^*}{\otimes} \overline{B}(n)^*$$

as a $\overline{B}(n)^*$-algebra. If X is an arbitrary complex, $\{X_\alpha\}$ the set of all finite subcomplexes of X, 1.8. implies that

1.10. $$\overline{B}(n)^*(X) \cong \varprojlim_{\alpha} (B(n)^*(X_\alpha) \underset{B(n)^*}{\otimes} \overline{B}(n)) .$$

REMARK: It should be observed, that 1.3. does not depend on theorem 1.1. See [17],6.19. for a different proof which also includes the case p=2.

Note that the obvious multiplicative completion map c_n: $B(n)^*(X) \to \overline{B}(n)^*(X)$ is injective (for all X). Let $\overline{B}(n)^*(-)$ be $\mathbb{C}$-oriented by $u_n = c_n(e^{B(n)}(\eta))$. Then $F_{\overline{B}(n)}$ is just the extension of $F_{B(n)}$ to $\overline{B}(n)^*$. Let $\phi_n(x)$ be the isomorphism of theorem 1.5..

1.11. THEOREM. Suppose p>2. There is a unique multiplicative and stable operation of degree 0

$$\Omega_n: \overline{B}(n)^*(-) \to \overline{B}(n)^*(-)$$

such that $\Omega_n(u_n) = \phi_n(u_n)$. Ω_n is idempotent and agrees on the coefficient ring with the composition

$$\overline{B}(n)^* \to \overline{B}(n)^*/\overline{J}_n \cong K(n)^* \subset \overline{B}(n)^*.$$

Moreover, there is a canonical isomorphism

$$\mathrm{im}\{\Omega_n:\overline{B}(n)^*(X) \to \overline{B}(n)^*(X)\} \cong K(n)^*(X)$$

1.12. COROLLARY. There are canonical isomorphisms of multiplicative cohomology theories over W_f

$$\overline{B}(n)^*(X) \cong \overline{B}(n)^* \underset{K(n)^*}{\otimes} K(n)^*(X)$$

$$K(n)^*(X) \cong K(n)^* \underset{\overline{B}(n)^*}{\otimes} \overline{B}(n)^*(X) .$$

REMARK. The second isomorphism in 1.12. is just a version of the Conner-Floyd theorem mod I_n and does not depend on theorem 1.11. (see [5][15]). Both isomorphisms of 1.12. may be extended to $\underline{W}$ (see 1.10.) and similar equivalences hold for homology.

From 1.11. it follows that there is a commutative diagram of ring spectra and morphisms of ring spectra

1.13.
$$\begin{array}{ccc} \bar{B}(n) & \xrightarrow{\Omega_n} & \bar{B}(n) \\ & \searrow^{\pi_n} \quad \nearrow_{\iota_n} & \\ & K(n) & \end{array} \quad , \quad \pi_n \cdot \iota_n = id_{K(n)}$$

and corollary 1.12. is an immediate consequence of the existence of the maps π_n and ι_n, using the comparison theorem for cohomology theories.

Theorem 1.11. is our main result. It gives some new information concerning the question how the Morava K-theories are related to $BP^*(-)$ and, if one likes, a new definition of $K(n)^*(-)$. The proof of 1.11. will be given in section 3 . Section 4 contains some consequences of 1.11. and additional remarks and section 2 is devoted to the proof of theorem 1.5.

2. On formal groups of finite height over $\mathbb{F}_p$-algebras

Let F be a formal group over the graded $\mathbb{F}_p$-algebra A . Recall (see for example [4]) that the height of F, ht(F), is defined as follows: ht(F) = ∞ if $[p]_F(x) = 0$ and ht(F) = n if p^n is the highest power of p such that $[p]_F(x) = f(x^{p^n})$ for some $f(x) \in A[\![x]\!]$. Every formal group over an $\mathbb{F}_p$-algebra has a well-defined height. If ht(F) = n,

2.1. $\quad [p]_F(x) \equiv a \cdot x^{p^n} \bmod(\text{degree } p^n+1)$, $\quad a \neq 0$.

DEFINITION: F is of strict height n, if a is a unit of A.

We denote the formal group of a complex-oriented ring theory $E^*(-)$ by $F_E(x,y)$. As is well known (see [11][4][3]) F_{BP} is universal for p-typical formal groups over $\mathbb{Z}_{(p)}$-

algebras. From the relation (see [14])

2.2. $$[p]_{F_{BP}}(x) \equiv \sum_{i>0}^{F_{BP}} v_i \cdot x^{p^i} \pmod p$$

one immediatly sees that $F_{P(n)}$ is universal for p-typical formal groups of height $\geq$ n and that $F_{B(n)}$ is universal for p-typical formal groups of strict height n over $\mathbb{F}_p$-algebras.

Now let us consider a p-typical formal group F of strict height n over the graded $\mathbb{F}_p$-algebra A, with classifying ring homomorphism $f: B(n)_* \to A$. f gives A the structure of a $B(n)_*$-algebra. The composition $\tilde{f}: K(n)_* \subset B(n)_* \xrightarrow{f} A$ defines a new formal group $\tilde{F} = \tilde{f}_* F_{K(n)}$ over A which has the property that

2.3. $$[p]_{\tilde{F}}(x) = ax^{p^n}$$

if $[p]_F(x)$ is as in 2.1..

<u>DEFINITION</u>. In the situation above, we define the F-completion $\bar{A}_F$ of A as the $\bar{B}(n)^*$-algebra $\bar{A}_F = A \otimes_f \bar{B}(n)^*$. A is called F-complete, if the obvious completion homomorphism $c_F: A \to \bar{A}_F$ is an isomorphism.

2.4. THEOREM. *Let F be a p-typical formal group of strict height n over the graded $\mathbb{F}_p$-algebra A . There exists a canonical isomorphism*

$$\phi_F: (c_F)_* F \xrightarrow{\sim} (c_F)_* \tilde{F}$$

over the F-completion $\bar{A}_F$ of A.

<u>Proof</u>: 2.4. is an obvious consequence of theorem 1.5. using the universality of $F_{B(n)}$.

<u>REMARKS</u>. (1) Because over an $\mathbb{F}_p$-algebra, every formal group is canonically isomorphic to a p-typical one [3][4] the assumtion that F has to be p-typical is not essential.
(2) 2.4. should be compared with the fact that any formal

group over a torsion free ring A is isomorphic to the additive formal group x + y over $A \otimes \mathbb{Q}$.

For the proof of 1.5.(and also for the next section) we need some preparation. Recall that a groupoid is a small category in which every morphism is an isomorphism. Let k be a commutative ring, $\underline{Alg}_k$ the category of k-algebras. By a groupoidscheme over k we mean a representable functor $G: \underline{Alg}_k \to \underline{Groupoids}$ from $\underline{Alg}_k$ to the category of groupoids. Here representable simply means that the two set-valued functors $A \mapsto ob(G(A))$ and $A \mapsto mor(G(A))$ are representable. For all A we have morphisms (natural in A)

$$2.5. \quad mor\, G(A) \cong Hom_{\underline{Alg}_k}(C,A) \overset{\rightarrow}{\underset{\leftarrow}{\rightarrow}} Hom_{\underline{Alg}_k}(B,A) \cong ob\, G(A)$$

which are induced by the maps source, target and identity of the category G(A). 2.5. gives rise to k-algebra homomorphisms $\eta_L, \eta_R: B \to C$ and $\varepsilon: C \to B$. Furthermore, the composition of morphisms in G(A) is represented by a map $\psi: C \to C \otimes_B C$ and all these data together make (B,C) into a Hopf algebroid (see [9][10]).

For any $\mathbb{F}_p$-algebra A consider the set $TI_n(A)$ of triples (F,G,ϕ) where F,G are p-typical formal groups of height $m \geq n$ over A and $\phi: G \to F$ is an isomorphism. $TI_n(A)$ is a groupoid in an obvious sense and we get a functor

$$TI_n(-): \underline{Alg}_{\mathbb{F}_p} \to \underline{Groupoids} \quad .$$

$TI_n(-)$ is just the height $\geq n$ analog of Landweber's functor TI(-) of [8] and we put $TI_0(A) := TI(A) = \{(F,G,\phi)\}$, $\phi: G \to F$ an isomorphism between arbitrary p-typical formal groups over the $\mathbb{Z}_{(p)}$-algebra A.

2.6. THEOREM. $TI_n(-)$ _is a groupoidscheme over_ $\mathbb{F}_p$ (_resp._ $\mathbb{Z}_{(p)}$ _if_ n = 0) _which is represented by the Hopf algebroid_ $(BP_*/I_n, BP_*(BP)/I_n)$.

Stated more explicitly we see in particular that if

$(F,G,\phi) \in TI_n(A)$, there exist unique ring homomorphisms $f\colon BP_*/I_n \to A$ and $g\colon \mathbb{F}_p[t_1,t_2,\ldots] \to A$ with the following properties. Consider the diagram

$$2.7.\quad BP_*/I_n \overset{\eta_L}{\underset{\eta_R}{\rightrightarrows}} BP_*(BP)/I_n \cong BP_*/I_n \otimes \mathbb{F}_p[t_1,t_2,..] \overset{f\otimes g}{\to} A$$

Then F is represented by $(f\otimes g)\circ\eta_L$, G by $(f\otimes g)\circ\eta_R$ and $\phi(x) = \sum_{i\geq 0}^F g(t_i)x^{p^i}$.

<u>Proof of 2.6.</u>: For n = 0, this is just a reformulation of the combination of theorem 1 and theorem 2 of [8]. The assertion for n > 0 is a consequence of the case n = 0, because the ideal I_n is invariant.

We will need the following lemma:

<u>2.8. LEMMA. Let</u> $b \in \bar{J}_n \subset \bar{B}(n)_*$ <u>be a homogeneous element and</u> i <u>an arbitrary natural number. Then the equation</u>

$$2.9.\qquad b - v_n^{p^i}x + v_n x^{p^n} = o$$

<u>has a (homogeneous) solution in</u> $\bar{J}_n \subset \bar{B}(n)_*$

<u>Proof</u>: Define $z = \Sigma_{j=1}^{j=\infty} z_j \in \bar{J}_n \subset \bar{B}(n)_*$ recursively by $z_1 = v_n^{-p^i} b$, $z_{j+1} = v_n^{1-p^i} z_j^{p^n}$. We show that z is a solution of 2.9.. Because we are working mod p, one sees that for all $r \geqslant 1$, $\Sigma_{j=r}^{j=\infty} z_j$ is a solution of the equation

$$\text{(a)}\qquad v_n^{p^i} z_r - v_n^{p^i} x + v_n x^{p^n} = o$$

iff $\Sigma_{j=r+1}^{j=\infty} z_j$ solves

$$\text{(b)}\qquad v_n^{p^i} z_{r+1} - v_n^{p^i} x + v_n x^{p^n} = o \quad .$$

But $v_n^{p^i} z_{r+1} \equiv 0 \bmod \bar{J}_n^{p^{rn}}$ by the definition of z, so $x = o$ solves (b) over $\bar{B}(n)_*/\bar{J}_n^{p^{rn}}$. Using the above observation, one sees that $\Sigma_{j=1}^{j=r} z_j$ solves 2.9. over the ring $\bar{B}(n)_*/\bar{J}_n^{p^{rn}}$. Because $\bar{B}(n)_* = \varprojlim_k B(n)_*/J_n^k \cong \varprojlim_k \bar{B}(n)_*/\bar{J}_n^k$ and the construction is compatible with the reduction maps, the result follows.

We are now ready for the

<u>Proof of theorem 1.5.:</u> (A) Existence of an isomorphism $\phi_n: F_{B(n)} \to F_{K(n)}$ over $\bar{B}(n)_*$. For $k \geq 0$ we will inductively construct a sequence of formal groups F_k and isomorphisms $\psi_k: F_k \to F_{k+1}$ over $\bar{B}(n)_*$ such that $F_0 = F_{B(n)}$ and the following conditions are satisfied:

$(i)_k$ $\psi_k(x) \equiv x \bmod (\deg p^k)$

$(ii)_k$ Let $f_k: BP_*/I_n \to \bar{B}(n)_*$ be the classifying homomorphism of F_k. Then $f_k(v_n) = v_n$, $f_k(v_{n+1}) = f_k(v_{n+2}) = \ldots = f_k(v_{n+k}) = 0$ and $f_k(v_{n+k+1}) \in \bar{J}_n$.

Assuming this proved for the moment, an isomorphism ϕ_n is obtained as follows. From $(i)_k$ we see that the sequence of compositions

$$\phi^{(m)} = \psi_{m-1} \circ \ldots \circ \psi_0 : F_0 \xrightarrow{\sim} F_m$$

converges (in the power series topology) to some power series $\phi_n(x) \in \bar{B}(n)_*[\![x]\!]$. If we put

$$F_\infty = \phi_n F_{B(n)}(\phi_n^{-1}(x), \phi_n^{-1}(y)) \quad ,$$

$\phi_n: F_{B(n)} \to F_\infty$ is by definition an isomorphism. From the definition of $\phi_n(x)$ and condition $(ii)_k$ one sees that the classifying map f_∞ of F_∞ is given by $f_\infty(v_{n+i}) = v_n$ if $i=0$ and 0 otherwise. This shows that $F_\infty = F_{K(n)}$.

To construct the F_k and ψ_k we proceed as follows. Suppose $m \geq 0$ and assume inductively that a formal group F_m which satisfies condition $(ii)_m$ has been constructed (remember $F_0 = F_{B(n)}$). Consider the equation

$$2.10. \qquad f_m(v_{n+m+1}) - v_n^{p^{m+1}} x + v_n x^{p^n} = 0 \quad .$$

Because $f_m(v_{n+m+1}) \in \bar{J}_n$ by our hypothesis, it follows from lemma 2.8. that 2.10. has a solution $a_{m+1} \in \bar{J}_n \subset \bar{B}(n)_*$. We define a homomorphism $g_{m+1}: \mathbb{F}_p[t_1, t_2, \ldots] \to \overline{B}(n)_*$ of $\mathbb{F}_p$-algebras by $g_{m+1}(t_{m+1}) = a_{m+1}$ and $g(t_i) = 0$ if $i \neq m+1$. Then we put

$$f_{m+1} := (f_m \otimes g_{m+1}) \circ \eta_R : BP_*/I_n \to \overline{B}(n)_*$$

$$\psi_m(x) := \{x +^{F_m} a_{m+1} x^{p^{m+1}}\}^{-1}$$

$$F_{m+1} := (f_{m+1})_* F_{BP/I_n}$$

From theorem 2.6. we see that $\psi_m : F_m \to F_{m+1}$ is an isomorphism. Clearly, $\psi_m(x) \equiv x \bmod (\text{degree } p^{m+1})$, so to finish the induction it suffices to show that f_{m+1} has the property $(ii)_{m+1}$. Because $f_m(v_n) = v_n$ and $\eta_R(v_n) = v_n$ one has $f_{m+1}(v_n) = v_n$. Now recall the relation ([12])

2.11. $\quad \eta_R(v_{n+i}) \equiv v_{n+i} - v_n^{p^i} t_i + v_n t_i^{p^n} \bmod A_{n+i}$

where A_{n+i} denotes the ideal $(v_{n+1}, \ldots, v_{n+i-1}, t_1, \ldots, t_{i-1})$ of $BP_*(BP)/I_n$. From the relation 2.11., the fact that f_m satisfies the condition $(ii)_m$ and the definition of g_{m+1} it follows that $f_{m+1}(v_{n+1}) = \ldots = f_{m+1}(v_{n+m+1}) = 0$. Because both v_{n+m+2} and $a_{m+1} = g(t_{m+1})$ lie in $\overline{J}_n$, 2.11. also implies $f_{m+1}(v_{n+m+2}) \in \overline{J}_n$. This ends the induction and the existence proof for ϕ_n.

(B) Uniqueness of ϕ_n. Clearly, the reduction mod $\overline{J}_n$ of $F_{B(n)}$ is just $F_{K(n)}$. The uniqueness statement of theorem 1.5. is proved if we can show that the homomorphism of abelian groups induced by reduction mod $\overline{J}_n$

$$\alpha : \mathrm{Hom}_{\overline{B}(n)_*}(F_{K(n)}, F_{B(n)}) \to \mathrm{Hom}_{K(n)_*}(F_{K(n)}, F_{K(n)})$$

is injective. Suppose $f: F_{K(n)} \to F_{B(n)}$ is a homomorphism such that $\alpha(f) = 0$. Then $f(x) \in \overline{J}_n^r[\![x]\!]$ for some $r \geq 1$. Now

$$f(F_K(x,y)) = F_B(f(x), f(y)) = f(x) + f(y) + \sum_{i,j \geq 1} a_{ij} f(x)^i f(y)^j$$

so

$$f([p]_{F_K}(x)) = f(v_n x^{p^n}) \equiv 0 \bmod \bar{J}_n^{r+1}$$

which implies that $f(x) \equiv 0 \bmod \bar{J}_n^{r+1}$. By induction one sees that $f(x) \in \bar{J}_n^r[\![x]\!]$ for all r, so the coefficients of f lie in $\cap_r \bar{J}_n^r$ but this is 0 because $\bar{B}(n)_*$ is Hausdorff. This ends the proof of theorem 1.5..

2.12. REMARK. The formal groups $F_{K(n)}$ and $F_{B(n)}$ are not isomorphic over $B(n)_*$. This may be seen as follows. Suppose $\psi: F_{K(n)} \to F_{B(n)}$ is an isomorphism over $B(n)_*$. This means (see theorem 2.6.) that there exists a ring homomomorphism $g: \mathbb{F}_p[t_1, t_2, \ldots] \to B(n)_*$ such that $\alpha = (id \otimes g) \circ \eta_R : B(n)_* \to B(n)_*$ represents $F_{K(n)}$. In particular, $\alpha(v_{n+1}) = 0$. Using 2.11. this leads to

$$\alpha(v_{n+1}) = v_{n+1} - v_n^p g(t_1) + v_n g(t_1)^{p^n} = 0.$$

But this is impossible, because otherwise one would get an algebraic dependence between polynomial generators of $B(n)_*$ which is seen by inspecting $B(n)_{2(p-1)}$.

3. Proof of theorem 1.11.

From the definition of the spectrum $\bar{B}(n)$ given in the introduction we know that

$$\bar{B}(n)_*(X) \cong B(n)_*(X) \otimes_{B(n)_*} \bar{B}(n)_* \quad .$$

From [15], lemma 2.5. and the mod I_n version ([9][16] etc) of Landweber's exact functor theorem [7] we see that

3.1. $$\bar{B}(n)_*\bar{B}(n) \cong \bar{B}(n)_* \otimes_{P(n)_*} P(n)_*P(n) \otimes_{P(n)_*} \bar{B}(n)_*$$

as a Hopf algebroid. Clearly, $\bar{B}(n)_*(-)$ takes values in the category of $\bar{B}(n)_*\bar{B}(n)$-comodules ([1]). Moreover, maps of ring spectra $\bar{B}(n) \to \bar{B}(n)$ (of degree 0) are in 1-1-correspondence with morphisms of $\overline{B}(n)_*$-algebras $\overline{B}(n)_*\overline{B}(n) \to \overline{B}(n)_*$. Using 3.1. we see that

$$\mathrm{Hom}_{\overline{B}(n)_*\text{-}\underline{alg}}(\overline{B}(n)_*\overline{B}(n), \overline{B}(n)_*) \cong \mathrm{Hom}_{P(n)_*\text{-}\underline{alg}}(P(n)_*P(n), \overline{B}(n)_*) \ .$$

From [16] 2.13. we know that

3.2. $$P(n)_*P(n) \cong BP_*BP/I_n \underset{P(n)_*}{\otimes} \Lambda(a_0, \ldots, a_{n-1})$$

as an algebra where degree $a_i = 2p^i - 1$. Moreover, $BP_*(BP)/I_n$ is a sub-Hopf-algebroid of $P(n)_*P(n)$. Because $\overline{B}(n)^{odd} = 0$ it follows from 3.1. and the isomorphism above that maps of ring spectra $\overline{B}(n) \to \overline{B}(n)$ are in 1-1-correspondence with homomorphisms of $P(n)_*$-algebras

$$BP_*(BP)/I_n \cong P(n)_*[t_1, t_2, \ldots] \to \overline{B}(n)_* \quad .$$

Let $\phi_n : F_{B(n)} \to F_{K(n)}$ be the canonical isomorphism of theorem 1.5. and put

$$\phi_n^{-1}(x) = \sum_{i \geq 0}{}^{F_{B(n)}} c_i x^{p^i} \quad .$$

If $g: \mathbf{F}_p[t_1, t_2, \ldots] \to \overline{B}(n)_*$ denotes the ring homomorphism defined by $g(t_i) = c_i$ we denote by Ω_n the map of ring spectra $\overline{B}(n) \to \overline{B}(n)$ which corresponds to $id \otimes g$: $BP_*BP/I_n \to \overline{B}(n)_*$. Then $\Omega_n(u_n) = \phi_n(u_n)$ by definition and Ω_n is obviously uniquely determined by this condition. From theorem 2.6. and the definition of g we see that on the coefficients, Ω_n is just the composition $\overline{B}(n)^* \to \overline{B}(n)^*/\overline{J}_n \cong K(n)^* \subset \overline{B}(n)^*$. Next, we must show that $\Omega_n^2 = \Omega_n$. Ω_n^2 is represented by the composition

$$\alpha: BP_*BP/I_n \xrightarrow{\psi} BP_*BP/I_n \underset{BP_*/I_n}{\otimes} BP_*BP/I_n \xrightarrow{\beta} \overline{B}(n)_*$$

where $\beta = (id \otimes g) \cdot (id \otimes g)$. Observe that $\psi \circ \eta_R(x) = 1 \otimes \eta_R(x)$. So it follows from theorem 2.6. and the definition of g that α represents an isomorphism of formal groups $F_{K(n)} \to F_{B(n)}$. Because $\alpha(t_i) \in \overline{J}_n$ for $i > 0$, the uniqueness statement of theorem 1.5. implies that $\alpha = (id \otimes g)$, so $\Omega_n^2 = \Omega_n$.

From the properties of Ω_n proved till now we see that $\mathrm{im}\{\Omega_n:\overline{B}(n)^*(X) \to \overline{B}(n)^*(X)\}$ is a multiplicative and complex-oriented cohomology theory with coefficient ring isomorphic to $K(n)^*$ and formal group $F_{K(n)}$. But this implies (see [15],1.8.) that there is a canonical isomorphism

$$\mathrm{im}\{\Omega_n:\overline{B}(n)^*(X) \to \overline{B}(n)^*(X)\} \cong K(n)^*(X)$$

of complex-oriented ring theories. This completes the proof of theorem 1.11..

4. Miscellaneous remarks and applications

4.1. We will first describe how theorem 1.11. leads to a simple model for the Hopf algebroid $\overline{B}(n)_*\overline{B}(n)$ (p>2). Let us write Σ_n for the Hopf algebra $K(n)_*K(n)$ whose structure maps we denote by ε_K, ψ_K and c_K. We consider $(\overline{B}(n)_*, \overline{B}(n)_* \otimes_{K(n)_*} \Sigma_n \otimes_{K(n)_*} \overline{B}(n)_*)$ as a Hopf algebroid with structure maps $\tilde{\eta}_L, \tilde{\eta}_R, \tilde{c}_n, \tilde{\varepsilon}_n$ and $\tilde{\psi}_n$ given by $\tilde{\eta}_L(u) = u\otimes 1\otimes 1$, $\tilde{\eta}_R(u) = 1\otimes 1\otimes u$, $\tilde{\varepsilon}_n(u\otimes x\otimes v) = u\cdot\varepsilon_K(x)\cdot v$, $\tilde{c}_n(u\otimes x\otimes v) = v\otimes c_K(x)\otimes u$ and $\tilde{\psi}_n$ the composition

$$\tilde{\psi}_n: \overline{B}(n)_* \otimes_{K(n)_*} \Sigma_n \otimes_{K(n)_*} \overline{B}(n)_* \to \overline{B}(n)_* \otimes_{K(n)_*} \Sigma_n \otimes_{K(n)_*} \Sigma_n \otimes_{K(n)_*} \overline{B}(n)_*$$

$$\to (\overline{B}(n)_* \otimes_{K(n)_*} \Sigma_n \otimes_{K(n)_*} \overline{B}(n)_*) \otimes_{\overline{B}(n)_*} (\overline{B}(n)_* \otimes_{K(n)_*} \Sigma_n \otimes_{K(n)_*} \overline{B}(n)_*)$$

From theorem 1.11. we know that there is a canonical map of ring spectra $\iota_n: K(n) \to \overline{B}(n)$.

<u>4.1.1.PROPOSITION</u>. <u>There is an isomorphism of Hopf algebroids</u>

$$\Phi: \overline{B}(n)_* \otimes_{K(n)_*} \Sigma_n \otimes_{K(n)_*} \overline{B}(n)_* \to \overline{B}(n)_*\overline{B}(n)$$

<u>where</u> $\Phi(u\otimes x\otimes v) = u\cdot(\iota_n \wedge \iota_n)_*(x)\cdot v$.

Proof: Apply [15], lemma 2.5..

4.2. It is possible to give a functorial interpretation of the isomorphism 4.1.1.from the point of view of formal groups, at least if one neglects the exterior parts of the Hopf algebroids considered. Set $\Gamma_n = \overline{B}(n)_*\overline{B}(n)/(a_o,\ldots,a_{n-1})$ and $\Lambda_n = K(n)_*K(n)/(a_o,\ldots,a_{n-1})$. For any $\mathbb{F}_p$-algebra A consider the groupoid $\overline{\overline{TI}}_n(A)$ of triples (F,G,ϕ) where F and G are p-typical formal groups of strict height n over A and ϕ is an isomorphism $c_*G \to c_*F$ over the ring ${}_FA_G := \overline{B}(n)_* \otimes_f A \otimes_g \overline{B}(n)_*$ ($c\colon A \to {}_FA_G$ denotes the obvious map and f,g are the classifying homomorphisms of F resp. G). Using theorem 2.6. it is easy to see that $\overline{\overline{TI}}_n(-)$ is a groupoidscheme represented by the Hopf algebroid Γ_n. From theorem 2.4. we see that every isomorphism $\phi\colon c_*G \to c_*F$ may be written in the form (with a slight abuse of notation)

$$c_*G \xrightarrow{\phi_G} F_{K(n)} \xrightarrow{\Theta} F_{K(n)} \xrightarrow{\phi_F^{-1}} c_*F$$

where Θ is an automorphism of $F_{K(n)}$. Because as a Hopf algebra, $\Lambda_n \cong K(n)_* \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} K(n)_*$, one sees using 2.6. that for every $K(n)_*$-algebra A, $\mathrm{Aut}_A(F'_{K(n)}) \cong \mathrm{Hom}_{K(n)_*\text{-alg}}(\Lambda_n, A)$ where $F'_{K(n)}$ is the formal group over A induced from $F_{K(n)}$ via $K(n)_* \to A$. So one gets a natural isomorphism of groupoid-valued functors

$$\mathrm{Hom}_{\mathrm{rings}}(\overline{B}(n)_* \underset{K(n)_*}{\otimes} \Lambda_n \underset{K(n)_*}{\otimes} \overline{B}(n)_*, A) \cong \overline{\overline{TI}}_n(A)$$

and it follows that $\Gamma_n \cong \overline{B}(n)_* \underset{K(n)_*}{\otimes} \Lambda_n \underset{K(n)_*}{\otimes} \overline{B}(n)_*$ as Hopf algebroids. This may be used to give a simple proof of Morava's "structure theorem for cobordism comodules" (see [10] for a treatment of these questions in a somewhat different context).

4.3. There is a $\overline{B}(n)_*(-)$-analog of theorem 1.1., i.e. there is an equivalence of multiplicative homology theories with values in the category of $\overline{B}(n)_*\overline{B}(n)$-comodules

4.3.1. $\overline{\omega}_X: \overline{B}(n)_*(X) \xrightarrow{\sim} \overline{B}(n)_*(K(n)) \mathop{\square}_{K(n)_*K(n)} K(n)_*(X)$

The proof goes as in [15]. The fact that one has a morphism of ring spectra $\iota_n: K(n) \to \overline{B}(n)$ implies that there is an isomorphism of rings, $(\overline{B}(n)_*, K(n)_*)$-bimodules and right-$K(n)_*K(n)$-comodules

4.3.2. $\Theta: \overline{B}(n)_*(K(n)) \cong \overline{B}(n)_* \otimes_{K(n)_*} K(n)_*K(n)$

whose inverse is given by $u \otimes x \mapsto u \cdot (\iota_n \wedge id)_*(x)$ (see[15], 2.5. and 2.6.). (note that the isomorphism 1.2. for $B(n)$ instead of $\overline{B}(n)$ is not multiplicative). 4.3.2. may be used to give an explicit description of $\iota_n^X: K(n)_*(X) \to \overline{B}(n)_*(X)$. Let $i: K(n)_* \to \overline{B}(n)_*$ be the inclusion and denote by Δ_X the coaction map of $K(n)_*(X)$. The image of the map $\alpha = (\Theta^{-1} \otimes id) \circ (i \otimes id \otimes id) \circ \Delta_X: K(n)_*(X) \to \overline{B}(n)_*K(n) \otimes_{K(n)_*} K(n)_*X$ is contained in $\overline{B}(n)_*(K(n)) \mathop{\square}_{K(n)_*K(n)} K(n)_*(X)$ and one obtains $\iota_n^X(x) = \omega_X^{-1} \alpha_X(x)$.

4.4. It is possible to generalise slightly the isomorphism of corollary 1.12.. Let $E^*(-)$ be a multiplicative cohomology theory with coefficient ring E^* of characteristic $p>2$, $\mathbb{C}$-oriented by $u \in E^2(\mathbb{C}P_\infty)$. Because E^* is an $\mathbb{F}_p$-algebra, there is a canonical change of orientation, $u^\xi = \xi(u)$, such that the formal group F associated to u^ξ is p-typical ([11][3][4]). Now assume that F is of strict height n. Using the notations of section 2 we define $\overline{E}_F^*(-) := E^*(-) \otimes_{E^*} \overline{E}_F^*$.

<u>4.4.1. PROPOSITION</u>. *$\overline{E}_F^*(-)$ is a cohomology theory over W_f and there is a natural and multiplicative isomorphism*

$$\chi_E: K(n)^*(-) \otimes_{K(n)^*} \overline{E}_F^* \xrightarrow{\sim} \overline{E}_F^*(-)$$

over W_f such that $\chi_E(u^K \otimes 1) = \phi_F(u^\xi)$ where ϕ_F is as in 2.4.

If $E^{1-2p^k}=0$ *for* $k=0,1,\dots,n-1$, χ_E *is uniquely determined by this condition.*

Proof: From [16], proposition 6.8., we see that there is a multiplicative transformation $\rho: P(n)^*(-) \to E^*(-)$ such that $\rho(u^P)=u^\xi$, unique if $E^{1-2p^k}=0$ for $k=0,1,\dots,n-1$. Because F is of strict height n it follows from the mod I_n version of Landweber's exact functor theorem that ρ extends uniquely to a multiplicative equivalence $B(n)^*(-)\otimes_{B(n)^*}E^* \tilde{\to} E^*(-)$ which, after tensoring with $\bar{B}(n)^*$, leads to an isomorphism $\bar{B}(n)^*\otimes_{\bar{B}(n)^*}\bar{E}^*_F \tilde{\to} \bar{E}^*_F(-)$ which we call $\bar{\rho}_n$. The isomorphism $\tilde{\iota}_n: \bar{B}(n)^*\otimes_{K(n)^*}K(n)^*(-) \tilde{\to} \bar{B}(n)^*(-)$ of corollary 1.12. has the property $\tilde{\iota}_n(1\otimes u^K)=\phi_n(u^B)$ and, using again the universal property of $P(n)^*(-)$ one sees that it is the only such isomorphism. Then $\chi_E=\bar{\rho}_n\circ\tilde{\iota}_n$ is the desired multiplicative equivalence.

4.4.2.EXAMPLE. Let $MU^*(-,\mathbb{F}_p)$ be complex cobordism theory with coefficients $\mathbb{F}_p$, $u\in MU^2(\mathbb{CP}_\infty,\mathbb{F}_p)$ the usual orientation class and F the p-typical formal group associated to u^ξ. Recall that $MU^*(S^0,\mathbb{F}_p)\cong\mathbb{F}_p[x_1,x_2,\dots]$. The polynomial generators x_i may be chosen such that $[p]_F(T)=\Sigma^F_{i>0}x_{p^i-1}T^{p^i}$ (see [3], §6 and 2.2.). Let G denote the multiplicative summand in the Adams splitting [1] of $K^*(-,\mathbb{Z}_{(p)})$. Then ([15],1.9.) $G(-,\mathbb{F}_p)\cong K(1)^*(-)$ and from 4.4.1. we see that there is a canonical multiplicative isomorphism

$$\chi: G^*(X,\mathbb{F}_p)\otimes_{\mathbb{F}_p[v_1,v_1^{-1}]}\Lambda \tilde{\to} \overline{MU}^*_F(X,\mathbb{F}_p)[\frac{1}{x_{p-1}}]$$

where $\Lambda=\overline{MU}^*_F(S^0,\mathbb{F}_p)[1/x_{p-1}]\cong\mathbb{F}_p[x_{p-1},x^{-1}_{p-1}][x_i\mid i\neq p^k-1][\![x_j\mid j=p^k-1,k>1]\!]$.

4.5. In all topological parts of this paper we assumed that p>2. For p= 2, the products in $\bar{B}(n)_*(-)$ and $K(n)_*(-)$ are not commutative (this is a non-trivial fact!) and the description of $\bar{B}(n)_*\bar{B}(n)$ resp. $K(n)_*K(n)$ is not so easy as in the case p odd. However, using a slightly different method, it is possible to prove a version of theorem 1.11. also in the case p= 2. We will perhaps come back to this and related questions concerning products in the case p=2 somewhere else.

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