

MONOIDAL BOUSFIELD LOCALIZATIONS AND ALGEBRAS OVER OPERADS

DAVID WHITE

ABSTRACT. We give conditions on a monoidal model category \mathcal{M} and on a set of maps C so that the Bousfield localization of \mathcal{M} with respect to C preserves the structure of algebras over various operads. This problem was motivated by an example due to Mike Hill which demonstrates that for the model category of equivariant spectra, preservation does not come for free, even for cofibrant operads. We discuss this example in detail and provide a general theorem regarding when localization preserves P -algebra structure for an arbitrary operad P .

We characterize the localizations which respect monoidal structure and prove that all such localizations preserve algebras over cofibrant operads. As a special case we recover numerous classical theorems about preservation of algebraic structure under localization, and we recover a recent result of Hill and Hopkins regarding preservation for equivariant spectra. To demonstrate our preservation result for non-cofibrant operads, we work out when localization preserves commutative monoids and the commutative monoid axiom. Finally, we provide conditions so that localization preserves the monoid axiom.

1. INTRODUCTION

Bousfield localization was originally introduced as a method to better understand the interplay between homology theories and the categories of spaces and spectra (see [7] and [8]). Thanks to the efforts of [14] and [23], Bousfield localization can now be understood as a process one may apply to general model categories, and the classes of maps which are inverted can be far more general than homology isomorphisms. Bousfield localization allows for the passage from levelwise model structures to stable model structures (see [26]), allows for the construction of point-set models for numerous ring spectra, and provides a powerful computational tool. Bousfield localization in the context of monoidal model categories has played a crucial role in a number of striking results. The reader is encouraged to consult [13], [22], [29], and [32] for examples.

Nowadays, structured ring spectra are often thought of as algebras over operads acting in any of the monoidal model categories for spectra. It is therefore natural to ask the extent to which Bousfield localization preserves such algebraic structure. For Bousfield localizations at homology isomorphisms this question is answered in [13] and [32]. The case for spaces is subtle and is addressed in [10] and [14]. More general Bousfield localizations are considered in [9].

The preservation question may also be asked in the context of equivariant and motivic spectra, and it turns out the answer is far more subtle. Mike Hill found an example of a naturally occurring Bousfield localization of equivariant spectra which preserves the type of algebraic structure considered in [13] but which fails to preserve the equivariant commutativity needed for the landmark results in [22]. Hill's example is the motivation behind this paper, and is expounded in Section 5.

In order to understand this and related examples, we find conditions on a model category \mathcal{M} and on a class of maps C so that the left Bousfield localization L_C with respect to C preserves the structure of algebras over various operads. After a review of the pertinent terminology in Section 2 we give our general preservation result in Section 3. In Section 4 we provide conditions on C so that the model category $L_C(\mathcal{M})$ is a monoidal model category. We then apply our general preservation results in such categories in Section 5, obtaining preservation results for Σ -cofibrant operads such as A_∞ and E_∞ in model categories of spaces, spectra, and chain complexes.

In Section 5 we also provide an in-depth study of the case of equivariant spectra. We highlight precisely what is failing in Hill's example and how to prohibit this behavior. We then discuss the connection between our work and the theorem of Hill and Hopkins presented in [21] which guarantees preservation of equivariant commutativity. Finally, we introduce a collection of operads which interpolate between naive E_∞ and genuine E_∞ , and we apply our results to determine which localizations preserve the type of algebraic structure encoded by these operads. These operads and the model structures in which they are cofibrant are of independent interest and will be pursued further in joint work of the author with Javier Gutiérrez [18].

In the latter half of the paper we turn to preservation of structure over non-cofibrant operads. An example is preservation of commutative monoids. For categories of spectra the phenomenon known as rectification means that preservation of strict commutativity is equivalent to preservation of E_∞ -structure, but in general there can be Bousfield localizations which preserve the latter type of structure and not the former. In the companion paper [44] we introduced a condition on a monoidal model category called the *commutative monoid axiom*, which guarantees that the category of commutative monoids inherits a model structure. We build on this work in Section 6 by providing conditions on the maps in C so that Bousfield localization preserves the commutative monoid axiom. We then apply our general preservation results from Section 3 to deduce preservation results for commutative monoids in Section 7. Numerous applications are given. Finally, in Section 8 we provide conditions so that $L_C(\mathcal{M})$ satisfies the monoid axiom when \mathcal{M} does.

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2. PRELIMINARIES

We assume the reader is familiar with basic facts about model categories. Excellent introductions to the subject can be found in [12], [23], and [25]. Throughout the paper we will assume \mathcal{M} is a cofibrantly generated model category, i.e. there is a set I of cofibrations and a set J of trivial cofibrations which permit the small object argument (with respect to some cardinal κ), and a map is a (trivial) fibration if and only if it satisfies the right lifting property with respect to all maps in J (resp. I).

Let I -cell denote the class of transfinite compositions of pushouts of maps in I , and let I -cof denote retracts of such. In order to run the small object argument, we will assume the domains K of the maps in I (and J) are κ -small relative to I -cell (resp. J -cell), i.e. given a regular cardinal $\lambda \geq \kappa$ and any λ -sequence $X_0 \rightarrow X_1 \rightarrow \dots$ formed of maps $X_\beta \rightarrow X_{\beta+1}$ in I -cell, then the map of sets $\text{colim}_{\beta < \lambda} \mathcal{M}(K, X_\beta) \rightarrow \mathcal{M}(K, \text{colim}_{\beta < \lambda} X_\beta)$ is a bijection. An object is small if there is some κ for which it is κ -small. See Chapter 10 of [23] for a more thorough treatment of this material. For any object X we have a cofibrant replacement $QX \rightarrow X$ and a fibrant replacement $X \rightarrow RX$.

We will at times also need the hypothesis that \mathcal{M} is *tractable*, i.e. that the domains (hence codomains) of the maps in I and J are cofibrant. This term was first introduced in [3]. The author only knows one example where this hypothesis is not satisfied, and that example was constructed precisely to demonstrate that tractability does not come for free. This example is discussed in Remark 4.16.

Our model category \mathcal{M} will also be a monoidal category with product \otimes and unit $S \in \mathcal{M}$. In order to ensure that the monoidal structure interacts nicely with the model structure (e.g. to guarantee it passes to a monoidal structure on the homotopy category $\text{Ho}(\mathcal{M})$ whose unit is given by S) we must assume

- (1) Unit Axiom: For any cofibrant X , the map $QS \otimes X \rightarrow S \otimes X \cong X$ is a weak equivalence.
- (2) Pushout Product Axiom: Given any $f : X_0 \rightarrow X_1$ and $g : Y_0 \rightarrow Y_1$ cofibrations, $f \square g : X_0 \otimes Y_1 \amalg_{X_0 \otimes Y_0} X_1 \otimes Y_0 \rightarrow X_1 \otimes Y_1$ is a cofibration. Furthermore, if either f or g is trivial then $f \square g$ is trivial.

If these hypotheses are satisfied then \mathcal{M} is called a *monoidal model category*. Note that the pushout product axiom is equivalent to the statement that $-\otimes-$ is a Quillen bifunctor. Furthermore, it is sufficient to check the pushout product axiom on the generating maps I and J , by Proposition 4.2.5 of [25]. Similar axioms define what

it means for \mathcal{M} to be a simplicial model category. In this context the pushout product axiom is called the SM7 axiom. It retains the same statement but where one of the maps f, g is in \mathcal{M} and the other is a morphism in $sSet$, the category of simplicial sets. For a topological model category the same axiom is used again but with one of the maps in \mathcal{M} and one in Top . We refer the reader to Definition 4.2.18 in [25] for details.

We will at times also need to assume that *cofibrant objects are flat* in \mathcal{M} , i.e. that whenever X is cofibrant and f is a weak equivalence then $f \otimes X$ is a weak equivalence. Finally, we remind the reader of the monoid axiom which appeared as Definition 3.3 in [39].

Given a class of maps C in \mathcal{M} , let $C \otimes \mathcal{M}$ denote the class of maps $f \otimes id_X$ where $f \in C$ and $X \in \mathcal{M}$. A model category is said to satisfy the *monoid axiom* if every map in $(\text{Trivial-Cofibrations} \otimes \mathcal{M})\text{-cell}$ is a weak equivalence.

We will be discussing preservation of algebraic structure as encoded by an operad. Let P be an operad valued in \mathcal{M} (for a general discussion of the interplay between operads and homotopy theory see [5]). Let $P\text{-alg}(\mathcal{M})$ denote the category whose objects are P -algebras in \mathcal{M} (i.e. admit an action of P) and whose morphisms are P -algebra homomorphisms (i.e. respect the P -action). The free P -algebra functor from \mathcal{M} to $P\text{-alg}(\mathcal{M})$ is left adjoint to the forgetful functor. We will say that $P\text{-alg}(\mathcal{M})$ *inherits* a model structure from \mathcal{M} if the model structure is transferred across this adjunction, i.e. if a P -algebra homomorphism is a weak equivalence (resp. fibration) if and only if it is so in \mathcal{M} . In Section 4 of [5], an operad P is said to be *admissible* if $P\text{-alg}(\mathcal{M})$ inherits a model structure in this way.

Finally, we remind the reader about the process of Bousfield localization as discussed in [23]. This is a general machine that starts with a (nice) model category \mathcal{M} and a set of morphisms C and produces a new model structure $L_C(\mathcal{M})$ on the same category in which maps in C are now weak equivalences. Furthermore, this is done in a universal way, introducing the smallest number of new weak equivalences as possible. When we say Bousfield localization we will always mean left Bousfield localization. So the cofibrations in $L_C(\mathcal{M})$ will be the same as the cofibrations in \mathcal{M} .

Bousfield localization proceeds by first constructing the fibrant objects of $L_C(\mathcal{M})$ and then constructing the weak equivalences. In both cases this is done via simplicial mapping spaces $\text{map}(-, -)$. If \mathcal{M} is a simplicial or topological model category then one can use the hom-object in $sSet$ or Top . Otherwise a framing is required to construct the simplicial mapping space. We refer the reader to [25] or [23] for details on this process.

An object N is said to be *C-local* if fibrant in \mathcal{M} and if for all $g : X \rightarrow Y$ in C , $\text{map}(g, N) : \text{map}(Y, N) \rightarrow \text{map}(X, N)$ is a weak equivalence in $sSet$. These objects are precisely the fibrant objects in $L_C(\mathcal{M})$. A map $f : A \rightarrow B$ is a *C-local equivalence* if for all N as above, $\text{map}(f, N) : \text{map}(B, N) \rightarrow \text{map}(A, N)$ is a weak equivalence. These maps are precisely the weak equivalences in $L_C(\mathcal{M})$.

Throughout this paper we assume C is a set of cofibrations between cofibrant objects. This can always be guaranteed in the following way. For any map f let Qf denote the cofibrant replacement and let \tilde{f} denote the left factor in the cofibration-trivial fibration factorization of Qf . Then \tilde{f} is a cofibration between cofibrant objects and we may define $\tilde{C} = \{\tilde{f} \mid f \in C\}$. Localization with respect to \tilde{C} yields the same result as localization with respect to C , so our assumption that the maps in C are cofibrations between cofibrant objects loses no generality.

We also assume everywhere that the model category $L_C(\mathcal{M})$ exists. This can be guaranteed by assuming \mathcal{M} is left proper and either combinatorial (as discussed in [3]) or cellular (as discussed in [23]). A model category is *left proper* if pushouts of weak equivalences along cofibrations are again weak equivalences. We will make this a standing hypothesis on \mathcal{M} . However, as we have not needed the cellularity or combinatoriality assumptions for our work we have decided not to assume them. In this way if a Bousfield localization is known to exist for some reason other than the theory in [23] then our results will be applicable.

3. GENERAL PRESERVATION RESULT

In this section we provide a general result regarding when Bousfield localization preserves P -algebras. We must first provide a precise definition for this concept. Throughout this section, let \mathcal{M} be a monoidal model category and let C be a class of maps in \mathcal{M} such that Bousfield localization $L_C(\mathcal{M})$ is also monoidal model category. On the model category level the functor L_C is the identity. So when we write L_C as a functor we shall mean the composition of derived functors $\mathrm{Ho}(\mathcal{M}) \rightarrow \mathrm{Ho}(L_C(\mathcal{M})) \rightarrow \mathrm{Ho}(\mathcal{M})$, i.e. $E \rightarrow L_C(E)$ is the unit map of the adjunction $\mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(L_C(\mathcal{M}))$. In particular, for any E in \mathcal{M} , $L_C(E)$ is weakly equivalent to R_CQE where R_C is a choice of fibrant replacement in $L_C(\mathcal{M})$ and Q is a cofibrant replacement in \mathcal{M} .

Let P be an operad valued in \mathcal{M} . Because the objects of $L_C(\mathcal{M})$ are the same as the objects of \mathcal{M} , P is also valued in $L_C(\mathcal{M})$. Thus, we may consider P -algebras in both categories and these classes of objects agree (because the P -algebra action is independent of the model structure). We denote the categories of P -algebras by $P\text{-alg}(\mathcal{M})$ and $P\text{-alg}(L_C(\mathcal{M}))$. These are identical as categories, but in a moment they will receive different model structures.

Definition 3.1. Assume that \mathcal{M} and $L_C(\mathcal{M})$ are monoidal model categories. Then L_C is said to *preserve P -algebras* if

- (1) When E is a P -algebra there is some P -algebra \tilde{E} which is weakly equivalent in \mathcal{M} to $L_C(E)$.
- (2) In addition, when E is a cofibrant P -algebra, then there is a choice of \tilde{E} and a lift of the localization map $E \rightarrow L_C(E)$ to a P -algebra homomorphism $E \rightarrow \tilde{E}$.

The notion of preservation was also considered in [9], but only for cofibrant E .

Recall that when we say $P\text{-alg}(\mathcal{M})$ inherits a model structure from \mathcal{M} we mean that this model structure is transferred by the free-forgetful adjunction. In particular, a map of P -algebras f is a weak equivalence (resp. fibration) if and only if f is a weak equivalence (resp. fibration) in \mathcal{M} .

Theorem 3.2. *Let \mathcal{M} be a monoidal model category such that the Bousfield localization $L_C(\mathcal{M})$ exists and is a monoidal model category. Let P be an operad valued in \mathcal{M} . If the categories of P -algebras in \mathcal{M} and in $L_C(\mathcal{M})$ inherit model structures from \mathcal{M} and $L_C(\mathcal{M})$ then L_C preserves P -algebras.*

Proof. Let R_C denote fibrant replacement in $L_C(\mathcal{M})$, let $R_{C,P}$ denote fibrant replacement in $P\text{-alg}(L_C(\mathcal{M}))$, and let Q_P denote cofibrant replacement in $P\text{-alg}(\mathcal{M})$. We will prove the first form of preservation and our method of proof will make it clear that in the special case where E is a cofibrant P -algebra then in fact we may deduce the second form of preservation.

In our proof, \tilde{E} will be $R_{C,P}Q_P(E)$. Because Q is the left derived functor of the identity adjunction between \mathcal{M} and $L_C(\mathcal{M})$, and R_C is the right derived functor of the identity, we know that $L_C(E) \simeq R_C Q(E)$. We must therefore show $R_C Q(E) \simeq R_{C,P}Q_P(E)$.

The map $Q_P E \rightarrow E$ is a trivial fibration in $P\text{-alg}(\mathcal{M})$, hence in \mathcal{M} . The map $Q E \rightarrow E$ is also a weak equivalence in \mathcal{M} . Consider the following lifting diagram in \mathcal{M} :

$$(1) \quad \begin{array}{ccc} \emptyset & \longrightarrow & Q_P E \\ \downarrow & \nearrow & \downarrow \simeq \\ Q E & \longrightarrow & E \end{array}$$

The lifting axiom gives the map $Q E \rightarrow Q_P E$ and it is necessarily a weak equivalence in \mathcal{M} by the 2 out of 3 property.

Since $Q_P E$ is a P -algebra in \mathcal{M} it must also be a P -algebra in $L_C(\mathcal{M})$, since the monoidal structure of the two categories is the same. We may therefore construct a lift:

$$\begin{array}{ccc} Q_P E & \longrightarrow & R_{C,P}Q_P E \\ \downarrow & \nearrow & \downarrow \\ R_C Q_P E & \longrightarrow & * \end{array}$$

In this diagram the left vertical map is a weak equivalence in $L_C(\mathcal{M})$ and the top horizontal map is a weak equivalence in $P\text{-alg}(L_C(\mathcal{M}))$. Because the model category $P\text{-alg}(L_C(\mathcal{M}))$ inherits weak equivalences from $L_C(\mathcal{M})$, this map is a weak equivalence in $L_C(\mathcal{M})$. Therefore, by the 2 out of 3 property, the lift is a weak

equivalence in $L_C(\mathcal{M})$. We make use of this map as the horizontal map in the lower right corner of the diagram below.

The top horizontal map $QE \rightarrow Q_P E$ in the following diagram is the first map we constructed, which was proven to be a weak equivalence in \mathcal{M} . The square in the diagram below is then obtained by applying R_C to that map. In particular, $R_C QE \rightarrow R_C Q_P E$ is a weak equivalence in $L_C(\mathcal{M})$:

$$\begin{array}{ccccc} QE & \longrightarrow & Q_P E & & \\ \downarrow & & \downarrow & & \\ R_C QE & \longrightarrow & R_C Q_P E & \longrightarrow & R_{C,P} Q_P E \end{array}$$

We have shown that both of the bottom horizontal maps are weak equivalences in $L_C(\mathcal{M})$. Thus, by the 2 out of 3 property, their composite $R_C QE \rightarrow R_{C,P} Q_P E$ is a weak equivalence in $L_C(\mathcal{M})$. All the objects in the bottom row are fibrant in $L_C(\mathcal{M})$, so these C -local equivalences are actually weak equivalences in \mathcal{M} .

As E was a P -algebra and Q_P and $R_{C,P}$ are endofunctors on categories of P -algebras, it is clear that $R_{C,P} Q_P E$ is a P -algebra. We have just shown that $L_C(E)$ is weakly equivalent to this P -algebra, so we are done.

We turn now to the case where E is assumed to be a cofibrant P -algebra. We have seen that there is an \mathcal{M} -weak equivalence $R_C QE \rightarrow R_{C,P} Q_P E$, and above we took $R_{C,P} Q_P E$ in \mathcal{M} as our representative for $L_C(E)$ in $\text{Ho}(\mathcal{M})$. Because E is a cofibrant P -algebra, there are weak equivalences $E \simeq Q_P(E)$ in $P\text{-alg}(L_C(\mathcal{M}))$. This is because all cofibrant replacements of a given object are weakly equivalent, e.g. by diagram (1). So passage to $Q_P(E)$ is unnecessary when E is cofibrant, and we take $R_{C,P} E$ as our representative for $L_C(E)$. We may then lift the localization map $E \rightarrow L_C(E)$ in $\text{Ho}(\mathcal{M})$ to the fibrant replacement map $E \rightarrow R_{C,P} E$ in \mathcal{M} . As this fibrant replacement is taken in $P\text{-alg}(L_C(\mathcal{M}))$, this map is a P -algebra homomorphism, as desired.

□

This theorem alone would not be a satisfactory answer to the question of when L_C preserves P -algebras, because there is no clear way to check the hypotheses. For this reason, in the coming sections we will discuss conditions on \mathcal{M} and P so that P -algebras inherit model structures, and then we will discuss which localizations L_C preserve these conditions (so that $P\text{-alg}(L_C(\mathcal{M}))$ inherits a model structure from $L_C(\mathcal{M})$). One such condition on \mathcal{M} is the monoid axiom. In Section 8 we discuss which localizations L_C preserve the monoid axiom. However, it will turn out that the monoid axiom is not necessary in order for our preservation results to apply. This is because the work in [24] and [42] produces semi-model structures on P -algebras and these will be enough for our proof above to go through.

Observe that in the proof above we only used formal properties of fibrant and cofibrant replacement functors, and the fact that the model structures on P -algebras

were inherited from \mathcal{M} and $L_C(\mathcal{M})$. So it should not come as a surprise to experts that the same proof works when P -algebras only form semi-model categories. For completeness, we remind the reader of the definition of a semi-model category. The motivating example is when \mathcal{D} is obtained from \mathcal{M} via the general transfer principle for transferring a model structure across an adjunction (see Lemma 2.3 in [39] or Theorem 12.1.4 in [15]) when not all the conditions needed to get a full model structure are satisfied.

In particular, the reader should imagine that weak equivalences and fibrations in \mathcal{D} are maps which forget to weak equivalences and fibrations in \mathcal{M} , and that the generating (trivial) cofibrations of \mathcal{D} are maps of the form $F(I)$ and $F(J)$ where $F : \mathcal{M} \rightarrow \mathcal{D}$ is the free algebra functor and I and J are the generating (trivial) cofibrations of \mathcal{M} . The following is Definition 1 from [42] and Definition 12.1.1 in [15]. Cofibrant should be taken to mean cofibrant in \mathcal{D} .

Definition 3.3. A *semi-model category* is a bicomplete category \mathcal{D} , an adjunction $F : \mathcal{M} \rightleftarrows \mathcal{D} : U$ where \mathcal{M} is a model category, and subcategories of weak equivalences, fibrations, and cofibrations in \mathcal{D} satisfying the following axioms:

- (1) U preserves fibrations and trivial fibrations.
- (2) \mathcal{D} satisfies the 2 out of 3 axiom and the retract axiom.
- (3) Cofibrations in \mathcal{D} have the left lifting property with respect to trivial fibrations. Trivial cofibrations in \mathcal{D} whose domain is cofibrant have the left lifting property with respect to fibrations.
- (4) Every map in \mathcal{D} can be functorially factored into a cofibration followed by a trivial fibration. Every map in \mathcal{D} whose domain is cofibrant can be functorially factored into a trivial cofibration followed by a fibration.
- (5) The initial object in \mathcal{D} is cofibrant.
- (6) Fibrations and trivial fibrations are closed under pullback.

\mathcal{D} is said to be *cofibrantly generated* if there are sets of morphisms I' and J' in \mathcal{D} such that I' -inj is the class of trivial fibrations and J' -inj the class of fibrations in \mathcal{D} , if the domains of I' are small relative to I' -cell, and if the domains of J' are small relative to maps in J' -cell whose domain becomes cofibrant in \mathcal{M} .

Note that the only difference between a semi-model structure and a model structure is that one of the lifting properties and one of the factorization properties requires the domain of the map in question to be cofibrant. Because fibrant and cofibrant replacements are constructed via factorization, (4) implies that every object has a cofibrant replacement and that objects with cofibrant domain have fibrant replacements. So one could construct a fibrant replacement functor which first does cofibrant replacement and then does fibrant replacement. These functors behave as they would in the presence of a full model structure.

We are now prepared to state our preservation result in the presence of only a semi-model structure on P -algebras. Again, when we say P -algebras inherit a semi-model structure we mean with weak equivalences and fibrations reflected and preserved by the forgetful functor.

Corollary 3.4. *Let \mathcal{M} be a monoidal model category such that the Bousfield localization $L_C(\mathcal{M})$ exists and is a monoidal model category. Let P be an operad valued in \mathcal{M} . If the subcategories of P -algebras in \mathcal{M} and in $L_C(\mathcal{M})$ inherit semi-model structures from \mathcal{M} and $L_C(\mathcal{M})$ then L_C preserves P -algebras.*

Proof. The proof proceeds exactly as the proof of the theorem above. We highlight where care must be taken in the presence of semi-model categories. As remarked above, the cofibrant replacement Q_P in the semi-model category $P\text{-alg}(\mathcal{M})$ exists and $Q_P E \rightarrow E$ is a weak equivalence in $P\text{-alg}(\mathcal{M})$, hence in \mathcal{M} . Diagram (1) is a lifting diagram in \mathcal{M} , so still yields a weak equivalence $Q_E \rightarrow Q_P E$.

Next, the fibrant replacement $R_C Q_P E$ is a replacement in $L_C(\mathcal{M})$, which is a model category. The fibrant replacement $Q_P E \rightarrow R_{C,P} Q_P E$ is a fibrant replacement in the semi-model category $P\text{-alg}(L_C(\mathcal{M}))$, and exists because the domain of $Q_P E \rightarrow *$ is cofibrant in $P\text{-alg}(L_C(\mathcal{M}))$. The resulting object $R_{C,P} Q_P E$ is fibrant in $P\text{-alg}(L_C(\mathcal{M}))$ hence in $L_C(\mathcal{M})$. The lift in the next diagram is a lift in $L_C(\mathcal{M})$, and again by the 2 out of 3 property in $L_C(\mathcal{M})$ the diagonal map is a C -local equivalence:

$$\begin{array}{ccc} Q_P E & \longrightarrow & R_{C,P} Q_P E \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ R_C Q_P E & \longrightarrow & * \end{array}$$

Finally, the last diagram is fibrant replacement in the model category $L_C(\mathcal{M})$, and so the argument that $R_C Q_E \rightarrow R_C Q_P E$ is a C -local equivalence remains unchanged.

$$\begin{array}{ccccc} Q_E & \longrightarrow & Q_P E & & \\ \downarrow & & \downarrow & & \\ R_C Q_E & \longrightarrow & R_C Q_P E & \longrightarrow & R_{C,P} Q_P E \end{array}$$

The composite across the bottom $R_C Q_E \rightarrow R_{C,P} Q_P E$ is a weak equivalence between fibrant objects in $L_C(\mathcal{M})$ and so is a weak equivalence in \mathcal{M} , as in the proof of the theorem.

Finally, for the case of E cofibrant in the semi-model category $P\text{-alg}(\mathcal{M})$, note that the localization map $E \rightarrow L_C(E)$ is again fibrant replacement $E \rightarrow R_{C,P} E$ in $P\text{-alg}(L_C(\mathcal{M}))$. This exists because the domain is cofibrant by assumption. By construction, this map is a P -algebra homomorphism, as desired.

□

4. MONOIDAL BOUSFIELD LOCALIZATIONS

In both Theorem 3.2 and Corollary 3.4 we assumed that $L_C(\mathcal{M})$ is a monoidal model category. In this section we provide conditions on \mathcal{M} and C so that this occurs. First, we provide an example demonstrating that the pushout product axiom can fail for $L_C(\mathcal{M})$, even if it holds for \mathcal{M} .

Example 4.1. It is not true that every Bousfield localization of a monoidal model category is a monoidal model category. Let $R = \mathbb{F}_2[\Sigma_3]$. An R module is simply an \mathbb{F}_2 vector space with an action of the symmetric group Σ_3 . Because R is a Frobenius ring, we may pass from R -mod to the *stable module category* $StMod(R)$ by identifying any two morphisms whose difference factors through a projective module.

Section 2.2 of [25] introduces a model category \mathcal{M} of R -modules whose homotopy category is $StMod(R)$. Furthermore, a series of propositions in Section 2.2 demonstrate that \mathcal{M} is a finitely generated, combinatorial, stable model category in which all objects are cofibrant (hence, \mathcal{M} is also tractable and left proper). Proposition 4.2.15 of [25] proves that for $R = \mathbb{F}_2[\Sigma_3]$, this model category is a monoidal model category because R is a Hopf algebra over \mathbb{F}_2 . The monoidal product of two R -modules is $M \otimes_{\mathbb{F}_2} N$ where R acts via its diagonal $R \rightarrow R \otimes_{\mathbb{F}_2} R$.

We now check that cofibrant objects are flat in \mathcal{M} . By the pushout product axiom, $X \otimes -$ is left Quillen. Since all objects are cofibrant, all weak equivalences are weak equivalences between cofibrant objects. So Ken Brown's lemma implies $X \otimes -$ preserves weak equivalences.

Let $f : 0 \rightarrow \mathbb{F}_2$, where the codomain has the trivial Σ_3 action. We'll show that the Bousfield localization with respect to f cannot be a monoidal Bousfield localization. First observe that being f -locally trivial is equivalent to having no Σ_3 -fixed points, and this is equivalent to failing to admit Σ_3 -equivariant maps from \mathbb{F}_2 (the non-identity element would need to be taken to a Σ_3 -fixed point because the Σ_3 -action on \mathbb{F}_2 is trivial).

If the pushout product axiom held in $L_f(\mathcal{M})$ then the pushout product of two f -locally trivial cofibrations g, h would have to be f -locally trivial. We will now demonstrate an f -locally trivial object N for which $N \otimes_{\mathbb{F}_2} N$ is not f -locally trivial, so $(\emptyset \rightarrow N) \square (\emptyset \rightarrow N)$ is not a trivial cofibration in $L_f(\mathcal{M})$.

Define $N \cong \mathbb{F}_2 \oplus \mathbb{F}_2$ where the element (12) sends $a = (1, 0)$ to $b = (0, 1)$ and the element (123) sends a to b and b to $c = a + b$. The reader can check that (12)(123) acts the same as (123)²(12), so that this is a well-defined Σ_3 -action. This object N is f -locally trivial. It does not admit any maps from \mathbb{F}_2 because it has no Σ_3 -fixed points. However, $N \otimes_{\mathbb{F}_2} N$ is not f -locally trivial because $N \otimes_{\mathbb{F}_2} N$ does admit a map from \mathbb{F}_2 which takes the non-identity element of \mathbb{F}_2 to the Σ_3 -invariant element $a \otimes a + b \otimes b + c \otimes c$. Thus, $L_f(\mathcal{M})$ is not a monoidal model category.

In order to get around examples such as the above we must place hypotheses on the maps C which we are inverting. A similar program was conducted in [9], where localizations of stable model categories were assumed to commute with suspension. Similarly, a condition on a stable localization to ensure that it is additionally monoidal was given in Definition 5.2 of [2] and the same condition appeared in Theorem 4.46 of [3]. This condition states that $C \square I$ is contained in the C -local equivalences.

Remark 4.2. The counterexample above fails to satisfy the condition that $C \square I$ is contained in the C -local equivalences. If this condition were satisfied then I would be contained in the f -local equivalences and this would imply all cofibrant objects (hence all objects) are f -locally trivial. But $0 \rightarrow N \otimes_{\mathbb{F}_2} N$ is not f -locally trivial. Thus, this counterexample has no bearing on the work of [2] or [3].

Remark 4.3. The counterexample demonstrates a general principle which we now highlight. In any G -equivariant world, there are multiple spheres due to the different group actions. In the example above, one can suspend by representations of Σ_n , i.e. copies of \mathbb{F}_2 on which Σ_n acts. The 1-point compactification of such an object is a sphere S^n on which Σ_n acts. A localization which kills a representation sphere should not be expected to respect the monoidal structure, because not all acyclic cofibrant objects can be built from one of the representation spheres alone. In particular, $N \otimes N$ will not be in the smallest thick subcategory generated by \mathbb{F}_2 . The point is that the homotopy categories of stable model categories in an equivariant context are not monogenic axiomatic stable homotopy categories in the sense of [28].

Note that this example also demonstrates that the monoid axiom can fail on $L_C(\mathcal{M})$. The author does not know an example of a model category satisfying the pushout product axiom but failing the monoid axiom.

In our applications we will need to know that $L_C(\mathcal{M})$ satisfies the pushout product axiom, the unit axiom, and the axiom that cofibrant objects are flat. We therefore give a name to such localizations, and then we characterize them.

Definition 4.4. A Bousfield localization L_C is said to be a *monoidal Bousfield localization* if $L_C(\mathcal{M})$ satisfies the pushout product axiom, the unit axiom, and the axiom that cofibrant objects are flat.

Theorem 4.5. *Suppose \mathcal{M} is a tractable monoidal model category in which cofibrant objects are flat. Let I denote the generating cofibrations of \mathcal{M} . Then L_C is a monoidal Bousfield localization if and only if every map of the form $f \otimes id_K$, where f is in C and K is a domain or codomain of a map in I , is a C -local equivalence.*

Theorem 4.6. *Suppose \mathcal{M} is a cofibrantly generated monoidal model category in which cofibrant objects are flat. Then L_C is a monoidal Bousfield localization if and only if every map of the form $f \otimes id_K$, where f is in C and K is cofibrant, is a C -local equivalence.*

We shall prove Theorem 4.5 in Subsection 4.1 and we shall prove Theorem 4.6 in Subsection 4.2. These theorems demonstrate precisely what must be done if one wishes to invert a given set of morphisms C and ensure that the resulting model structure is a monoidal model structure.

Definition 4.7. Suppose \mathcal{M} is tractable, left proper, and either cellular or combinatorial. The *smallest monoidal Bousfield localization* which inverts a given set of morphisms C is the Bousfield localization with respect to the set $C' = \{C \otimes id_K\}$ where K runs through the domains and codomains of the generating cofibrations I .

This notion has already been used in [30]. The reason for the tractability hypothesis is to ensure that C' is a set. Requiring left properness and either cellularity or combinatoriality ensures that $L_{C'}$ exists. The smallest Bousfield localization has a universal property, which we now highlight.

Proposition 4.8. *Suppose C' is the smallest monoidal Bousfield localization inverting C , and let $j : \mathcal{M} \rightarrow L_{C'}(\mathcal{M})$ be the left Quillen functor realizing the localization. Suppose \mathcal{N} is a monoidal model category with cofibrant objects flat. Suppose $F : \mathcal{M} \rightarrow \mathcal{N}$ is a monoidal left Quillen functor such that $\mathbb{L}F$ takes the images of C in $\text{Ho}(\mathcal{M})$ to isomorphisms in $\text{Ho}(\mathcal{N})$. Then there is a unique monoidal left Quillen functor $\delta : L_{C'}\mathcal{M} \rightarrow \mathcal{N}$ such that $\delta j = F$.*

Proof. Suppose $F : \mathcal{M} \rightarrow \mathcal{N}$ is a monoidal left Quillen functor, that \mathcal{N} has cofibrant objects flat, and that $\mathbb{L}F$ takes the images of C in $\text{Ho}(\mathcal{M})$ to isomorphisms in $\text{Ho}(\mathcal{N})$. Then F also takes the images of maps in C' to isomorphisms in $\text{Ho}(\mathcal{N})$, because for any $f \in C$ and any cofibrant K , $F(f \otimes K) \cong F(f) \otimes F(K)$ is a weak equivalence in \mathcal{N} . This is because $F(K)$ is cofibrant in \mathcal{N} (as F is left Quillen), cofibrant objects are flat in \mathcal{N} , and $F(f)$ is a weak equivalence in \mathcal{N} by hypothesis.

The universal property of the localization $L_{C'}$ then provides a unique left Quillen functor $\delta : L_{C'}\mathcal{M} \rightarrow \mathcal{N}$ which is the same as F on objects and morphisms (c.f. Theorem 3.3.18 and Theorem 3.3.19 in [23], which also provide uniqueness for δ). In particular, δ is a monoidal functor and $\delta q = Fq : F(QS) \rightarrow F(S)$ is a weak equivalence in \mathcal{N} because the cofibrant replacement $QS \rightarrow S$ is the same in $L_{C'}(\mathcal{M})$ as in \mathcal{M} . So δ is a unique monoidal left Quillen functor as required, and the commutativity $\delta j = F$ follows immediately from the definition of δ . \square

4.1. Proof of Theorem 4.5. In this section we will prove Theorem 4.5. We first prove that under the hypotheses of Theorem 4.5, cofibrant objects are flat in $L_C(\mathcal{M})$.

Proposition 4.9. *Let \mathcal{M} be a tractable monoidal model category in which cofibrant objects are flat. Let I denote the generating cofibrations of \mathcal{M} . Suppose that every map of the form $f \otimes id_K$, where f is in C and K is a domain or codomain of a map in I , is a C -local equivalence. Then cofibrant objects are flat in $L_C(\mathcal{M})$.*

Proof. We must prove that the class of maps $\{g \otimes X \mid g \text{ is a } C\text{-local equivalence and } X \text{ is a cofibrant object}\}$ is contained in the C -local equivalences. Let X be a cofibrant

object in $L_C(\mathcal{M})$ (equivalently, in \mathcal{M}). Let $g : A \rightarrow B$ be a C -local equivalence. To prove $-\otimes X$ preserves C -local equivalences, it suffices to show that it takes $L_C(\mathcal{M})$ trivial cofibrations between cofibrant objects to weak equivalences. This is because we can always do cofibrant replacement on g to get $Qg : QA \rightarrow QB$. While Qg need not be a cofibration in general, we can always factor it into $QA \hookrightarrow Z \xrightarrow{\cong} QB$. By abuse of notation we will continue to use the symbol QB to denote Z , and we will rename the cofibration $QA \rightarrow Z$ as Qg since Z is cofibrant and maps via a trivial fibration to B . Smashing with X gives:

$$\begin{array}{ccc} QA \otimes X & \longrightarrow & QB \otimes X \\ \downarrow & & \downarrow \\ A \otimes X & \longrightarrow & B \otimes X \end{array}$$

If we prove that $Qg \otimes X$ is a C -local equivalence, then $g \otimes X$ must also be by the 2-out-of-3 property, since the vertical maps are weak equivalences in \mathcal{M} due to X being cofibrant and cofibrant objects being flat in \mathcal{M} . So we may assume that g is an $L_C(\mathcal{M})$ trivial cofibration between cofibrant objects. Since X is built as a transfinite composition of pushouts of maps in I , we proceed by transfinite induction. For the rest of the proof, let K, K_1 , and K_2 denote domains/codomains of maps in I . These objects are cofibrant in \mathcal{M} by hypothesis, so they are also cofibrant in $L_C(\mathcal{M})$.

For the base case $X = K$ we appeal to Theorem 3.3.18 in [23]. The composition $F = id \circ K \otimes - : \mathcal{M} \rightarrow \mathcal{M} \rightarrow L_C(\mathcal{M})$ is left Quillen because K is cofibrant. F takes maps in C to weak equivalences by hypothesis. So Theorem 3.3.18 implies F induces a left Quillen functor $K \otimes - : L_C(\mathcal{M}) \rightarrow L_C(\mathcal{M})$. Thus, $K \otimes -$ takes C -local equivalences between cofibrant objects to C -local equivalences and in particular takes Qg to a C -local equivalence. Note that this is the key place in this proof where we use the hypothesis that L_C is a monoidal Bousfield localization. This theorem is the primary tool when one wishes to get from a statement about C to a statement about all C -local equivalences.

For the successor case, suppose X_α is built from K as above and is flat in $L_C(\mathcal{M})$. Suppose $X_{\alpha+1}$ is built from X_α and a map in I via a pushout diagram:

$$\begin{array}{ccc} K_1 & \xrightarrow{i} & K_2 \\ \downarrow & & \downarrow \\ X_\alpha & \longrightarrow & X_{\alpha+1} \end{array}$$

We smash this diagram with $g : A \rightarrow B$ and note that smashing a pushout square with an object yields a pushout square.

$$\begin{array}{ccccc}
A \otimes K_1 & \xrightarrow{A \otimes i} & A \otimes K_2 & & \\
\downarrow & \searrow^{g \otimes K_1} & \downarrow & \searrow^{g \otimes K_2} & \\
& B \otimes K_1 & \xrightarrow{\quad} & B \otimes K_2 & \\
& \downarrow & & \downarrow & \\
A \otimes X_\alpha & \xrightarrow{B \otimes i} & A \otimes X_{\alpha+1} & & \\
\downarrow & \searrow^{g \otimes X_\alpha} & \downarrow & \searrow^{g \otimes X_{\alpha+1}} & \\
& B \otimes X_\alpha & \xrightarrow{\quad} & B \otimes X_{\alpha+1} & \\
& \downarrow & & \downarrow &
\end{array}$$

Because g is a cofibration of cofibrant objects, A and B are cofibrant. Because pushouts of cofibrations are cofibrations, $X_\alpha \hookrightarrow X_{\alpha+1}$ for all α . Because X_0 is cofibrant, X_α is cofibrant for all α . So all objects above are cofibrant. Furthermore, $g \otimes K_i = g \square (0 \hookrightarrow K_i)$. Thus, by the Pushout Product axiom on \mathcal{M} and the fact that cofibrations in \mathcal{M} match those in $L_C(\mathcal{M})$, these maps are cofibrations.

Finally, the maps $g \otimes K_i$ are weak equivalences in $L_C(\mathcal{M})$ by the base case above, while $g \otimes X_\alpha$ is a weak equivalence in $L_C(\mathcal{M})$ by the inductive hypothesis. Thus, by Dan Kan's Cube Lemma (Lemma 5.2.6 in [25]), the map $g \otimes X_{\alpha+1}$ is a weak equivalence in $L_C(\mathcal{M})$.

For the limit case, suppose we are given a cofibrant object $X = \operatorname{colim}_{\alpha < \beta} X_\alpha$ where each X_α is cofibrant and flat in $L_C(\mathcal{M})$. Because each X_α is cofibrant, $g \otimes X_\alpha = g \square (0 \hookrightarrow X_\alpha)$ is still a cofibration. By the inductive hypothesis, each $g \otimes X_\alpha$ is also a \mathcal{C} -local equivalence, hence a trivial cofibration in $L_C(\mathcal{M})$. Since trivial cofibrations are always closed under transfinite composition, $g \otimes X = g \otimes \operatorname{colim} X_\alpha = \operatorname{colim}(g \otimes X_\alpha)$ is also a trivial cofibration in $L_C(\mathcal{M})$. \square

We now pause for a moment to extract the key point in the proof above, where we applied the universal property of Bousfield localization. This is a reformulation Theorem 3.3.18 in [23] which will be helpful to us in the sequel.

Lemma 4.10. *A left Quillen functor $F : \mathcal{M} \rightarrow \mathcal{M}$ induces a left Quillen functor $L_C F : L_C(\mathcal{M}) \rightarrow L_C(\mathcal{M})$ if and only if for all $f \in \mathcal{C}$, $F(f)$ is \mathcal{C} -local equivalence.*

We turn now to the unit axiom.

Proposition 4.11. *If \mathcal{M} satisfies the unit axiom then any Bousfield localization $L_C(\mathcal{M})$ satisfies the unit axiom. If cofibrant objects are flat in \mathcal{M} then the map $QS \otimes Y \rightarrow Y$ which is induced by cofibrant replacement $QS \rightarrow S$ is a weak equivalence for all Y , not just cofibrant Y . Furthermore, for any weak equivalence $f : K \rightarrow L$ between cofibrant objects, $f \otimes Y$ is a weak equivalence.*

Proof. Since $L_C(\mathcal{M})$ has the same cofibrations as \mathcal{M} , it must also have the same trivial fibrations. Thus, it has the same cofibrant replacement functor and the same

cofibrant objects. Thus, the unit axiom on $L_C(\mathcal{M})$ follows directly from the unit axiom on \mathcal{M} , because a weak equivalence in \mathcal{M} is in particular a C -local equivalence.

We now assume cofibrant objects are flat and that Y is an object of \mathcal{M} . Consider the following diagram:

$$\begin{array}{ccc} QS \otimes QY & \longrightarrow & QY \\ \downarrow & & \downarrow \\ QS \otimes Y & \longrightarrow & Y \end{array}$$

The top map is a weak equivalence by the unit axiom for the cofibrant object QY . The left vertical map is a weak equivalence because cofibrant objects are flat and QS is cofibrant. The right vertical is a weak equivalence by definition of QY . Thus, the bottom arrow is a weak equivalence by the 2-out-of-3 property.

For the final statement we again apply cofibrant replacement to Y and we get

$$\begin{array}{ccc} K \otimes QY & \longrightarrow & L \otimes QY \\ \downarrow & & \downarrow \\ K \otimes Y & \longrightarrow & L \otimes Y \end{array}$$

Again the top horizontal map and the vertical maps are weak equivalences because cofibrant objects are flat (for the first use that QX is cofibrant, for the second use that K and L are cofibrant). \square

We turn now to proving Theorem 4.5. As mentioned in the proof of Proposition 4.9, if h and g are $L_C(\mathcal{M})$ -cofibrations then they are \mathcal{M} -cofibrations and so $h \square g$ is a cofibration in \mathcal{M} (hence in $L_C(\mathcal{M})$) by the pushout product axiom on \mathcal{M} . To verify the rest of the pushout product axiom on $L_C(\mathcal{M})$ we must prove that if h is a trivial cofibration in $L_C(\mathcal{M})$ and g is a cofibration in $L_C(\mathcal{M})$ then $h \square g$ is a weak equivalence in $L_C(\mathcal{M})$.

Proposition 4.12. *Let \mathcal{M} be a tractable monoidal model category in which cofibrant objects are flat. Let I denote the generating cofibrations of \mathcal{M} . Suppose that every map of the form $f \otimes id_K$, where f is in C and K is a domain or codomain of a map in I , is a C -local equivalence. Then $L_C(\mathcal{M})$ satisfies the pushout product axiom.*

Proof. We have already remarked that the cofibration part of the pushout product axiom on $L_C(\mathcal{M})$ follows from the pushout product axiom on \mathcal{M} , since the two model categories have the same cofibrations. By Proposition 4.2.5 of [25] it is sufficient to check the pushout product axiom on generating (trivial) cofibrations. So suppose $h : X \rightarrow Y$ is an $L_C(\mathcal{M})$ trivial cofibration and $g : K \rightarrow L$ is a generating cofibration in $L_C(\mathcal{M})$ (equivalently, in \mathcal{M}). Then we must show $h \square g$ is an $L_C(\mathcal{M})$ trivial cofibration

By hypothesis on \mathcal{M} , K and L are cofibrant. Because h is a cofibration, $K \otimes h$ and $L \otimes h$ are cofibrations by the pushout product axiom on \mathcal{M} (because $K \otimes h = (\emptyset \hookrightarrow K) \square h$). By Proposition 4.9, cofibrant objects are flat in $L_C(\mathcal{M})$. So $K \otimes h$ and $L \otimes h$ are also weak equivalences. In particular, $K \otimes -$ and $L \otimes -$ are left Quillen functors. Consider the following diagram:

$$\begin{array}{ccc}
 K \otimes X \hookrightarrow & \xrightarrow{\cong} & K \otimes Y \\
 \downarrow & & \downarrow \\
 L \otimes X & \xrightarrow{\cong} & (K \otimes Y) \amalg_{K \otimes X} (L \otimes X) \\
 & \searrow & \downarrow h \square g \\
 & & L \otimes Y
 \end{array}$$

$\xrightarrow{\cong} L \otimes Y$

The map $L \otimes X \rightarrow (K \otimes Y) \amalg_{K \otimes X} (L \otimes X)$ is a trivial cofibration because it is the pushout of a trivial cofibration. Thus, by the 2-out-of-3 property for the lower triangle, $h \square g$ is a weak equivalence. Since we already knew it was a cofibration (because it is so in \mathcal{M}), this means it is a trivial cofibration. \square

We are now ready to complete the proof of Theorem 4.5.

Proof of Theorem 4.5. We begin with the forwards direction. Suppose $L_C(\mathcal{M})$ satisfies the pushout product axiom and has cofibrant objects flat. Let f be any map in \mathcal{C} . Note that in particular, f is a \mathcal{C} -local equivalence. Because cofibrant objects are flat, the map $f \otimes K$ is a \mathcal{C} -local equivalence for any cofibrant K . So the collection $\mathcal{C} \otimes K$ is contained in the \mathcal{C} -local equivalences, where K runs through the class of cofibrant objects, i.e. L_C is a monoidal Bousfield localization.

For the converse, we apply our three previous propositions. That cofibrant objects are flat in $L_C(\mathcal{M})$ is the content of Proposition 4.9. The unit axiom on $L_C(\mathcal{M})$ follows from Proposition 4.11 applied to $L_C(\mathcal{M})$. That the pushout product axiom holds on $L_C(\mathcal{M})$ is Proposition 4.12. \square

4.2. Proof of Theorem 4.6. We will now prove Theorem 4.6, following the outline above. The proof that cofibrant objects are flat in $L_C(\mathcal{M})$ will proceed just as it did in Proposition 4.9. Proposition 4.11 again implies the unit axiom in $L_C(\mathcal{M})$. Deducing the pushout product axiom on $L_C(\mathcal{M})$ will be more complicated without the tractability hypothesis. For this reason, we need the following lemma. First, let I' be obtained from the generating cofibrations I by applying any cofibrant replacement Q to all $i \in I$ and then taking the left factor in the cofibration-trivial fibration factorization of Q_i . So I' consists of cofibrations between cofibrant objects.

Lemma 4.13. *Suppose \mathcal{M} is a left proper model category cofibrantly generated by sets I and J in which the domains of maps in J are small relative to I -cell. Then the sets $I' \cup J$ and J cofibrantly generate \mathcal{M} .*

Proof. We verify the conditions given in Definition 11.1.2 of [23]. We have not changed J , so the fibrations are still precisely the maps satisfying the right lifting property with respect to J and the maps in J still permit the small object argument because the domains are small relative to J -cell.

Any map which has the right lifting property with respect to all maps in I is a trivial fibration, so will in particular have the right lifting property with respect to all cofibrations, hence with respect to maps in $I' \cup J$. Conversely, suppose p has the right lifting property with respect to all maps in $I' \cup J$. We are faced with the following lifting problem:

$$\begin{array}{ccccc} A' & \longrightarrow & A & \longrightarrow & X \\ \downarrow i' & & \downarrow i & & \downarrow p \\ B' & \longrightarrow & B & \longrightarrow & Y \end{array}$$

Because p has lifting with respect to $I' \cup J$, it has the right lifting property with respect to J . This guarantees us that p is a fibration. Now because \mathcal{M} is left proper, Proposition 13.2.1 in [23] applies to solve the lifting diagram above. In particular, because p has the right lifting property with respect to I' , p must have the right lifting property with respect to I . Thus, p is a trivial fibration as desired.

We now turn to smallness. Any domain of a map in J is small relative to J -cell, but in general this would not imply smallness relative to I -cell. We have assumed the domains of maps in J are small relative to I -cell, so they are small relative to $(J \cup I')$ -cell because $J \cup I'$ is contained in I -cell.

Any domain of a map in I' is of the form QA for A a domain of a map in I . We will show QA is small relative to I -cell. As $J \cup I'$ is contained in I -cell this will show QA is small relative to $J \cup I'$. Consider the construction of QA as the left factor in

$$\begin{array}{ccc} & QA & \\ \nearrow & & \searrow \simeq \\ \emptyset & \longrightarrow & A \end{array}$$

The map $\emptyset \rightarrow QA$ is in I -cell, so QA is a colimit of cells (let us say κ_A many), each of which is κ -small where κ is the regular cardinal associated to I by Proposition 11.2.5 of [23]. So for any λ greater than the cofinality of $\max(\kappa, \kappa_A)$, a map from QA to a λ -filtered colimit of maps in I -cell must factor through some stage of the colimit because all the cells making up QA will factor in this way. One can find a uniform λ for all objects QA by an appeal to Lemma 10.4.6 of [23].

□

Remark 4.14. In a combinatorial model category no smallness hypothesis needs to be made because all objects are small. In a cellular model category, the assumption

that the domains of J are small relative to cofibrations is included. As these hypotheses are standard when working with left Bousfield localization, we shall say no more about the additional smallness hypothesis placed on J above.

Corollary 4.15. *Suppose \mathcal{M} is a left proper model category cofibrantly generated by sets I and J in which the domains of maps in J are small relative to I -cell and are cofibrant. Then \mathcal{M} can be made tractable.*

Remark 4.16. Note that this corollary does not say that any left proper, cofibrantly generated model category can be made tractable. There is an example due to Carlos Simpson (found on page 199 of [41]) of a left proper, combinatorial model category which cannot be made tractable. In this example the cofibrations and trivial cofibrations are the same, so cannot be leveraged against one another in the way we have done above.

We are now prepared to prove Theorem 4.6.

Proof of Theorem 4.6. First, if L_C is a monoidal Bousfield localization then every map of the form $f \otimes id_K$, where $f \in C$ and K is cofibrant, is a C -local equivalence. This is because f is a C -local equivalence and cofibrant objects are flat in $L_C(\mathcal{M})$. We turn now to the converse.

Assume every map of the form $f \otimes id_K$, where $f \in C$ and K is cofibrant, is a C -local equivalence. Then cofibrant objects are flat in $L_C(\mathcal{M})$. To see this, let X be cofibrant and define $F(-) = X \otimes -$. Then Lemma 4.10 implies F is left Quillen when viewed as a functor from $L_C(\mathcal{M})$ to $L_C(\mathcal{M})$. So F takes C -local equivalences between cofibrant objects to C -local equivalences. By the reduction at the beginning of the proof of Proposition 4.9, this implies F takes all C -local equivalences to C -local equivalences.

Next, the unit axiom on $L_C(\mathcal{M})$ follows from the unit axiom on \mathcal{M} , by Proposition 4.11. Finally, we must prove the pushout product axiom holds on $L_C(\mathcal{M})$. As in the proof of Proposition 4.12, Proposition 4.2.5 of [25] reduces the problem to checking the pushout product axiom on a set of generating (trivial) cofibrations. We apply Lemma 4.13 to \mathcal{M} and check the pushout product axiom with respect to this set of generating maps.

As in the tractable case, let $h : X \rightarrow Y$ be a trivial cofibration in $L_C(\mathcal{M})$ and let $g : K \rightarrow L$ be a generating cofibration. By the lemma, the map g is either a cofibration between cofibrant objects or a trivial cofibration in \mathcal{M} . If the former, then the proof of Proposition 4.12 goes through verbatim and proves that $h \square g$ is an $L_C(\mathcal{M})$ -trivial cofibration, since cofibrant objects are flat in $L_C(\mathcal{M})$. If the latter, then because g is a trivial cofibration in \mathcal{M} and h is a cofibration in \mathcal{M} we may apply the pushout product axiom on \mathcal{M} to see that $h \square g$ is a trivial cofibration in \mathcal{M} (hence in $L_C(\mathcal{M})$ too). This completes the proof of the pushout product axiom on $L_C(\mathcal{M})$. \square

Remark 4.17. The use of the lemma demonstrates that this proposition proves something slightly more general. Namely, if \mathcal{M} is cofibrantly generated, left proper, has cofibrant objects flat, and the class of cofibrations is closed under pushout product then \mathcal{M} satisfies the pushout product axiom.

Additionally, one could also prove the forwards direction in the theorem using only that $L_C(\mathcal{M})$ satisfies the pushout product axiom. For any cofibrant K we have a cofibration $\phi_K : \emptyset \hookrightarrow K$. Note that for any $f \in \mathcal{C}$, $f \otimes K = f \square \phi_K \subset \mathcal{C}$ -local equivalences, because f is a trivial cofibration in $L_C(\mathcal{M})$.

We record this remark because in the future we hope to better understand the connection between monoidal Bousfield localizations and the closed localizations which appeared in [9], and this remark may be useful.

5. PRESERVATION OF ALGEBRAS OVER COFIBRANT OPERADS

In this section we will provide several applications of the results in the previous section. We remind the reader that for operads valued in \mathcal{M} , a map of operads $A \rightarrow B$ is said to be a trivial fibration if $A_n \rightarrow B_n$ is a trivial fibration in \mathcal{M} for all n . An operad P is said to be *cofibrant* if the map from the initial operad into P has the left lifting property in the category of operads with respect to all trivial fibrations of operads. An operad P is said to be Σ -*cofibrant* if it has this left lifting property only in the category of symmetric sequences. The E_∞ -operads considered in [33] are Σ -cofibrant precisely because the n^{th} space is assumed to be an $E\Sigma_n$ space.

We begin with a theorem due to Markus Spitzweck, proven as Theorem 5 in [42] and as Theorem A.8 in [17], which makes it clear that the hypotheses of Corollary 3.4 are satisfied when L_C is a monoidal Bousfield localization and P is a cofibrant operad.

Theorem 5.1. *Suppose P is a Σ -cofibrant operad and \mathcal{M} is a monoidal model category. Then P -alg is a semi-model category.*

This theorem, applied to both \mathcal{M} and $L_C(\mathcal{M})$ (if the localization is monoidal), endows the categories of P -algebras in \mathcal{M} and $L_C(\mathcal{M})$ with inherited semi-model structures. By Corollary 3.4, monoidal Bousfield localizations preserve algebras over Σ -cofibrant operads. In particular, monoidal localizations preserve A_∞ and E_∞ algebras in \mathcal{M} , where A_∞ and E_∞ are any operads which are Σ -cofibrant and weakly equivalent to *Ass* and *Com* in the category $\text{Coll}(\mathcal{M})$. When \mathcal{M} is a category of spectra we are free to work with operads valued in spaces because the Σ^∞ functor will take a (Σ -cofibrant) space-valued operad to a (Σ -cofibrant) spectrum-valued operad with the same algebras.

5.1. Spaces and Spectra. For topological spaces the situation is especially nice. We always work in the context of pointed spaces.

Proposition 5.2. *Let \mathcal{M} be the model category of (pointed) simplicial sets or k -spaces. Every Bousfield localization of \mathcal{M} is a monoidal Bousfield localization.*

Proof. For a review of the monoidal model structures on spaces and simplicial sets see Chapter 4 of [25]. Both are tractable, left proper, monoidal model categories with cofibrant objects flat.

For $\mathcal{M} = sSet$ we can simply rely on Theorem 4.1.1 of [23], which guarantees that $L_C(\mathcal{M})$ is a simplicial model category. The pushout product axiom is equivalent to the SM7 axiom for $sSet$, so this proves $L_C(\mathcal{M})$ is a monoidal model category and hence that L_C is monoidal. There is also an elementary proof of this fact which is obtained from the proof below by replacing $F(-, -)$ everywhere by $\text{map}(-, -)$.

We turn now to $\mathcal{M} = Top$. By definition, any Bousfield localization L_C will be a monoidal Bousfield localization as soon as we show $C \wedge S_+^n$ is contained in the C -local equivalences (the codomains of the generating cofibrations are contractible, so do not matter). As remarked in the discussion below Definition 4.1 in [30], for topological model categories Bousfield localization with respect to a set of cofibrations can be defined using topological mapping spaces rather than simplicial mapping spaces (at least when all maps in C are cofibrations). Let $F(X, Y)$ denote the space of based maps $X \rightarrow Y$.

We will make use of Proposition 3.2 in [26], a version of which states that because Top is left proper and cofibrantly generated, a map f is a weak local equivalence if and only if $F(T, f)$ is a weak equivalence of topological spaces for all T in the (co)domains of the generating cofibrations I in Top .

Now consider the following equivalent statements, where T runs through the domains and codomains of generating cofibrations.

f is a C -local equivalence
iff $F(f, Z)$ is a weak equivalence for all C -local Z
iff $F(T, F(f, Z))$ is a weak equivalence for all C -local Z and all T (by Prop. 3.2)
iff $F(T \wedge f, Z)$ is a weak equivalence for all C -local Z (by adjointness)
iff $T \wedge f$ is a C -local equivalence

This proves that the class of C -local equivalences is closed under smashing with a domain or codomain of a generating cofibration, so L_C is a monoidal Bousfield localization.

□

The reader may wonder whether all Bousfield localizations preserve algebras over cofibrant operads in general model categories \mathcal{M} , i.e. whether all Bousfield localizations are monoidal. This is false, as demonstrated by the following example, which can be found at the end of Section 6 in [9].

Example 5.3. Let \mathcal{M} be symmetric spectra, orthogonal spectra, or \mathbb{S} -modules.

Recall that in topological spaces, the n^{th} Postnikov section functor P_n is the Bousfield localization L_f corresponding to the map Σf where $f : S^n \rightarrow *$. Applying Σ^∞ gives a map of spectra and we again denote by P_n the Bousfield localization with respect to this map.

The Bousfield localization P_{-1} on \mathcal{M} does not preserve A_∞ -algebras. If R is a non-connective A_∞ -algebra then the unit map $\nu : S \rightarrow P_{-1}R$ is null because $\pi_0(P_{-1}R) = 0$. Thus, $P_{-1}R$ cannot admit a ring spectrum structure (not even up to homotopy) because $S \wedge P_{-1}R \rightarrow P_{-1}R \wedge P_{-1}R \rightarrow P_{-1}R$ is not a homotopy equivalence as it would have to be for $P_{-1}R$ to be a homotopy ring.

In [9], examples of the sort above are prohibited by assuming that L -equivalences are closed under the monoidal product. It is then shown in Theorem 6.5 that for symmetric spectra this property is implied if the localization is stable, i.e. $L \circ \Sigma \simeq \Sigma \circ L$. We now compare our requirement that L_C be a monoidal Bousfield localization to existing results regarding preservation of monoidal structure.

Proposition 5.4. *Every monoidal Bousfield localization is stable. In a monogenic setting such as spectra, every stable localization is monoidal.*

This is clear, since suspending is the same as smashing with the suspension of the unit sphere.

The Postnikov section is clearly not stable, and indeed the counterexample above hinges on the fact that the section has truncated the spectrum by making trivial the degree in which the unit must live. Under the hypothesis that localization respects the monoidal product, Theorem 6.1 of [9] proves that cofibrant algebras over a cofibrant colored operad valued in $sSet_*$ or Top_* are preserved. Theorem 3.2 recovers this result in the case of operads, and improves on it by extending the class of operads so that they do not need to be valued in $sSet_*$ or Top_* , by discussing preservation of non-cofibrant algebras, by weakening the cofibrancy required of the operad to Σ -cofibrancy (using Theorem 5.1 above), and by potentially weakening the hypothesis on the localization. A different generalization of [9] has been given in [17].

Proposition 5.5. *Every Bousfield localization for which the local equivalences are closed under \otimes is a monoidal Bousfield localization, but the converse fails.*

Proof. To see why this fact is true, consider the maps id_K as L -equivalences when testing whether or not $id_K \otimes C$ is a C -local equivalence. To see that the converse fails, take C to be the generating trivial cofibrations of any cofibrantly generated model category in which the weak equivalences are not closed under \otimes . \square

Thus, our hypothesis on a monoidal Bousfield localization is strictly weaker than requiring L -equivalences to be closed under \otimes . Theorems 4.5 and 4.6 demonstrate that the hypothesis that $C \otimes id_K$ is contained in the C -local equivalences is best-possible, since it L_C is a monoidal Bousfield localization if and only if this

property holds, and without the pushout product axiom on $L_C(\mathcal{M})$ the question of preservation of algebras under localization is not even well-posed.

Remark 5.6. In light of the Postnikov Section example, the argument of Proposition 5.2 must break down for spectra. The precise place where the argument fails is the passage through $\text{map}(T, \text{map}(f, Z))$. In spectra, this expression has no meaning, because T is a spectrum but $\text{map}(f, Z)$ is a space. So the argument of Proposition 5.2 relies precisely on the fact that $\mathcal{M} = \text{set}$ (or $\mathcal{M} = \text{Top}$ in the topological case), so that the SM7 axiom for \mathcal{M} is precisely the same as the pushout product axiom.

Theorem 3.2 and Theorem 5.1 combine to prove that any monoidal Bousfield localization of spectra preserves A_∞ and E_∞ algebras. In particular, A_∞ and E_∞ algebras are preserved by stable Bousfield localizations such as L_E where E is a homology theory. So our results recover Theorems VIII.2.1 and VIII.2.2 of [13].

5.2. Equivariant Spectra. We turn now to the example which motivated Theorem 3.2, in the case where \mathcal{M} is the category of equivariant orthogonal spectra. The author learned this example from a talk given by Mike Hill at Oberwolfach (the proceedings can be found in [20]). A similar example appeared in [34]. Before presenting this motivating example, we must introduce some new notation.

Let G be a compact Lie group and let $\mathcal{M} = \mathcal{S}^G$ be the positive stable model structure on equivariant orthogonal spectra. Given a G -space X and a closed subgroup H , one may restrict the G action to H and obtain an H -space denoted $\text{res}_H(X)$. This association is functorial and lifts to a functor $\text{res}_H : \mathcal{S}^G \rightarrow \mathcal{S}^H$. This *restriction functor* has a left adjoint $G_+ \wedge_H (-)$, the *induction functor*. We refer the reader to Section 2.2.4 of [22] for more details. If one shifts focus to commutative monoids Comm_G in \mathcal{S}^G (equivalently to genuine E_∞ -algebras) then there is again a restriction functor $\text{res}_H : \text{Comm}_G \rightarrow \text{Comm}_H$ and it again has a left adjoint functor $N_H^G(-)$ called the *norm*. This functor is discussed in Section 2.3.2 of [22].

Example 5.7. There are localizations which destroy genuine commutative structure but which preserve naive E_∞ -algebra structure. For this example, let G be a (non-trivial) finite group.

Consider the reduced real regular representation $\bar{\rho}$ obtained by taking the quotient of the real regular representation ρ by the trivial representation, which is a summand. We write $\bar{\rho}_G = \rho_G - 1$ where 1 means the trivial representation $\mathbb{R}[e]$. Taking the one-point compactification of this representation yields a representation sphere $S^{\bar{\rho}}$. There is a natural inclusion $a_{\bar{\rho}} : S^0 \rightarrow S^{\bar{\rho}}$ induced by the inclusion of the trivial representation into $\bar{\rho}$. Consider the spectrum $E = \mathbb{S}[a_{\bar{\rho}}^{-1}]$ obtained from the unit \mathbb{S} (certainly a commutative algebra in \mathcal{S}^G) by localization with respect to $a_{\bar{\rho}}$. We will show that this spectrum does not admit maps from the norms of its restrictions, and hence cannot be commutative.

First, $\rho_G|_H = [G : H]\rho_H$, so $\bar{\rho}_G|_H = [G : H]\bar{\rho}_H + ([G : H]1 - 1)$. We will use this to prove that for all proper $H < G$, $res_H(E)$ is contractible. Because $[G : H] - 1$ is a number k greater than 0 we have $res_H S^{\bar{\rho}_G} = (S^{\bar{\rho}_H})^{\#[G:H]} \wedge S^k$. This means that as an H -spectrum it is contractible, because there is enough space in the S^k part to deform it to a point. Note, however, that E itself is not locally trivial. Thinking of S^0 as $\{0, \infty\}$ we see that the only fixed points of $a_{\bar{\rho}}$ are 0 and ∞ , so the map $a_{\bar{\rho}}$ is not equivariantly trivial.

If E were a commutative algebra in \mathcal{S}^G then the counit of the norm-restriction adjunction would provide a ring homomorphism $N_H^G res_H(E) \rightarrow E$. But the domain is contractible for every proper subgroup H because $res_H(E)$ is contractible. This cannot be a ring map unless E to be contractible, and we know E is not contractible because $a_{\bar{\rho}}$ fixes 0 and ∞ .

This example would have been a hole in the proof of the Kervaire Invariant One Theorem (because the spectrum $\Omega = D^{-1}MU^{(4)}$ needed to be commutative) if not for the following theorem from [21].

Theorem 5.8. *Let G be a finite group. Let L be a localization of equivariant spectra. If for all L -acyclics Z and for all subgroups H , $N_H^G Z$ is L -acyclic, then for all commutative G -ring spectra R , $L(R)$ is a commutative G -ring spectrum.*

The hypothesis in this theorem is designed so that the proof in [13] regarding preservation of E_∞ structure under localization (i.e. via the skeletal filtration) may go through. We wish to understand how our general preservation result relates to this example and theorem, so we now specialize Corollary 3.4 to the case of $\mathcal{M} = \mathcal{S}^G$, where G is a compact Lie group.

We must first understand the generating cofibrations. For Top^G , the (co)domains of maps in I take the form $((G/H) \times S^{n-1})_+$ and $((G/H) \times D^n)_+$ for H a closed subgroup of G , by Definition 1.1 in [31]. For \mathcal{S}^G , we first need a new piece of notation. For any finite dimensional orthogonal G -representation W there is an evaluation functor $Ev_W : \mathcal{S}^G \rightarrow Top^G$. This functor has a left adjoint F_W (see Proposition 3.1 in [30] for more details). The (co)domains of maps in I take the form $F_W((G/H)_+ \wedge S_+^{n-1})$ and $F_W((G/H)_+ \wedge D_+^n)$ by Definition 1.11 in [31], where W runs through some fixed G -universe \mathcal{U} . The latter are contractible, and so smashing with them does not make a difference. Observe that these are tractable model structures. That \mathcal{S}^G is a monoidal model category with cofibrant objects flat is verified in [31], and may also be deduced from Corollary 4.4 in [30]. Thus, for $\mathcal{M} = \mathcal{S}^G$, our preservation result (Corollary 3.4 together with Theorem 4.5) becomes:

Theorem 5.9. *Let G be a compact Lie group. In \mathcal{S}^G , a Bousfield localization L_C is monoidal if and only if $C \wedge F_W((G/H)_+ \wedge S_+^{n-1})$ is a C -local equivalence for all closed subgroups H of G , for all W in the universe, and for all n . Furthermore, such localizations preserve genuine equivariant commutativity.*

Ignoring suspensions, monoidal Bousfield localizations are precisely the ones for which L_C respects smashing with $(G/H)_+$ for all subgroups H . We think of these localizations as the ones which can ‘see’ the information of all subgroups. We now discuss Hill’s example in more detail, in light of this theorem. First, it is clear that Hill’s example fails to be a monoidal Bousfield localization because $E \wedge (G/H)_+$ is contractible for all proper H (see Section 2.3.2 in [22]), but as we have remarked E itself is not contractible (not even locally).

Example 4.1 has already demonstrated that localizations that kill a representation sphere should not be expected to be monoidal. The presence of $S^{\bar{P}}$ demonstrates that Hill’s example is analogous, but the example can also be viewed in another way. In Hill’s example, smashing with G/H for a non-trivial proper H is equivalent to suspending with respect to the representation sphere corresponding to H . In this light, Hill’s example is demonstrating that a monoidal Bousfield localization of spectra must be stable with respect to all representation spheres, and it can be seen as an equivariant analogue of Example 5.3. Hill’s example *is* stable with respect to the monoidal unit, so naive E_∞ algebras are preserved. The failure only manifests when a non-trivial, proper H is considered. However, because *any* such H will lead to a failure of $L(\mathbb{S}) = E$ to be commutative, Hill’s example is in some sense maximally bad. In light of this, it is natural to ask what happens when the localization respects some, but not all, of the subgroups of G . We will now answer this question. The following model structures are considered in Theorem 6.3 in [31] (on spectra either the stable or positive stable model structure can be used):

Definition 5.10. Let \mathcal{F} be a family of closed subgroups of G , i.e. a non-empty set of subgroups closed under conjugation and taking subgroups. Then the \mathcal{F} -fixed point model structure on pointed G -spaces is a cofibrantly generated model structure in which a map f is a weak equivalence (resp. fibration) if and only if f^H is a weak equivalence (resp. fibration) in Top for all $H \in \mathcal{F}$. We will denote this model structure by $Top^{\mathcal{F}}$. The generating (trivial) cofibrations are $(G/H \times g)_+$, where g is a generating (trivial) cofibration of topological spaces, and $H \in \mathcal{F}$.

The corresponding cofibrantly generated model structure on G -spectra will be denoted $\mathcal{S}^{\mathcal{F}}$. Again, weak equivalences (resp. fibrations) are maps f such that f^H is a weak equivalence (resp. fibration) of orthogonal spectra for all $H \in \mathcal{F}$. The generating (trivial) cofibrations are $F_W((G/H)_+ \wedge g)$ as H runs through \mathcal{F} , g runs through the generating (trivial) cofibrations of spaces, and W runs through some G -universe \mathcal{U} .

With the generating cofibrations in hand, Theorem 4.5 implies that monoidal Bousfield localizations in $\mathcal{S}^{\mathcal{F}}$ are characterized by the property that $C \wedge (G/H)_+$ is a C -local equivalence for all $H \in \mathcal{F}$ (again, ignoring suspensions). One can also define \mathcal{F} -fixed point semi-model structures $Oper^{\mathcal{F}}$ on the category of G -operads, e.g. by applying the general machinery of Theorem 12.2.A in [15].

Definition 5.11. Let $E_\infty^{\mathcal{F}}$ be the cofibrant replacement for the operad Com in the \mathcal{F} -fixed point semi-model structure on G -operads.

These operads form a lattice (ordered by family inclusion) interpolating between naive E_∞ (which corresponds to the family $\mathcal{F} = \{e\}$) and genuine E_∞ (which corresponds to the family $\mathcal{F} = \{All\}$ and which is denoted E_∞^G). These $E_\infty^{\mathcal{F}}$ operads isolate the difference between norm, restriction, and transfer. An $E_\infty^{\mathcal{F}}$ -algebra X has a multiplicative structure on $res_H(X)$ (compatible with the transfers) for every $H \in \mathcal{F}$. However, $N_H^G(res_H(X))$ need not have a multiplicative structure. The author feels these operads $E_\infty^{\mathcal{F}}$ are worthy of study in their own right, and so they will be considered further in joint work between the author and Javier Gutiérrez. In this work we prove that there is an \mathcal{F} -model structure on operads (rather than only a semi-model structure), we discuss rectification between these $E_\infty^{\mathcal{F}}$ operads and Com , and we provide a comparison to the N_∞ -operads recently studied in [6]. For now we will focus on how $E_\infty^{\mathcal{F}}$ -algebra structure interacts with Bousfield localization. We first re-formulate Example 5.7. This is the form of Hill's example which was presented in [20] for \mathcal{P} the family of proper subgroups.

Example 5.12. If X is an $E_\infty^{\mathcal{F}}$ -algebra then there is a localization L sending X to a naive E_∞ -algebra. Consider the cofiber sequence $E\mathcal{P}_+ \rightarrow S^0 \rightarrow \widetilde{E}\mathcal{P}$ for any family $\mathcal{P} \supseteq \mathcal{F}$ which does not contain G . Recall the fixed-point property of the space $E\mathcal{P}$ (discussed very nicely in Section 7 of [37]) and deduce:

$$(E\mathcal{P}_+)^H \simeq \begin{cases} *_{+} = S^0 & \text{if } H \in \mathcal{P} \\ \emptyset_{+} = * & \text{if } H \notin \mathcal{P} \end{cases}$$

For all H , the H -fixed points of S^0 are S^0 . So that the cofiber obtained by mapping this space into S^0 satisfies the following fixed-point property

$$(\widetilde{E}\mathcal{P})^H \simeq \begin{cases} * & \text{if } H \in \mathcal{P} \\ S^0 & \text{if } H \notin \mathcal{P} \end{cases}$$

Now apply Σ_+^∞ to the map $S^0 \rightarrow \widetilde{E}\mathcal{P}$. If G is a finite group then the resulting map $\mathbb{S} \rightarrow E$ is the same localization map considered in Example 5.7 (see Section 7 of [37]). This E is not contractible because $E\mathcal{P}_+$ is not homotopy equivalent to S^0 (since \mathcal{P} doesn't contain G), though $res_H(E\mathcal{P}_+)$ is homotopy equivalent to $res_H(S^0)$ for any $H \in \mathcal{P}$.

In this formulation it is clear that the map $\mathbb{S} \rightarrow E$ is a nullification which kills all maps out of the induced cells $G_+ \wedge_H (H/K)_+ = (G/H)_+$ for all $H \in \mathcal{P}$. With the characterization of monoidal Bousfield localizations in $\mathcal{S}^{\mathcal{F}}$, we can see that in order to produce a localization which sends $E_\infty^{\mathcal{F}}$ -algebras to naive E_∞ -algebras one need only apply the localization $S^0 \rightarrow \widetilde{E}\mathcal{F}$ rather than the localization $S^0 \rightarrow \widetilde{E}\mathcal{P}$ for the full family \mathcal{P} of proper subgroups of G .

The presentation in Example 5.13 for finite G makes it clear that this localization is simply killing a homotopy element (namely: the Euler class $a_{\bar{p}}$ discussed in Section 2.6.3 of [22]). The presentation in Example 5.12 has several benefits of its own: it generalizes to compact Lie groups G , it demonstrates that a smaller localization

is needed to destroy $E_\infty^{\mathcal{F}}$ -algebra structure rather than E_∞^G -algebra structure, and it provides a generalization of Hill's example in which localization can reduce one's place in the lattice $E_\infty^{\mathcal{F}}$ without reducing it all the way down to naive E_∞ .

Example 5.13. Localization can take an $E_\infty^{\mathcal{F}}$ -algebra E to a $E_\infty^{\mathcal{K}}$ -algebra for $\mathcal{K} \subseteq \mathcal{F}$. To define such a localization L we need to kill some, but not all, maps from induced cells corresponding to $H \in \mathcal{F}$. This can be done by inverting a wedge of maps which kills whichever induced cells one desires to kill, as long as this localization does not kill any induced cells for $K \in \mathcal{K}$. This can be accomplished by inverting only cells corresponding to H which intersect \mathcal{K} in the identity subgroup. Then because maps from induced cells corresponding to $K \in \mathcal{K}$ have not been killed, the resulting object LE has $E_\infty^{\mathcal{K}}$ -algebra structure inherited from E .

This example demonstrates once again that the key property of a localization L_C so that it preserves $E_\infty^{\mathcal{F}}$ -algebra structure is a compatibility condition governing the behavior of the maps C after the functor $-\wedge(G/H)_+$ is applied (as H runs through the family \mathcal{F}). We formalize this by another application of Corollary 3.4 and Theorem 4.5. First, observe that both $Oper^{\mathcal{F}}$ and $\mathcal{S}^{\mathcal{F}}$ are $Top^{\mathcal{F}}$ -model structures (in the sense of Definition 4.2.18 in [25]) and the cofibrancy of $E_\infty^{\mathcal{F}}$ is relative to the \mathcal{F} -model structure. Thus, from a model category theoretic standpoint, $E_\infty^{\mathcal{F}}$ -algebras are best viewed in $\mathcal{S}^{\mathcal{F}}$.

Theorem 5.14. *Let $\mathcal{M} = \mathcal{S}^G$ and let \mathcal{F} be a family of closed subgroups of G . Assume $F_W((G/H)_+ \wedge S_+^{n-1}) \wedge C$ is contained in the C -local equivalences for all $H \in \mathcal{F}$, for all n , and for all W in the universe. Then L_C takes any E_∞^G -algebra to an $E_\infty^{\mathcal{F}}$ -algebra.*

Localizations of the form above are \mathcal{F} -monoidal but not necessarily G -monoidal. This is why $L_C(X)$ for $X \in E_\infty^G\text{-alg}$ has $E_\infty^{\mathcal{F}}$ -algebra structure but may not have E_∞^G -algebra structure, as demonstrated by Example 5.13. More generally, we have the following result, which encodes the fact that if we work in $\mathcal{S}^{\mathcal{K}}$ rather than \mathcal{S}^G then localizations should be compatible with both \mathcal{K} and \mathcal{F} . Because there are now two families involved, the localization will preserve algebraic structure corresponding to the meet of these two families in the lattice of families.

Theorem 5.15. *Let \mathcal{M} be the \mathcal{K} -fixed point model structure on G -spectra and let \mathcal{K}' be a subfamily of \mathcal{K} . Assume $F_W((G/H)_+ \wedge S_+^{n-1}) \wedge C$ is contained in the C -local equivalences for all $H \in \mathcal{K}'$, for all n , and for all W in the universe. Then L_C takes any $E_\infty^{\mathcal{F}}$ -algebra to an $E_\infty^{\mathcal{F} \cap \mathcal{K}'}$ -algebra.*

Proof. In order to apply Corollary 3.4, first forget to the model structure $\mathcal{S}^{\mathcal{F} \cap \mathcal{K}'}$ and observe that any $E_\infty^{\mathcal{F}}$ -algebra is sent to a $E_\infty^{\mathcal{F} \cap \mathcal{K}'}$ -algebra. The hypothesis on L_C guarantees that L_C is a monoidal Bousfield localization with respect to the $\mathcal{F} \cap \mathcal{K}'$ model structure, and so $E_\infty^{\mathcal{F} \cap \mathcal{K}'}$ is preserved. \square

This theorem also explains why LE has $E_\infty^{\mathcal{K}}$ -algebra structure in Example 5.13. The localization L described in Example 5.13 is a monoidal Bousfield localization with respect to the $\mathcal{S}^{\mathcal{K}}$ model structure.

Together, Theorem 5.15 and Example 5.13 resolve the question of preservation for the lattice $E_\infty^{\mathcal{F}}$. As expected, preservation of lesser algebraic structure comes down to requiring a less stringent condition on the Bousfield localization. The least stringent condition is for $\mathcal{F} = \{e\}$ and recovers the notion of a stable localization (i.e. one which is monoidal on the category of spectra after forgetting the G -action). Thus, our preservation theorem is a generalization of the result in [21] that any such localization takes commutative equivariant ring spectra to spectra with an action of an E_∞ operad. We will discuss Theorem 5.8 more in Section 7 after developing the theory of preservation for commutative monoids.

6. BOUSFIELD LOCALIZATION AND COMMUTATIVE MONOIDS

In this section we turn to the interplay between monoidal Bousfield localizations and commutative monoids, i.e. algebras over the (non-cofibrant) operad Com . In [44], the following theory is developed as Definition 3.1, Theorem 3.2, and Corollary 3.8.

Definition 6.1. A monoidal model category \mathcal{M} is said to satisfy the *commutative monoid axiom* if whenever h is a trivial cofibration in \mathcal{M} then $h^{\square n}/\Sigma_n$ is a trivial cofibration in \mathcal{M} for all $n > 0$.

If, in addition, the class of cofibrations is closed under the operation $(-)^{\square n}/\Sigma_n$ then \mathcal{M} is said to satisfy the *strong commutative monoid axiom*.

Theorem 6.2. *Let \mathcal{M} be a cofibrantly generated monoidal model category satisfying the commutative monoid axiom and the monoid axiom, and assume that the domains of the generating maps I (resp. J) are small relative to $(I \otimes \mathcal{M})$ -cell (resp. $(J \otimes \mathcal{M})$ -cell). Let R be a commutative monoid in \mathcal{M} . Then the category $CAlg(R)$ of commutative R -algebras is a cofibrantly generated model category in which a map is a weak equivalence or fibration if and only if it is so in \mathcal{M} . In particular, when $R = S$ this gives a model structure on commutative monoids in \mathcal{M} .*

Corollary 6.3. *Let \mathcal{M} be a cofibrantly generated monoidal model category satisfying the commutative monoid axiom, and assume that the domains of the generating maps I (resp. J) are small relative to $(I \otimes \mathcal{M})$ -cell (resp. $(J \otimes \mathcal{M})$ -cell). Then for any commutative monoid R , the category of commutative R -algebras is a cofibrantly generated semi-model category in which a map is a weak equivalence or fibration if and only if it is so in \mathcal{M} .*

While these results only make use of the commutative monoid axiom, in practice we usually desire the strong commutative monoid axiom so that in the category of commutative R -algebras cofibrations with cofibrant domains forget to cofibrations

in \mathcal{M} . This is discussed further in [44] and numerous examples of model categories satisfying these axioms are given.

In order to apply the corollary above to verify the hypotheses of Corollary 3.4 we must give conditions on the maps \mathcal{C} so that if \mathcal{M} satisfies the commutative monoid axiom then so does $L_{\mathcal{C}}(\mathcal{M})$. As for the pushout product axiom, our method will be to apply Lemma 4.10, which is just the universal property of Bousfield localization. However, $(-)^{\square n}/\Sigma_n$ is not a functor on \mathcal{M} , but rather on $\text{Arr}(\mathcal{M})$. The following lemma lets us instead work with the functor $\text{Sym}^n : \mathcal{M} \rightarrow \mathcal{M}$ defined by $\text{Sym}^n(X) = X^{\otimes n}/\Sigma_n$.

Lemma 6.4. *Assume that for every $g \in I$, $g^{\square n}/\Sigma_n$ is a cofibration. Suppose f is a trivial cofibration between cofibrant objects and $f^{\square n}/\Sigma_n$ is a cofibration for all n . Then $f^{\square n}/\Sigma_n$ is a trivial cofibration for all n if and only if $\text{Sym}^n(f)$ is a trivial cofibration for all n .*

Proof. Appendix A of [44] proves that if all maps g in I (resp. J) have the property that $g^{\square n}/\Sigma_n$ is a (trivial) cofibration, then the same holds for all (trivial) cofibrations g . Thus, the hypothesis implies that the class of cofibrations is closed under the operation $(-)^{\square n}/\Sigma_n$.

Part of the proof in [44] involves an induction on subdiagrams of the n -dimensional cube whose vertices are words $C_1 \otimes \cdots \otimes C_n$ in which each C_i is either X or Y . The initial vertex of the cube is $X^{\otimes n}$ and the terminal vertex is $Y^{\otimes n}$. Let Q_q^n be the colimit of the subdiagram consisting of the vertices of distance $\leq q$ from $Q_0^n = X^{\otimes n}$. These objects Q_q^n inherit a Σ_n -action from the cube, and the Σ_n -equivariant maps $Q_{q-1}^n \rightarrow Q_q^n$ may be equivalently defined by the following pushout diagram.

$$(2) \quad \begin{array}{ccc} \Sigma_n \cdot_{\Sigma_{n-q} \times \Sigma_q} X^{\otimes(n-q)} \otimes Q_{q-1}^q & \longrightarrow & Q_{q-1}^n \\ \downarrow & \searrow \cong & \downarrow \\ \Sigma_n \cdot_{\Sigma_{n-q} \times \Sigma_q} X^{\otimes(n-q)} \otimes Y^{\otimes q} & \longrightarrow & Q_q^n \end{array}$$

Observe that the pushout diagram above remains a pushout diagram if we apply $(-)/\Sigma_n$ to all objects and morphisms in the diagram, because $(-)/\Sigma_n$ is a left adjoint and so commutes with colimits. We obtain the diagram

$$(3) \quad \begin{array}{ccc} \text{Sym}^{n-q}(X) \otimes Q_{q-1}^q/\Sigma_q & \longrightarrow & Q_{q-1}^n/\Sigma_n \\ \downarrow & \searrow \cong & \downarrow \\ \text{Sym}^{n-q}(X) \otimes \text{Sym}^q(Y) & \longrightarrow & Q_q^n/\Sigma_n \end{array}$$

We have assumed X is cofibrant, so $\text{Sym}^k(X)$ is cofibrant for all k because the map $\emptyset \rightarrow \text{Sym}^k(X)$ is simply the k^{th} iterated pushout product of the map $\emptyset \hookrightarrow X$. Thus, the left vertical map above is a trivial cofibration as soon as $f^{\square q}$ is a trivial cofibration, by the pushout product axiom.

We are now ready to prove the forwards direction in the lemma. Fix n and realize $\text{Sym}^n(f)$ as a composite of maps $\mathcal{Q}_{q-1}^n/\Sigma_n \rightarrow \mathcal{Q}_q^n/\Sigma_n$ as above. Assume $f^{\square q}$ is a trivial cofibration for all q and deduce that each $\mathcal{Q}_{q-1}^n/\Sigma_n \rightarrow \mathcal{Q}_q^n/\Sigma_n$ is a trivial cofibration, because trivial cofibrations are closed under pushout. Furthermore, because trivial cofibrations are closed under composite, this proves $\text{Sym}^n(f)$ is a trivial cofibration.

To prove the converse, assume that $\text{Sym}^k(f)$ is a trivial cofibration for all k . We will prove $f^{\square n}/\Sigma_n$ is a trivial cofibration for all n by induction. For $n = 1$ the map is f , which we have assumed to be a trivial cofibration. Now assume $f^{\square i}/\Sigma_i$ is a trivial cofibration for all $i < n$. As in the proof in [44] we may again prove $f^{\square n}/\Sigma_n$ is a trivial cofibration via the filtration in diagram (3). By our inductive hypothesis, we know that for all $i < n$, $\mathcal{Q}_{i-1}^n/\Sigma_n \rightarrow \mathcal{Q}_i^n/\Sigma_n$ is a trivial cofibration. We therefore have a composite:

$$\text{Sym}^n(X) = \mathcal{Q}_0^n/\Sigma_n \rightarrow \mathcal{Q}_1^n/\Sigma_n \rightarrow \cdots \rightarrow \mathcal{Q}_{n-1}^n/\Sigma_n \rightarrow \mathcal{Q}_n^n/\Sigma_n = \text{Sym}^n(Y)$$

in which each map except the last is a trivial cofibration. However, we have assumed $\text{Sym}^n(X) \rightarrow \text{Sym}^n(Y)$ is a trivial cofibration, so by the two out of three property the map $\mathcal{Q}_{n-1}^n/\Sigma_n \rightarrow \mathcal{Q}_n^n/\Sigma_n$ is in fact a weak equivalence. This map is $f^{\square n}/\Sigma_n$, and is a cofibration by hypothesis, so it is a trivial cofibration. This completes the induction.

□

With this lemma in hand, we are ready to prove the main result of this section, regarding preservation of the commutative monoid axiom by Bousfield localization.

Theorem 6.5. *Assume \mathcal{M} is a tractable monoidal model category satisfying the strong commutative monoid axiom. Suppose that $L_C(\mathcal{M})$ is a monoidal Bousfield localization with generating trivial cofibrations J_C . If $\text{Sym}^n(f)$ is a C -local equivalence for all $n \in \mathbb{N}$ and for all $f \in J_C$, then $L_C(\mathcal{M})$ satisfies the strong commutative monoid axiom.*

Proof. It is proven in Appendix A of [44] that if $(-)^{\square n}/\Sigma_n$ takes generating (trivial) cofibrations to (trivial) cofibrations, then it takes all (trivial) cofibrations to (trivial) cofibrations. The generating cofibrations of $L_C(\mathcal{M})$ are the same as those in \mathcal{M} and \mathcal{M} satisfies the strong commutative monoid axiom, so the class of cofibrations of $L_C(\mathcal{M})$ is closed under the operation $(-)^{\square n}/\Sigma_n$.

Suppose now that $f : X \rightarrow Y$ is a generating trivial cofibration of $L_C(\mathcal{M})$. Because \mathcal{M} is tractable and tractability is preserved by Bousfield localization (see Proposition 4.3 in [27]), we may assume f has cofibrant domain and codomain. In particular, the proof of Lemma 6.4 implies $\text{Sym}^n(f)$ is a cofibration, because $f^{\square k}/\Sigma_k$ is a cofibration for all k and the domain X of f is cofibrant.

By hypothesis, $\text{Sym}^n(f)$ is a trivial cofibration of $L_C(\mathcal{M})$ for all n . We are therefore in the situation of Lemma 6.4 and may conclude that $f^{\square n}/\Sigma_n$ is a trivial cofibration for all n . We now apply the result from Appendix A of [44] to conclude that all trivial cofibrations of $L_C(\mathcal{M})$ are closed under the operation $(-)^{\square n}/\Sigma_n$. \square

Remark 6.6. It is tempting to try to prove the theorem using Lemma 4.10, i.e. using the universal property of Bousfield localization. After all, just as in Theorem 4.5 we are assuming the property we need on the maps in C and trying to deduce this property for all C -local equivalences between cofibrant objects. However, Sym^n is not a left adjoint. One could attempt to get around this by applying Lemma 4.10 with the functor $\text{Sym} : \mathcal{M} \rightarrow \text{CMon}(\mathcal{M})$, but this would require the existence of a model structure on $\text{CMon}(\mathcal{M})$ in which the weak equivalences are C -local equivalences. As this is what we're trying to prove by obtaining the commutative monoid axiom on $L_C(\mathcal{M})$, this approach is doomed to fail.

If we know more about \mathcal{M} in the statement of the theorem above then we can in fact get a sharper condition regarding the generating trivial cofibrations J_C . One way to better understand the trivial cofibrations in $L_C(\mathcal{M})$ is via the theory of framings. Definition 4.2.1 of [23] defines the *full class of horns on C* to be the class

$$\Lambda(C) = \{\tilde{f} \square i_n \mid f \in C, n \geq 0\}$$

where $i_n : \partial\Delta[n] \rightarrow \Delta[n]$ and $\tilde{f} : \tilde{A} \rightarrow \tilde{B}$ is a Reedy cofibration between cosimplicial resolutions. In the case where C is a set and \mathcal{M} is cofibrantly generated, Definition 4.2.2 of [23] defines an *augmented set of C -horns* to be $\overline{\Lambda(C)} = \Lambda(C) \cup J$. Finally, 4.2.5 defines a set $\widehat{\Lambda(C)}$ to be a set of relative I -cell complexes with cofibrant domains obtained from $\overline{\Lambda(C)}$ via cofibrant replacement.

We now advertise the surprising and powerful Theorem 3.11 in [2]. This result states that if \mathcal{M} is proper and stable, if the C -local objects are closed under Σ (such L_C are called *stable*), and if C consists of cofibrations between cofibrant objects then J_C is $J \cup \Lambda(C)$. The last hypothesis is a standing assumption for this paper. The key input to Theorem 3.11 is the observation that for such \mathcal{M} , a map is a C -fibration if and only if its fiber is C -fibrant.

Corollary 6.7. *Suppose \mathcal{M} is a stable, proper, simplicial model category satisfying the strong commutative monoid axiom. Suppose that L_C is a stable and monoidal Bousfield localization such that for all $n \in \mathbb{N}$ and $f \in C$, $\text{Sym}^n(f)$ is a C -local equivalence. Then $L_C(\mathcal{M})$ satisfies the strong commutative monoid axiom.*

Proof. By Theorem 6.5 we must only check that Sym^n takes maps in $J_C = J \cup \Lambda(C)$ to C -local equivalences. By the commutative monoid axiom on \mathcal{M} , maps in J are taken to weak equivalences, so we must only consider maps in $\Lambda(C)$.

The reason for the hypothesis that \mathcal{M} is simplicial is Remark 5.2.10 in [25], which states that the functor $\tilde{A}^m = A \otimes \Delta[m]$ is a cosimplicial resolution of A (at least, when A is cofibrant). We further observe that the model structure on $L_C(\mathcal{M})$ is

independent of the choice of cosimplicial resolution. Thus, we may take our map in $\Lambda(\mathcal{C})$ to be of the form $(f \otimes \Delta[m]) \square i_n$ where $f : A \rightarrow B$ is in \mathcal{C} .

The map $(f \otimes \Delta[m]) \square i_n$ can be realized as the corner map in the diagram

$$\begin{array}{ccc}
 A \otimes \Delta[m] \otimes \partial\Delta[n]_+ & \longrightarrow & B \otimes \Delta[m] \otimes \partial\Delta[n]_+ \\
 \downarrow & \Downarrow & \downarrow \\
 A \otimes \Delta[m] \otimes \Delta[n]_+ & \longrightarrow & \text{dom}((f \otimes \Delta[m]) \square i_n) \\
 & \searrow & \downarrow (f \otimes \Delta[m]) \square i_n \\
 & & B \otimes \Delta[m] \otimes \Delta[n]_+
 \end{array}$$

If we can prove that $(g \otimes K)^{\square n} / \Sigma_n$ is a \mathcal{C} -local trivial cofibration for any \mathcal{C} -local trivial cofibration g between cofibrant objects then we can apply the same reasoning from the proof of Proposition 4.12 to deduce that the corner map becomes a \mathcal{C} -local trivial cofibration after applying $(-)^{\square n} / \Sigma_n$. This reasoning goes by proving that after applying $(-)^{\square n} / \Sigma_n$ the lower curved map and the top horizontal map are \mathcal{C} -local trivial cofibrations, so the bottom horizontal map is as well (because it is a pushout), and hence the corner map is a weak equivalence by the two out of three property. This reasoning works because whenever f is a pushout of g then $f^{\square n} / \Sigma_n$ is a pushout of $g^{\square n} / \Sigma_n$ as shown in Appendix A of [44].

Because $g \otimes K$ is a \mathcal{C} -local trivial cofibration between cofibrant objects, we may apply Lemma 6.4 to reduce the final step to checking that if $\text{Sym}^n(g)$ is a \mathcal{C} -local trivial cofibration for all n then so is $\text{Sym}^n(g \otimes K)$. This is proven as Lemma 27 in [16]. \square

When the hypotheses of stability and properness are dropped one can no longer easily write down the set $J_{\mathcal{C}}$. However, Theorem 4.1.1 (and its proof, notably 4.3.1) in [23] demonstrate that the class of maps $X \rightarrow L_{\mathcal{C}}(X)$ are contained in $\widehat{\Lambda(\mathcal{C})}$ -cell. Given a \mathcal{C} -local trivial cofibration $g : X_1 \rightarrow X_2$ between cofibrant objects, applying fibrant replacement $L_{\mathcal{C}}$ results in a map $L_{\mathcal{C}}(g)$ which is a weak equivalence between cofibrant objects. An appeal to Ken Brown's lemma on the functor Sym^n and to the two out of three property reduces the verification that $(-)^{\square n} / \Sigma_n$ takes g to a \mathcal{C} -local equivalence to verifying that $(-)^{\square n} / \Sigma_n$ takes $X_i \rightarrow L_{\mathcal{C}}(X_i)$ to \mathcal{C} -local equivalences.

Since such maps are in $\widehat{\Lambda(\mathcal{C})}$ -cell, by Appendix A of [44] one must only show that maps in $\widehat{\Lambda(\mathcal{C})}$ are taken to \mathcal{C} -local equivalences by $(-)^{\square n} / \Sigma_n$ (that they are taken to cofibrations is immediate by the strong commutative monoid axiom on \mathcal{M}). This observation leads to the following result, which we have recently learned was independently discovered as Theorem 28 in version 3 of the preprint [16].

Theorem 6.8. *Suppose \mathcal{M} is a cofibrantly generated, tractable, simplicial model category satisfying the strong commutative monoid axiom. Suppose that for all*

$n \in \mathbb{N}$ and $f \in C$, $\text{Sym}^n(f)$ is a C -local equivalence. Then $L_C(\mathcal{M})$ satisfies the strong commutative monoid axiom.

As the proof of this Theorem appears in [16], we will content ourselves with the sketch of the proof given above and we refer the interested reader to [16] for details. With a careful analysis of $\widetilde{\Lambda}(C)$ the author believes one could remove the need for \mathcal{M} to be simplicial. However, lacking equations of the sort found in Remark 5.2.10 of [25], he does not know how to proceed.

We conclude this section by remarking that the commutative monoid axiom has a natural generalization to an arbitrary operad P . The proof of Proposition 7.6 in [19] demonstrates a precise hypothesis on \mathcal{M} so that P -algebras inherit a model structure, namely that for all $A \in P\text{-alg}$ and for all n , $P_A[n] \otimes_{\Sigma_n} (-)^{\square n}$ preserves trivial cofibrations (where P_A is the enveloping operad). If these hypotheses are only satisfied for cofibrant A then $P\text{-alg}$ inherits a semi-model structure. We hope in the future to study the types of localizations which preserve these axioms, so that Corollary 3.4 may be applied to deduce preservation results for arbitrary operads P . We conjecture that the correct condition on a localization is that for all $f \in C$, for all $A \in P\text{-alg}$, and for all n , then $P_A[n] \otimes_{\Sigma_n} f^{\square n}$ is contained in the C -local equivalences.

7. PRESERVATION OF COMMUTATIVE MONOIDS

We turn now to the question of preservation under Bousfield localization of commutative monoids. We will be applying Theorem 6.5 and Corollary 3.4 for this purpose in a moment, but we first remark on a simpler case where the hypotheses of Theorem 6.5 are not necessary.

7.1. Spectra. Preservation of commutative monoids by monoidal Bousfield localizations is easy in certain categories of spectra, because of the property that for all cofibrant X in \mathcal{M} , the map $(E\Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n} \rightarrow X^{\wedge n}/\Sigma_n$ is a weak equivalence. This property was first noticed in [13], and we will now discuss it more generally.

Recall that two operads O and P are said to satisfy *rectification* if $P\text{-alg}$ and $O\text{-alg}$ are Quillen equivalent model categories. In [44], we introduced the *rectification axiom*, which states that if $Q_{\Sigma_n}S \rightarrow S$ is cofibrant replacement for S in \mathcal{M}^{Σ_n} then for all cofibrant X in \mathcal{M} , the map $Q_{\Sigma_n}S \otimes_{\Sigma_n} X^{\otimes n} \rightarrow X^{\otimes n}/\Sigma_n$ is a weak equivalence (this is the natural generalization of the property from [13] mentioned above). Observe that this property automatically holds on $L_C(\mathcal{M})$ if it holds on \mathcal{M} , because the cofibrant objects are the same and the weak equivalences are contained in the C -local equivalences. We now prove that in the presence of the rectification axiom, preservation results for commutative monoids are particularly nice.

Theorem 7.1. *Let $QCom$ denote a Σ -cofibrant replacement of Com in \mathcal{M} . Let \mathcal{M} be a monoidal model category in which the rectification axiom implies that $QCom$*

and *Com* rectify. Let L_C be a monoidal Bousfield localization. Then L_C preserves commutative monoids.

In particular

- For positive symmetric spectra, positive orthogonal spectra, or \mathbb{S} -modules, $QCom$ is E_∞ and any monoidal Bousfield localization preserves strict commutative ring spectra.
- For positive G -equivariant orthogonal spectra, $QCom$ is E_∞^G and any monoidal Bousfield localization preserves strict commutative equivariant ring spectra.

Proof. Let E be a commutative monoid, so in particular E is a $QCom$ algebra via the map $QCom \rightarrow Com$. Because $QCom$ is Σ -cofibrant, $QCom$ -algebras in both \mathcal{M} and $L_C(\mathcal{M})$ inherit semi-model structures, so Corollary 3.4 implies $L_C(E)$ is weakly equivalent to some $QCom$ -algebra E_Q . The rectification axiom in $L_C(\mathcal{M})$ now implies E_Q is weakly equivalent to a commutative monoid \widehat{E} . \square

Currently, this result is only known to apply to the categories of spectra listed in the statement of the theorem. We conjectured in [44] that the rectification axiom implies rectification between $QCom$ and Com for general \mathcal{M} . If this conjecture is proven then the theorem will apply to all \mathcal{M} which satisfy the rectification axiom. Even if the conjecture is false, the following proposition demonstrates that when \mathcal{M} satisfies the rectification axiom then the conditions of Theorem 6.5 are satisfied and so any monoidal localization preserves commutative monoids.

Proposition 7.2. *Suppose \mathcal{N} is a monoidal model category satisfying the rectification axiom. Then $\text{Sym}^n(-)$ takes trivial cofibrations between cofibrant objects to weak equivalences.*

In particular, if $L_C(\mathcal{M})$ is a monoidal Bousfield localization and \mathcal{M} satisfies the rectification axiom, then L_C preserves commutative monoids.

Proof. The first part is proven as Proposition 4.6 in [44], and we refer the reader there for a proof. For the second part, we apply the first part with $\mathcal{N} = L_C(\mathcal{M})$, using our observation that the rectification axiom holds on $L_C(\mathcal{M})$ whenever it holds on \mathcal{M} . Thus, $\text{Sym}^n : L_C(\mathcal{M}) \rightarrow L_C(\mathcal{M})$ takes C -local trivial cofibrations between cofibrant objects to C -local equivalences. In particular, the hypotheses of Theorem 6.5 are satisfied and we may deduce from Corollary 3.4 that L_C preserves commutative monoids. \square

7.2. Spaces. We turn our attention now to simplicial sets and topological spaces. Rectification is known to fail (see Example 4.4 in [44]), so even though all localizations are monoidal we may not apply the result above. For spaces the path connected commutative monoids are weakly equivalent to generalized Eilenberg-Mac Lane spaces, i.e. products of Eilenberg-Mac Lane spaces. Preservation of

commutative monoids has been proven for pointed CW complexes as Theorem 1.4 in [10].

Theorem 7.3. *Let \mathcal{M} be the category of pointed CW complexes. Let C be any set of maps. Then $\text{Sym}(-)$ preserves C -local equivalences and L_C sends GEMs to GEMs.*

The proof of this theorem is based on work of Farjoun which appears in Chapter 4 of [14], so will also hold for $\mathcal{M} = sSet$. That work is generalized in [44] to hold for the category of k -spaces. So we may extend the theorem above to k -spaces as well. Observe that the theorem above implies both $sSet$ and k -spaces satisfy the conditions of Theorem 6.5 because Sym^n is a retract of Sym . We summarize

Theorem 7.4. *Let \mathcal{M} be either simplicial sets or k -spaces. Then every Bousfield localization preserves GEMs.*

Thus, we have extended the result above and Theorem 4.B.4 in [14] to a wider class of topological spaces than spaces having the homotopy type of a CW complex.

7.3. Chain Complexes.

Proposition 7.5. *Let k be a field of characteristic zero. The only Bousfield localizations of $Ch(k)_{\geq 0}$ are truncations.*

Proof. Over any PID, the homotopy type is determined by H_* , so this means adding weak equivalences is equivalent to killing some object. Thus, all localizations are nullifications. All objects are wedges of spheres, and killing k^2 in degree n is the same as killing k in degree n . Thus, the localization is completely determined by the lowest dimension in which the first nullification occurs. The localization is therefore equivalent to $0 \rightarrow V$ where V is the sphere on k in that dimension. \square

Corollary 7.6. *All Bousfield localizations of $Ch(k)_{\geq 0}$ are monoidal and hence preserve algebras over cofibrant operads.*

Remark 7.7. For unbounded chain complexes, truncations need not preserve algebraic structure. For example, if $f : S^{-2} \rightarrow D^{-3}$ gets inverted then just as with the Postnikov Section, an algebra will be taken to an object with no unit.

Quillen proved in Proposition 2.1 of Appendix B of [36] that bounded chain complexes over a field of characteristic zero satisfies the commutative monoid axiom. The proof that all quasi-isomorphisms are closed under Sym^n goes via cofiber and the 5-lemma on homology groups. The key observation is that $\text{Sym}^n(-)$ preserves group isomorphisms. The same proof demonstrates that Sym^n preserves C -local equivalences for all L_C as above. Hence, all Bousfield localizations of $Ch(k)_{\geq 0}$ preserve commutative differential graded algebras. Of course, this can also be seen directly from the description of L_C as a truncation.

7.4. Equivariant Spectra. We conclude this section by returning to the result of Hill and Hopkins (Theorem 5.8) which motivated this work. In [21], several equivalent conditions are given in order for a localization to preserve commutative structure. We may therefore restate Theorem 5.8 using the condition most related to our approach in Section 6.

Theorem 7.8. *Suppose L is a localization. If $\mathrm{Sym}^n(-)$ preserves L -acyclicity for all n then L preserves commutativity.*

Preservation of L -acyclics is the same as preservation of L -local equivalences as can be seen for example via the rectification axiom and the property that cofibrant objects are flat. So we see that when we specialize Theorem 6.5 to the model category of equivariant spectra and to localizations of the form L we precisely recover the theorem of Hill and Hopkins.

Recall that in Theorem 5.9 we gave minimal conditions for a Bousfield localization to preserve E_∞^G -structure. Theorem 7.1 implies the same conditions will guarantee preservation of strict commutative structure because equivariant spectra satisfy rectification (see the Appendix to [6]). Thus, we have improved on Theorem 5.8 and obtained sharper, easier to check conditions.

8. BOUSFIELD LOCALIZATION AND THE MONOID AXIOM

As shown by Corollary 3.4, our preservation results do not require $L_C(\mathcal{M})$ to satisfy the monoid axiom. However, having found conditions so that the pushout product axiom, commutative monoid axiom, and property that cofibrant objects are flat transfer to $L_C(\mathcal{M})$, we feel we should include a word on how to obtain the monoid axiom for $L_C(\mathcal{M})$ in case the reader is interested in studying the monoidal model category $L_C(\mathcal{M})$ for a purpose other than the preservation of operad-algebra structure.

We remark that Proposition 3.8 of [1] proves that $L_C(\mathcal{M})$ inherits the monoid axiom from \mathcal{M} if L_C takes a special form similar to localization at a homology theory. In contrast, our result will place no hypothesis on the maps in \mathcal{C} at all, beyond our standing hypothesis that these maps are cofibrations. We additionally remark that the preprint [35] has independently considered the question of when Bousfield localization preserves the monoid axiom, towards the goal of rectification results in general categories of spectra. This preprint should appear soon.

In order to understand when Bousfield localization will preserve the monoid axiom we must introduce a definition, taken from [4]. Note that this is a different usage of the term h -cofibration than the usage in [13] where it means ‘Hurewicz cofibration.’ The meaning here is for ‘homotopical cofibration’ for reasons which will become clear.

Definition 8.1. A map $f : X \rightarrow Y$ is called an h -cofibration if the functor $f_! : X/\mathcal{M} \rightarrow Y/\mathcal{M}$ given by cobase change along f preserves weak equivalences. Formally, this means that in any diagram as below, in which both squares are pushout

squares and w is weak equivalence, then w' is also a weak equivalence:

$$\begin{array}{ccccc} X & \longrightarrow & A & \xrightarrow{w} & B \\ f \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & A' & \xrightarrow{w'} & B' \end{array}$$

It is clear that any trivial cofibration is an h -cofibration, by the two out of three property. If \mathcal{M} is left proper then any cofibration is an h -cofibration (because $A \rightarrow A'$ is automatically a cofibration if f is). In fact, the converse holds as well, and is proven in Lemma 1.2 of [4]. Lemma 1.3 proves that h -cofibrations are closed under composition, pushout, and finite coproduct.

Now let \mathcal{M} be left proper. An equivalent characterization of an h -cofibration is as a map f such that every pushout along f is a homotopy pushout (this the version of the definition above was independently discovered in [45]). Proposition 1.5 in [4] proves that f is an h -cofibration if and only if there is a factorization of f into a cofibration followed by a *cofiber equivalence* $w : W \rightarrow Y$, i.e. for any map $g : W \rightarrow K$ the right-hand vertical map in the following pushout diagram is a weak equivalence:

$$\begin{array}{ccc} W & \longrightarrow & K \\ w \downarrow & & \downarrow \\ X & \longrightarrow & T \end{array} \quad \begin{array}{c} \Downarrow \\ \Downarrow \end{array}$$

We will make use of these various properties of h -cofibrations in this section. The purpose for introducing h -cofibrations is to make the following definition, which should be thought of as saying that the cofibrations in \mathcal{M} behave like inclusions of closed neighborhood deformation retracts of topological spaces.

Definition 8.2. \mathcal{M} is said to be *h -monoidal* if for each (trivial) cofibration f and each object Z , $f \otimes Z$ is a (trivial) h -cofibration.

We will find conditions so that Bousfield localization preserves h -monoidality, and we will then use this to deduce when Bousfield localization preserves the monoid axiom. In [4], h -monoidality is verified for the model categories of topological spaces, simplicial sets, chain complexes over a field (with the projective model structure), symmetric spectra (with the stable projective model structure), and several other model categories not considered in this paper. We now verify h -monoidality for the remaining model structures of interest in this paper. We remind the reader that an *injective model structure* has weak equivalences and cofibrations defined levelwise, and fibrations defined by the right lifting property.

Proposition 8.3. *The following model structures on symmetric spectra are h -monoidal:*

- (1) *The levelwise projective model structure (of Theorem 5.1.2 in [29]).*
- (2) *The positive model structure (of Theorem 14.1 in [32]).*

- (3) *The flat model structure (of Proposition 2.2 in [40], there called the S -model structure).*
- (4) *The positive flat model structure (obtained by redefining the cofibrations from the model structure above to be isomorphisms in level 0).*
- (5) *The stable projective model structure (this is proven to be h -monoidal in Proposition 1.14 of [4]).*
- (6) *The positive stable model structure (of Theorem 14.2 in [32]).*
- (7) *The flat stable model structure (of Theorem 2.4 in [40]).*
- (8) *The positive flat stable model structure (of Proposition 3.1 in [40]).*

Proof. We appeal to Proposition 1.9 in [4], and make use of the injective (or injective stable for (5)-(8)) model structure on symmetric spectra, introduced in Definition 5.1.1 (resp. after Definition 5.3.6) of [29]. The references above prove that all eight of the model structures above are monoidal and that both injective model structures are left proper (e.g. because all objects are cofibrant). The final condition in Proposition 1.9 is that for any (trivial) cofibration f and any object X , the map $f \otimes X$ is a (trivial) cofibration in the corresponding injective model structure. The cofibration part of this is Proposition 4.15(i) in version 3 of Stefan Schwede's book project [38], since for all eight of the model structures above the cofibrations are contained in the flat cofibrations and for any X the map $\emptyset \rightarrow X$ is an injective (a.k.a. levelwise) cofibration. The trivial cofibration part is Proposition 4.15(iv) in [38], which includes statements for both levelwise and stable weak equivalences. \square

We turn now to orthogonal and equivariant orthogonal spectra. We first need a lemma regarding the existence of injective model structures. Let Sp_{Δ}^O denote orthogonal spectra built on Δ -generated spaces (an overview of this category may be found in [11]). Let G be a compact Lie group and let $GS p_{\Delta}^O$ denote G -equivariant orthogonal spectra built on Δ -generated spaces.

Lemma 8.4. *The following model structures exist and are left proper and combinatorial: the levelwise injective model structure on Sp_{Δ}^O , the stable injective model structure on Sp_{Δ}^O , the levelwise injective model structure on $GS p_{\Delta}^O$, and the stable injective model structure on $GS p_{\Delta}^O$.*

Proof. Left properness will be inherited from Δ -generated spaces. For existence, we proceed as in Theorem 5.1.2 and Lemma 5.1.4 of [29]. Verification of lifting and factorization make use of a set C (resp. tC) containing a map from each isomorphism class of (trivial) cofibrations $i : X \rightarrow Y$ where Y is a countable spectrum. The use of Zorn's Lemma in Lemma 5.1.4 and the requisite countability from Lemmas 5.1.6 and 5.1.7 hold in this setting because of our decision to work with Δ -generated spaces. The rest of Lemma 5.1.4 goes through mutatis mutandis, using properties of topological fibrations and using Lemma 12.2 in [32] when

checking that injective cofibrations are closed under smashing with an arbitrary object.

The sets C and tC serve as generating (trivial) cofibrations. Together with the fact that a category of spectra built on a locally presentable category is again locally presentable, this proves the model structures are combinatorial. The stable injective structures are obtained by Bousfield localization in the usual way, which exists because the levelwise structures are left proper and combinatorial. \square

Proposition 8.5. *Work over Δ -generated spaces. Fix a compact Lie group G and fix a universe \mathcal{U} which we take to mean a G -universe when working equivariantly. The following model structures are h -monoidal:*

- (1) *The levelwise (projective) model structure on orthogonal spectra (of Theorem 6.5 in [32]).*
- (2) *The positive model structure on orthogonal spectra (of Theorem 14.1 in [32]).*
- (3) *The flat model structure on orthogonal spectra (of Proposition 1.3.5 in [43]).*
- (4) *The positive flat model structure on orthogonal spectra (of Proposition 1.3.10 in [43]).*
- (5) *The levelwise (projective) model structure on G -equivariant orthogonal spectra (of Theorem III.2.4 in [31]).*
- (6) *The positive model structure on G -equivariant orthogonal spectra (of Theorem III.2.10 in [31]).*
- (7) *The flat model structure on G -equivariant orthogonal spectra (of Theorem 2.3.13 of in [43]).*
- (8) *The positive flat model structure on G -equivariant orthogonal spectra (obtained by redefining the cofibrations from the model structure above to be isomorphisms in level 0).*
- (9) *The stable model structure on orthogonal spectra (of Theorem 9.2 in [32]).*
- (10) *The positive stable model structure on orthogonal spectra (of Theorem 14.2 in [32]).*
- (11) *The flat stable model structure on orthogonal spectra (of Theorem 2.3.27 in [43]).*
- (12) *The positive flat stable model structure on orthogonal spectra (of Theorem 2.3.27 in [43]).*
- (13) *The stable model structure on G -equivariant orthogonal spectra (of Theorem III.4.2 in [31]).*

- (14) *The positive stable model structure on G -equivariant orthogonal spectra (of Theorem III.5.3 in [31]).*
- (15) *The flat stable model structure on G -equivariant orthogonal spectra (of Theorem 2.3.13 of in [43]).*
- (16) *The positive flat stable model structure on G -equivariant orthogonal spectra (of Theorem 2.3.27 in [43]).*

Proof. The proof proceeds just as it does for Proposition 8.3, i.e. by comparison to the injective (stable) model structures in each of these settings. For the statement that for any cofibration f and any object X , the map $f \otimes X$ is a cofibration in the corresponding injective model structure, we appeal to Lemma 12.2 of [32] (which works equally well in the equivariant context). Finally, we turn to the statement that for any trivial cofibration f and any object X , the map $f \otimes X$ is a weak equivalence in the corresponding injective model structure. For the levelwise model structures above this property is inherited from spaces, e.g. by Lemma 12.2 in [32]. For the stable model structures we appeal to the monoid axiom on all of the model structures in the theorem and to the fact that projective (stable) equivalences are the same as injective (stable) equivalences. The monoid axiom has been verified in [43] for all these model structures by Theorems 1.2.54 and 1.2.57 (both originally proven in [32]), 1.3.10, 2.2.46 and 2.2.50 (both originally from [31]), and 2.3.27. \square

We return now to the question of the monoid axiom. It is proven in Proposition 2.5 of [4] that if \mathcal{M} is left proper, h -monoidal, and the weak equivalences in $(\mathcal{M} \otimes I)$ -cell are closed under transfinite composition, then \mathcal{M} satisfies the monoid axiom. We will use this to find conditions on \mathcal{M} so that $L_C(\mathcal{M})$ satisfies the monoid axiom. First, we improve Proposition 2.5 from [4] by replacing the third condition with the hypothesis that the (co)domains of I are finite relative to the class of h -cofibrations (in the sense of Section 7.4 of [25]).

Proposition 8.6. *Suppose \mathcal{M} is cofibrantly generated, left proper, h -monoidal, and the (co)domains of I are finite relative to the class of h -cofibrations. Then \mathcal{M} satisfies the monoid axiom.*

Proof. We follow the proof of Proposition 2.5 in [4]. Consider the class $\{f \otimes Z \mid Z \in \mathcal{M}, f \in J\}$. As \mathcal{M} is h -monoidal, this is a class of trivial h -cofibrations. By Lemma 1.3 in [4], h -cofibrations are closed under pushout. By Lemma 1.6 in [4], because \mathcal{M} is left proper, trivial h -cofibrations are closed under pushouts (e.g. because weak equivalences are closed under homotopy pushout). In order to prove $\{f \otimes Z \mid Z \in \mathcal{M}, f \in J\}$ -cell is contained in the weak equivalences of \mathcal{M} we must only prove that transfinite compositions of trivial h -cofibrations are weak equivalences.

Consider a λ -sequence $A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_\lambda$ of trivial h -cofibrations. Let j_β denote the map $A_\beta \rightarrow A_{\beta+1}$ in this λ -sequence. As in Proposition 17.9.4 of [23] we

may construct a diagram

$$\begin{array}{ccccccc}
 A'_0 & \longrightarrow & A'_1 & \longrightarrow & \dots & \longrightarrow & A'_\beta & \longrightarrow & \dots \\
 q_0 \downarrow & & q_1 \downarrow & & \downarrow & & q_\beta \downarrow & & \\
 A_0 & \longrightarrow & A_1 & \longrightarrow & \dots & \longrightarrow & A_\beta & \longrightarrow & \dots
 \end{array}$$

in which each A'_β is cofibrant, all the maps $A'_\beta \rightarrow A_\beta$ are trivial fibrations, and all the maps $A'_\beta \rightarrow A'_{\beta+1}$ are trivial cofibrations. Construction of this diagram proceeds by applying the cofibration-trivial fibration factorization iteratively to every composition $j_\beta \circ q_\beta : A'_\beta \rightarrow A_\beta \rightarrow A_{\beta+1}$ in order to construct $A'_{\beta+1}$. As j_β and q_β are both weak equivalences, so is their composite and so the cofibration $A'_\beta \rightarrow A'_{\beta+1}$ produced by the cofibration-trivial fibration factorization is a weak equivalence by the 2 out of 3 property.

We now show that the map $q_\lambda : A'_\lambda \rightarrow A_\lambda$ is a weak equivalence, following the approach of Lemma 7.4.1 in [25]. Consider the lifting problem

$$\begin{array}{ccc}
 X & \longrightarrow & A'_\lambda \\
 f \downarrow & & \downarrow q_\lambda \\
 Y & \longrightarrow & A_\lambda
 \end{array}$$

Where f is in the set I of generating cofibrations. Because the domains and codomains of maps in I are finitely presented we know that the map $X \rightarrow A'_\lambda$ factors through some finite stage A'_n . Similarly, $Y \rightarrow A_\lambda$ factors through some finite stage A_m . Let $k = \max(n, m)$. The map $A'_k \rightarrow A_k$ is a trivial fibration so there is a lift $g : Y \rightarrow A'_k$. Define $h : Y \rightarrow A'_\lambda$ as the composite with $A'_k \rightarrow A'_\lambda$.

$$\begin{array}{ccccc}
 X & \longrightarrow & A'_k & \longrightarrow & A'_\lambda \\
 f \downarrow & & \nearrow g & \downarrow & \downarrow q_\lambda \\
 Y & \longrightarrow & A_k & \longrightarrow & A_\lambda \\
 & & \searrow h & &
 \end{array}$$

Both triangles in the left-hand square commute by definition of lift. The triangle featuring g and h commutes because it is a composition. So the triangle featuring f and h commutes. The right-hand square commutes by construction of A'_λ and A_λ , so the trapezoid containing g and q_λ commutes. Thus, the triangle featuring h and q_λ commutes.

The existence of this lift h for all $f \in I$ proves that $A'_\lambda \rightarrow A_\lambda$ is a trivial fibration. Now consider that transfinite compositions of trivial cofibrations are always trivial cofibrations, so $A'_0 \rightarrow A'_\lambda$ is a weak equivalence. Furthermore, the vertical maps $q_0 : A'_0 \rightarrow A_0$ and $q_\lambda : A'_\lambda \rightarrow A_\lambda$ are trivial fibrations. So by the 2 out of 3 property, the map $A_0 \rightarrow A_\lambda$ is a weak equivalence as required. \square

It is shown in [4] that the compactness hypothesis of the proposition is satisfied for topological spaces, simplicial sets, equivariant and motivic spaces, and chain complexes. Similarly, it holds for all our categories of structured spectra because the sphere spectrum is \mathfrak{N}_0 -compact as a spectrum. Lastly, it holds for all the stable analogues of these structures because the compactness hypothesis is automatically preserved by any Bousfield localization (the set of generating cofibrations of $L_C(\mathcal{M})$ is simply I again).

Remark 8.7. The proof above only uses the fact that the maps j_β were h -cofibrations in order to factor $Y \rightarrow A_\lambda$ through some finite stage. So if the (co)domains of I are finite relative to the class of weak equivalences then the proof above demonstrates that weak equivalences are preserved under transfinite composition.

Proposition 8.8. *Suppose \mathcal{M} is tractable, left proper, h -monoidal, such that the (co)domains of I are finite relative to the class of h -cofibrations and cofibrant objects are flat. Let L_C be a monoidal Bousfield localization. Then $L_C(\mathcal{M})$ is h -monoidal.*

Proof. Suppose $f : A \rightarrow B$ is a cofibration in $L_C(\mathcal{M})$ and Z is any object of $L_C(\mathcal{M})$. We must show $f \otimes Z$ is an h -cofibration in $L_C(\mathcal{M})$. Because $L_C(\mathcal{M})$ is left proper, Proposition 1.5 in [4] reduces us to proving that there is a factorization of $f \otimes Z$ into a cofibration followed by a cofiber equivalence $w : X \rightarrow B \otimes Z$, i.e. for any map $g : X \rightarrow K$ the right-hand vertical map in the following pushout diagram is a C -local equivalence:

$$\begin{array}{ccc} X & \longrightarrow & K \\ w \downarrow & & \downarrow \\ B \otimes Z & \longrightarrow & T \end{array}$$

Because f is a cofibration in \mathcal{M} , the h -monoidality of \mathcal{M} guarantees us that $f \otimes Z$ is an h -cofibration in \mathcal{M} . Apply the cofibration-trivial fibration factorization in \mathcal{M} . Note that this is also a cofibration-trivial fibration factorization of $f \otimes Z$ in $L_C(\mathcal{M})$ because cofibrations and trivial fibrations in the two model categories agree. The resulting $w : X \rightarrow B \otimes Z$ is a trivial fibration in either model structure. Because \mathcal{M} is left proper we know that the map w is a cofiber equivalence in \mathcal{M} by Proposition 1.5 in [4] applied to the h -cofibration $f \otimes Z$. So in any pushout diagram as above the map $K \rightarrow T$ is a weak equivalence in \mathcal{M} , hence in $L_C(\mathcal{M})$. Thus, w is a cofiber equivalence in $L_C(\mathcal{M})$ and its existence proves $f \otimes Z$ is an h -cofibration in $L_C(\mathcal{M})$.

Now suppose f were a trivial cofibration in $L_C(\mathcal{M})$ to start. We must show that $f \otimes Z$ is a C -local equivalence. We do this first in the case where f is a generating trivial cofibration. Because $L_C(\mathcal{M})$ is tractable, A and B are cofibrant. Apply cofibrant

replacement to Z :

$$\begin{array}{ccc} A \otimes QZ & \longrightarrow & B \otimes QZ \\ \downarrow & & \downarrow \\ A \otimes Z & \longrightarrow & B \otimes Z \end{array}$$

The fact that cofibrant objects are flat in $L_C(\mathcal{M})$ implies the vertical maps are C -local equivalences (because A and B are cofibrant) and that the top horizontal map is a C -local equivalence (because QZ is cofibrant). By the 2 out of 3 property the bottom horizontal map is a C -local equivalence.

By Lemma 1.3 in [4], the class of h -cofibrations is closed under cobase change and retracts. By Lemma 1.6, the class of trivial h -cofibrations is closed under cobase change (because $L_C(\mathcal{M})$ is left proper). Weak equivalences are always closed under retract. Finally, by Proposition 8.6 the class of trivial h -cofibrations is closed under transfinite composition by our compactness hypothesis on \mathcal{M} (equivalently, on $L_C(\mathcal{M})$). So for a general f in the trivial cofibrations of $L_C(\mathcal{M})$, realize f as a retract of $g \in J_C$ -cell, so that $g \otimes Z$ is a transfinite composite of pushouts of maps of the form $j \otimes Z$ for $j \in J_C$. We have just proven that all $j \otimes Z$ are trivial h -cofibrations and closure properties imply $g \otimes Z$ and hence $f \otimes Z$ are trivial h -cofibrations as well.

□

Theorem 8.9. *Suppose \mathcal{M} is a tractable, left proper, h -monoidal model category such that the (co)domains of I are finite relative to the h -cofibrations and cofibrant objects are flat. Then for any monoidal Bousfield localization L_C , the model category $L_C(\mathcal{M})$ satisfies the monoid axiom.*

Proof. Apply Proposition 8.6 to the category $L_C(\mathcal{M})$. By the proposition just proven, $L_C(\mathcal{M})$ is h -monoidal. It is left proper because \mathcal{M} is left proper.

The argument of Proposition 8.6 is to be applied to λ -sequences of maps which are pushouts of maps in $\{f \otimes Z \mid f \text{ is a trivial cofibration in } L_C(\mathcal{M})\}$. Such maps are h -cofibrations in \mathcal{M} because \mathcal{M} is h -monoidal, f is a cofibration in \mathcal{M} , and h -cofibrations are closed under pushout. Thus, the hypothesis that the (co)domains of I are finite relative to the h -cofibrations in \mathcal{M} is sufficient to construct the lift in Proposition 8.6 and to prove the transfinite composition part of the proof of the monoid axiom.

□

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DEPARTMENT OF MATHEMATICS, WESLEYAN UNIVERSITY, MIDDLETOWN, CT 06459

E-mail address: dwhite03@wesleyan.edu