ON DEGENERATION OF ONE-DIMENSIONAL FORMAL GROUP LAWS AND APPLICATIONS TO STABLE HOMOTOPY THEORY

By Takeshi Torii

Abstract. In this note we study a certain formal group law over a complete discrete valuation ring $\mathbf{F}[\![u_{n-1}]\!]$ of characteristic $p > 0$ which is of height n over the closed point and of height $n-1$ over the generic point. By adjoining all coefficients of an isomorphism between the formal group law on the generic point and the Honda group law H_{n-1} of height $n-1$, we get a Galois extension of the quotient field of the discrete valuation ring with Galois group isomorphic to the automorphism group S_{n-1} of H_{n-1}. We show that the automorphism group S_n of the formal group over the closed point acts on the quotient field, lifting to an action on the Galois extension which commutes with the action of Galois group. We use this to construct a ring homomorphism from the cohomology of S_{n-1} to the cohomology of S_n with coefficients in the quotient field. Applications of these results in stable homotopy theory and relation to the chromatic splitting conjecture are discussed.

1. Introduction. The ring $MU_*(MU)$ of co-operations in complex cobordism theory has a well-known interpretation in terms of one-dimensional commutative formal group laws. The category $\mathbf{C}$ of p-local comodules over $MU_*(MU)$ has a filtration

$$\mathbf{C} = \mathbf{C}_0 \supset \mathbf{C}_1 \supset \cdots \supset \mathbf{C}_n \supset \cdots$$

by categories of submodules supported on formal group laws of height n over p-local rings. In [15] the quotient category $\mathbf{C}_n/\mathbf{C}_{n+1}$ is related to a category of discrete modules over some complete ring E_n with action of some profinite group G_n. Motivated by this work, Miller, Ravenel and Wilson [14] established a framework for organizing systematically the periodic phenomena on the E_2-term of the Adams-Novikov spectral sequence based on the cobordism theory MU. Then Ravenel [16] formulated his conjectures on the reflection of the algebraic structure of the Adams-Novikov E_2-term on the actual stable homotopy category. Devinatz, Hopkins and Smith [3, 7] have verified all of these conjectures except for the telescope conjecture. From these works, we get a filtration of full subcategories in the stable homotopy category $\mathcal{C}$ of p-local finite spectra

$$\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n \supset \cdots$$

Manuscript received September 12, 2001; revised January 15, 2003.
Research supported in part by JSPS Research Fellowships for Young Scientists.
American Journal of Mathematics 125 (2003), 1037–1077.

where n is related to the height of formal group laws. From Morava's point of view, the $K(n)$-local category of p-local finite spectra, which is in some sense the analogue of the quotient associated to the filtration, is studied through the Adams-Novikov spectral sequence by some algebraic category. This category consists of the discrete modules over the function ring E_n of the deformation space of the Honda group law H_n of height n, with compatible action of the automorphism group G_n of H_n. The next step to understand the stable homotopy category of p-local finite spectra may be to understand the "extensions," and for that it may be helpful to study the relation between formal group laws of neighboring height. In this note we study a certain one-dimensional commutative formal group law over a complete discrete valuation ring which is of height n over the closed point and of height $n-1$ over the generic point.

Let $\mathbf{F}$ be an algebraic extension of the prime field $\mathbf{F}_p$ which contains $\mathbf{F}_{p^n}$ and $\mathbf{F}_{p^{n-1}}$. There is the Honda group law H_n of height n over $\mathbf{F}$. The nth Morava stabilizer group S_n is the automorphism group of H_n which is isomorphic to the unit group of the maximal order of the central division algebra over the p-adic number field $\mathbf{Q}_p$ with invariant $1/n$. There is a universal deformation F_n of H_n. The formal group law F_n is defined over the formal power series ring $E_n = W(\mathbf{F})[[u_1, \ldots, u_{n-1}]]$ where $W(\mathbf{F})$ is the ring of Witt vectors with coefficients in $\mathbf{F}$. Then the action of $G_n = S_n \rtimes \Gamma$ on H_n lifts to the action on F_n which induces a continuous action of G_n on $W(\mathbf{F})[[u_1, \ldots, u_{n-1}]]$ where Γ is the Galois group of $\mathbf{F}$ over $\mathbf{F}_p$. Since the ideal generated by $p, u_1, \ldots, u_{n-2}$ is invariant under the action of G_n, there is an induced action of G_n on the quotient ring $\mathbf{F}[[u_{n-1}]]$. We denote by K the quotient field $\mathbf{F}((u_{n-1}))$. We consider that the formal group law F_n is defined over $\mathbf{F}[[u_{n-1}]]$. Then the formal group law F_n is of height n on the closed point $\mathbf{F}$ and of height $n-1$ on the generic point K. By the result of Lazard [10], the formal group laws over a separably closed field of characteristic $p > 0$ are classified up to isomorphism by their height. Hence there is an isomorphism between F_n and the Honda group law H_{n-1} of height $n-1$ over the separable closure K^{sep} of K. In [1] Ando, Morava and Sadofsky showed that there is a unique isomorphism between F_n and H_{n-1} over K^{sep} which satisfies certain conditions motivated from a geometric point of view. We would like to consider the above situation with the action of the nth Morava stabilizer group G_n.

Let Φ be an isomorphism between F_n and H_{n-1} over the separable closure K^{sep}. Let L be an extension of K obtained by adjoining all the coefficients of Φ. Hence we have a morphism of formal group laws from (F_n, L) to $(H_{n-1}, \mathbf{F})$. The main theorem of this note is as follows.

THEOREM 1.1. (cf. Theorem 2.9) *The group $(S_n \times S_{n-1}) \rtimes \Gamma$ acts on (F_n, L) where the action of $S_n \rtimes \Gamma$ is a lift of the action on (F_n, K) and the subgroup $S_{n-1} \rtimes \Gamma$ is identified with the Galois group of the extension $L/\mathbf{F}_p((u_{n-1}))$. If we consider that the group $(S_n \times S_{n-1}) \rtimes \Gamma$ acts on $(H_{n-1}, \mathbf{F})$ such that the subgroup S_n acts trivially, then there is a $(S_n \times S_{n-1}) \rtimes \Gamma$ equivariant morphism from (F_n, L) to $(H_{n-1}, \mathbf{F})$.*

In geometric terms $\mathrm{Spec}(\mathbf{F}[[u_{n-1}]])$ is an S_n-invariant 1-dimensional subspace of the formal deformation space of the Honda group law H_n. Let $U = \mathrm{Spec}(\mathbf{F}[[u_{n-1}]]) - \mathrm{Spec}(\mathbf{F})$ which is an analogue of a punctured disk. Then there is a Galois covering of U with Galois group isomorphic to S_{n-1}. The action of S_n lifts to the Galois covering which commutes with the action of the Galois group. Furthermore, if we consider that the product group $S_n \times S_{n-1}$ acts on H_{n-1} where the action of S_n is trivial, then there is an $S_n \times S_{n-1}$-equivariant morphism from the lift of F_n on the Galois covering to H_{n-1} on the point $\mathrm{Spec}(\mathbf{F})$.

The main application to the stable homotopy theory is as follows. By using Theorem 1.1, we construct as some kind of correspondence a ring homomorphism Θ from the cohomology of S_{n-1} with coefficients in $\mathbf{F}[w^{\pm 1}]$ to the cohomology of S_n with the coefficient in $K[u^{\pm 1}]$:

$$\Theta\colon\ H_c^*(S_{n-1}; \mathbf{F}[w^{\pm 1}])^{\Gamma} \longrightarrow H_c^*(S_n; K[u^{\pm 1}])^{\Gamma}$$

where w satisfies $w^{-(p^{n-1}-1)} = v_{n-1}$ (cf. (3.3)). If the Smith-Toda spectrum $V(n-2)$ exists, then $H_c^*(S_{n-1}; \mathbf{F}[w^{\pm 1}])^{\Gamma}$ is the E_2-term of the Adams-Novikov spectral sequence converging to $\pi_* L_{n-1} V(n-2)$, and $H_c^*(S_n; K[u^{\pm 1}])^{\Gamma}$ is the E_2-term of some Adams type spectral sequence converging to $\pi_* L_{n-1} L_{K(n)} V(n-2)$, where L_{n-1} (resp. $L_{K(n)}$) is the Bousfield localization functor with respect to $K(0) \vee K(1) \vee \cdots \vee K(n-1)$ (resp. $K(n)$). There is a natural map from $L_{n-1}V(n-2)$ to $L_{n-1}L_{K(n)}V(n-2)$ and the chromatic splitting conjecture contains the statement that this map is a split monomorphism. We show that the natural map lifts to a morphism of the spectral sequences and the morphism on E_2-term is given by Θ.

The organization of this note is as follows. In §2 we recall Lubin and Tate's deformation theory of one-dimensional formal group law of finite height over a field of characteristic $p > 0$. Then we recall a generalization of homomorphisms between formal group laws over possibly different ground rings. We study isomorphisms between two formal group laws F_n and H_{n-1} over the separable closure K^{sep} of K. We define an extension L of K by adjoining all coefficients of an isomorphism between F_n and H_{n-1}. Then we show that L is stable under any action on K^{sep} which is an extension of the action of nth Morava stabilizer group G_n on K. In particular, L is a Galois extension over K. We recall the result of Gross [4] that the Galois group of L/K is isomorphic to S_{n-1}, which is obtained as monodromy representation of F_n restricted to U. Then we define a group $\mathcal{G}$ which consists of all lifts of the action of G_n on K to the action on L. We prove that $\mathcal{G}$ is isomorphic to the profinite group $(S_n \times S_{n-1}) \rtimes \Gamma$ and this is a reformulation of the main theorem. In §3 we study the group cohomology based on continuous cochains and consider inflation maps under some conditions. Then we define quotient groups $\mathcal{G}(i)$ of the profinite group $\mathcal{G}$ which acts on the graded field $L_i[u^{\pm 1}]$ where L_i is a subfield of L obtained by adjoining some coefficients of an isomorphism between F_n and H_{n-1}. Then we show that the cohomology group $H_c^*(\mathcal{G}(i-1); L_{i-1}[u^{\pm 1}])$ is isomorphic to $H_c^*(\mathcal{G}(i); L_i[u^{\pm 1}])$ through the inflation map. Then we construct a ring homomorphism Θ from the cohomology of G_{n-1} with coefficients in $\mathbf{F}[w^{\pm 1}]$ to the cohomology of G_n with coefficients in $K[u^{\pm 1}]$. In §5 we recall the cohomology of comodules over the Hopf algebroid $BP_*(BP)$. In particular, we recall the relation between the cohomology of the comodule M^1_{n-1} and the cohomology of G_n with coefficients in $\mathbf{F}[\![u_{n-1}]\!][u^{\pm 1}]$. Then we introduce a filtration on the cohomology group $H^*_c(G_{n-1}; \mathbf{F}[w^{\pm 1}])$ using the homomorphism Θ and define a homomorphism Ξ of Bockstein type from the associated graded module of the filtration to some quotient of $H^*_c(G_n; \mathbf{F}[u^{\pm 1}])$. In §6 we compute the homomorphism Θ on some elements of H^1 and show the non-triviality of the image by using the homomorphism Ξ. In §7 we study the relation between the ring homomorphism Θ and the chromatic splitting conjecture.

Acknowledgments. The author would like to thank Professor Morava for discussions on the subject and Professor Wilson for his support during my stay at the Johns Hopkins University. He also would like to thank the referees for many valuable comments.

2. Isomorphisms between F_n and H_{n-1}. In this section we investigate isomorphisms between two formal group laws F_n and H_{n-1} over a separable closure K^{sep} and some Galois extension L of $K = \mathbf{F}((u_{n-1}))$. First, we recall basic facts on formal group laws and their deformation theory. Then we define a profinite group $\mathcal{G}$ which consists of all lifts of the action of G_n on (F_n, K) to (F_n, L) and prove the main theorem (Theorem 2.9).

2.1. Deformation of formal group laws. Let R be a complete Noetherian local ring with maximal ideal I such that the residue field $k = R/I$ is of characteristic $p > 0$. Let G be a one-dimensional commutative formal group law over k of height $n < \infty$. In this subsection we recall Lubin and Tate's deformation theory of formal group laws [11].

For a formal power series $f(X)$ over a ring A_1 and a ring homomorphism $\alpha\colon A_1 \to A_2$, we denote by $\alpha^* f(X)$ the formal power series over A_2 obtained by the base change α. For a local homomorphism α between local rings, we denote by $\overline{\alpha}$ the induced homomorphism on the residue fields.

Let A be a complete Noetherian local R-algebra with maximal ideal $\mathfrak{m}$. We denote by ι the canonical inclusion of residue fields $k \subset A/\mathfrak{m}$ induced by the R-algebra structure. A deformation of G to A is a formal group law $\widetilde{G}$ over A such that $\iota^* G = \pi^* \widetilde{G}$ where $\pi\colon A \to A/\mathfrak{m}$ is the canonical projection. Let $\widetilde{G}_1$ and $\widetilde{G}_2$ be two deformations of G to A. We define a $*$-isomorphism between $\widetilde{G}_1$ and $\widetilde{G}_2$ as an isomorphism $\widetilde{u}\colon \widetilde{G}_1 \to \widetilde{G}_2$ over A such that $\pi^*\widetilde{u}$ is the identity map between $\pi^*\widetilde{G}_1 = \iota^* G = \pi^*\widetilde{G}_2$.

LEMMA 2.1. (cf. [11]) *There is at most one $*$-isomorphism between $\widetilde{G}_1$ and $\widetilde{G}_2$.*

We denote by $\mathbf{D}(R)$ the category of complete Noetherian local R-algebras with local R-algebra homomorphisms as morphisms. For an object A of $\mathbf{D}(R)$, we let DEF(A) be the set of all $*$-isomorphism classes of the deformations of G to A. Then DEF defines a functor from $\mathbf{D}(R)$ to the category of sets.

Let $R[\![t]\!] = R[\![t_1, \ldots, t_{n-1}]\!]$ be a formal power series ring over R with $n-1$ indeterminates. Note that $R[\![t]\!]$ is an object of $\mathbf{D}(R)$. There is a one-to-one correspondence between a local R-algebra homomorphism from $R[\![t]\!]$ to A and an $(n-1)$-tuple $(a_1, \ldots, a_{n-1})$ of elements of the maximal ideal $\mathfrak{m}$ of A. Lubin and Tate showed that there is a formal group law $F(t) = F(t_1, \ldots, t_{n-1})$ over $R[\![t]\!]$ which satisfies the following conditions:

(1) $\pi^* F(0, \ldots, 0)(X, Y) = G(X, Y)$ where $\pi\colon R \to k$ is the projection.

(2) For each i $(1 \le i \le n-1)$,

$$F(0, \ldots, 0, t_i, \ldots, t_{n-1})(X, Y) \equiv X + Y + t_i C_{p^i}(X, Y) \bmod \deg\,(p^i + 1)$$

where $C_{p^i}(X, Y) = (X^{p^i} + Y^{p^i} - (X+Y)^{p^i})/p$.

We say that a formal group law $F(t)$ satisfying the above conditions is a universal deformation of G due to the following theorem.

THEOREM 2.2. (Lubin and Tate [11]) *Let A be an object of* $\mathbf{D}(R)$. *For every deformation $\widetilde{G}$ of G to A, there is a unique local R-algebra homomorphism $\alpha\colon R[\![t]\!] \to A$ such that $\alpha^* F(t)$ is $*$-isomorphic to $\widetilde{G}$. Hence the functor* DEF *is represented by $R[\![t]\!]$:*

$$\mathrm{DEF}(A) \cong \mathrm{Hom}_{\mathbf{D}(R)}(R[\![t]\!], A)$$

and $F(t)$ is a universal object.

Let $\mathbf{F}$ be an algebraic extension of the prime finite field $\mathbf{F}_p$. We consider the height n Honda formal group law H_n defined over $\mathbf{F}$. The formal group law H_n is p-typical with p-series

$$[p]^{H_n}(X) = X^{p^n}.$$

Let E_n be a formal power series ring over $W(\mathbf{F})$ with $(n-1)$ indeterminates

$$E_n = W(\mathbf{F})[\![u_1, \ldots, u_{n-1}]\!]$$

where $W(\mathbf{F})$ is the ring of Witt vectors with coefficients in $\mathbf{F}$. The ring E_n is a complete Noetherian local ring with residue field $\mathbf{F}$. There is a p-typical formal group law F_n defined over E_n with p-series

$$[p]^{F_n}(X) = pX +_{F_n} u_1 X^p +_{F_n} u_2 X^{p^2} +_{F_n} \cdots +_{F_n} u_{n-1} X^{p^{n-1}} +_{F_n} X^{p^n}.$$

The formal group law F_n is a deformation of H_n to E_n. The following lemma is well known.

LEMMA 2.3. *The formal group law F_n is a universal deformation of H_n.*

2.2. Homomorphisms of formal group laws. In this subsection we recall a generalization of homomorphisms between formal group laws over possibly different ground rings considered by several authors (cf. [24]).

Let A_1 and A_2 be two (topological) commutative rings. Let F_1 (resp. F_2) be a formal group law over A_1 (resp. A_2). We understand that a homomorphism from F_1 to F_2 is a pair (α, f) of a (topological) ring homomorphism $\alpha\colon A_2 \to A_1$ and a homomorphism of formal group laws $f\colon F_1 \to \alpha^*F_2$ in the usual sense. The composition of two homomorphisms $(\alpha, f)\colon F_1 \to F_2$ and $(\beta, g)\colon F_2 \to F_3$ is defined by $(\alpha\circ\beta, \alpha^*g\circ f)\colon F_1 \to F_3$:

$$F_1 \xrightarrow{f} \alpha^*F_2 \xrightarrow{\alpha^*g} \alpha^*(\beta^*F_3) = (\alpha\circ\beta)^*F_3.$$

A homomorphism $(\alpha, f)\colon F_1 \to F_2$ is an isomorphism if there exists a homomorphism $(\beta, g)\colon F_2 \to F_1$ such that $(\alpha, f)\circ(\beta, g) = (id, id)$ and $(\beta, g)\circ(\alpha, f) = (id, id)$. Then a homomorphism $(\alpha, f)\colon F_1 \to F_2$ is an isomorphism if and only if α is an isomorphism of (topological) rings and f is an isomorphism of formal group laws in the usual sense.

Let $\mathbf{F}$ be an algebraic extension of $\mathbf{F}_p$ which contains $\mathbf{F}_{p^n}$. Let S_n be the automorphism group of H_n over $\mathbf{F}$ in the usual sense. We denote by G_n the automorphism group of H_n over $\mathbf{F}$ in the above generalized sense. The following lemma is easy.

LEMMA 2.4. *G_n is isomorphic to the semidirect product $\Gamma \ltimes S_n$ where Γ is the Galois group $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$.*

Proof. An automorphism of H_n consists of a ring isomorphism $\alpha\colon \mathbf{F} \to \mathbf{F}$ and an isomorphism of formal group laws $f\colon H_n \to \alpha^*H_n$. Then $\alpha \in \Gamma$. Since H_n is defined over the prime field $\mathbf{F}_p$, $\alpha^*H_n = H_n$. Hence we get $f \in S_n$. We regard S_n as the subset of the power series ring $\mathbf{F}[\![X]\!]$. Then the action of the Galois group Γ induces an action on S_n. The semi-direct product $\Gamma \ltimes S_n$ with respect to this action is isomorphic to the automorphism group of H_n over $\mathbf{F}$. □

Let $\widetilde{G}_n$ be the automorphism group of the universal deformation F_n of H_n in the generalized sense. There is a natural homomorphism $\widetilde{G}_n \to G_n$. Then we obtain the following proposition by Lemma 2.1 and Theorem 2.2.

PROPOSITION 2.5. *The natural homomorphism $\widetilde{G}_n \to G_n$ is an isomorphism.*

2.3. Galois extension L/K. In this subsection we define a Galois extension L of K obtained by adjoining all coefficients of an isomorphism between F_n and

H_{n-1} on K^{sep}. Then we recall that the Galois group of L/K is identified with S_{n-1} (cf. [4]).

Let $n \geq 2$. Let $\mathbf{F}$ be an algebraic extension of $\mathbf{F}_p$ which contains $\mathbf{F}_{p^n}$ and $\mathbf{F}_{p^{n-1}}$. We denote by V the quotient ring of the universal deformation ring E_n by the ideal $(p, u_1, \ldots, u_{n-2})$. Then $V = \mathbf{F}[[u_{n-1}]]$ is a complete discrete valuation ring. Let $K = \mathbf{F}((u_{n-1}))$ be the quotient field of V. There is a $W(\mathbf{F})$-algebra homomorphism θ: $E_n \to V$ given by $\theta(u_i) = 0$ for $i = 1, \ldots, n-2$ and $\theta(u_{n-1}) = u_{n-1}$. Then we get a p-typical formal group law $\theta^* F_n$ over V. We abbreviate $\theta^* F_n$ to F_n. The formal group law F_n is p-typical with p-series

$$[p]^{F_n}(X) = u_{n-1}X^{p^{n-1}} +_{F_n} X^{p^n}.$$

Let K^{sep} be a separable closure of K. Then there is an isomorphism between F_n and H_{n-1} over K^{sep}, since the height of F_n is $n-1$ (cf. Appendix 2 [17]). We fix an isomorphism Φ from F_n to H_{n-1}. Since Φ is a homomorphism between p-typical formal group laws, Φ has the following form

$$\Phi(X) = \sum_{i \geq 0}{}^{H_{n-1}} \Phi_i X^{p^i}.$$

We define a subfield $L_i = K(\Phi_0, \Phi_1, \ldots, \Phi_i)$ for $i \geq -1$ and $L = \cup_{i \geq -1} L_i$.

We recall that S_n is isomorphic to the unit group of the maximal order of the central division algebra over the p-adic number field $\mathbf{Q}_p$ with invariant $1/n$. We write an element $h \in S_n$ by $h = h_0 + h_1 T + h_2 T^2 + \cdots$ where $h_i \in W(\mathbf{F}_{p^n}), h_i^{p^n} = h_i$ for $i \geq 0$ and $h_0 \neq 0$. Then h corresponds to the automorphism

$$h(X) = \sum_{i \geq 0}{}^{H_n} \overline{h}_i X^{p^i}$$

where $\overline{h}_i$ is the image of h_i under the reduction $W(\mathbf{F}_{p^n}) \to \mathbf{F}_{p^n}$. Let $S_n^{(0)} = S_n$. We define the subgroups $S_n^{(i)}$ for $i \geq 1$ by

$$S_n^{(i)} = \{h \in S_n \mid h_0 = 1, h_1 = 0, \ldots, h_{i-1} = 0\}.$$

Then $S_n^{(i+1)}$ is a normal subgroup of S_n and the quotient group $S_n/S_n^{(i+1)}$ is finite of order $(p^n - 1)p^{ni}$ for $i \geq 0$. The canonical homomorphism $S_n \to \varprojlim S_n/S_n^{(i+1)}$ is an isomorphism. Hence S_n and $G_n = \Gamma \ltimes S_n$ are profinite groups.

For $g \in \widetilde{G}_n$, we obtain a continuous automorphism of V and hence an automorphism of K. We abbreviate this automorphism by g. We note that this is a right action of $\widetilde{G}_n$ on K. We denote by $f^g(X)$ the base change of a power series $f(X) \in K[[X]]$ by g. Also, we obtain an isomorphism $t(g)$ from F_n to F_n^g for $g \in \widetilde{G}_n$. Let $\widehat{g}$ be a continuous automorphism of the separable closure K^{sep} which is an extension of the automorphism g on K. Then we obtain a commutative diagram:

$$
\begin{array}{ccc}
F_n & \xrightarrow{t(g)} & F_n^g \\
\Phi \big\downarrow & & \big\downarrow \Phi^{\widehat{g}} \\
H_{n-1} & \xrightarrow{h(g,\widehat{g})} & H_{n-1}.
\end{array} \tag{2.1}
$$

Note that $F_n^{\widehat{g}} = F_n^g$ (resp. $H_{n-1}^{\widehat{g}} = H_{n-1}$), since F_n is defined over $\mathbf{F}_p[\![u_{n-1}]\!]$ (resp. $\mathbf{F}_p$).

LEMMA 2.6. *For every $\widehat{g}$ ($g \in \widetilde{G}_n$) and i, $L_i^{\widehat{g}} = L_i$. In particular, $L_i/\mathbf{F}_p((u_{n-1}))$ is a Galois extension.*

Proof. From the commutative diagram (2.1), we have

$$\Phi^{\widehat{g}}(t(g)(X)) = h(g,\widehat{g})(\Phi(X)). \tag{2.2}$$

Here $t(g)(X) = \sum_{i\geq 0}^{F_n^g} t_i(g)X^{p^i}$ and t_i is a continuous function from S_n to V for all $i \geq 0$. Since the automorphism $h(g,\widehat{g})\colon H_{n-1} \to H_{n-1}$ is an element of S_{n-1}, $h(g,\widehat{g})(X)$ has the form $h(g,\widehat{g})(X) = \sum_{i\geq 0}^{H_{n-1}} h_i(g,\widehat{g})X^{p^i}$ where $h_i(g,\widehat{g}) \in \mathbf{F}_{p^{n-1}}$. From the left-hand side of (2.2), we get

$$\sum_{i,j\geq 0}^{H_{n-1}} \Phi_j^{\widehat{g}} t_i(g)^{p^j} X^{p^{i+j}}.$$

From the right-hand side of (2.2), we get

$$\sum_{i,j\geq 0}^{H_{n-1}} h_j(g,\widehat{g})\Phi_i^{p^j} X^{p^{i+j}}.$$

By comparing the coefficients of X, we obtain $\Phi_0{}^{\widehat{g}} t_0(g) = h_0(g,\widehat{g})\Phi_0$. Since $h_0(g,\widehat{g}) \in \mathbf{F}_{p^{n-1}}$, we get $\Phi_0{}^{\widehat{g}} = h_0(g,\widehat{g})\Phi_0 t_0(g)^{-1} \in K(\Phi_0) = L_0$. We assume that $\Phi_0{}^{\widehat{g}}, \ldots, \Phi_{i-1}{}^{\widehat{g}} \in L_{i-1}$. Then by comparing the coefficients of X^{p^i}, we obtain $\Phi_i{}^{\widehat{g}} t_0(g)^{p^i} - h_0(g,\widehat{g})\Phi_i \in L_{i-1}$. Hence we get $\Phi_i{}^{\widehat{g}} \in L_{i-1}(\Phi_i) = L_i$. This completes the proof. □

For $\sigma \in \mathrm{Gal}(L/\mathbf{F}_p((u_{n-1})))$, we consider the following diagram:

$$
\begin{array}{ccc}
F_n & \overset{id}{=} & F_n{}^{\sigma} \\
\Phi \big\downarrow & & \big\downarrow \Phi^{\sigma} \\
H_{n-1} & \xrightarrow{h'(\sigma)} & H_{n-1}^{\sigma}.
\end{array}
$$

We note that ${F_n}^{\sigma} = F_n$ and $H^{\sigma}_{n-1} = H_{n-1}$. This diagram defines a homomorphism

$$h'\colon\ \mathrm{Gal}(L/\mathbf{F}_p((u_{n-1}))) \to G_{n-1}.$$

THEOREM 2.7. (Gross [4]) *The homomorphism h' is an isomorphism.*

Proof. There is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
1 \to & \mathrm{Gal}(L/K) & \longrightarrow & \mathrm{Gal}(L/\mathbf{F}_p((u_{n-1}))) & \longrightarrow & \mathrm{Gal}(K/\mathbf{F}_p((u_{n-1}))) & \to 1 \\
 & \downarrow & & \downarrow{\scriptstyle h'} & & \downarrow & \\
1 \to & S_{n-1} & \longrightarrow & G_{n-1} & \longrightarrow & \Gamma & \to 1.
\end{array}$$

Since $K/\mathbf{F}_p((u_{n-1}))$ is an unramified extension, the right vertical arrow is an isomorphism. From Theorem 3.5 1) *a*) of [4] and its proof, the left vertical arrow is also an isomorphism. □

Let $\triangle$ be the set of all isomorphisms from F_n to H_{n-1} over K^{sep}. For $\Phi' \in \triangle$, we consider the following commutative diagram

$$\begin{array}{ccccc}
 & & F_n & & \\
 & {\scriptstyle \Phi}\swarrow & & \searrow{\scriptstyle \Phi'} & \\
H_{n-1} & & \xrightarrow{\ \ h\ \ } & & H_{n-1}.
\end{array}$$

Since the isomorphism $h\colon\ H_{n-1} \to H_{n-1}$ is defined over $\mathbf{F}_{p^{n-1}}$, we see that Φ' is defined over L. Then the Galois group $\mathrm{Gal}(L/K)$ acts on $\triangle$. The following corollary is easy.

COROLLARY 2.8. *The action of* $\mathrm{Gal}(L/K)$ *on $\triangle$ is simply transitive.*

2.4. Extension $\mathcal{G}$ of G_n. In this subsection we define a group $\mathcal{G}$ which consists of all lifts of the action of G_n on the formal group law (F_n, K) to (F_n, L) and show that $\mathcal{G}$ is isomorphic to the profinite group $\Gamma \ltimes (S_n \times S_{n-1})$.

Let $\mathrm{Aut}(K)$ be the automorphism group of the topological field K. Suppose that $\mathrm{Aut}(K)$ acts on K from the right. Note that there is a homomorphism $\widetilde{G}_n \to \mathrm{Aut}(K)$. Let $\mathrm{Aut}(L)$ be the automorphism group of the topological field L. We denote by $A(L/K)$ the subgroup of $\mathrm{Aut}(L)$ consisting of automorphisms which preserve the subfield K: $A(L/K) = \{\theta \in \mathrm{Aut}(L) |\ \theta(K) = K\}$. Then we have a restriction homomorphism $A(L/K) \to \mathrm{Aut}(K)$. We define $\mathcal{G} = \widetilde{G}_n \times_{\mathrm{Aut}(K)} A(L/K)$

to be the fibre product

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{p} & \widetilde{G}_n \\ \downarrow & & \downarrow \\ A(L/K) & \longrightarrow & \mathrm{Aut}(K). \end{array}$$

By Lemma 2.6, the natural projection $p\colon\ \mathcal{G} \to \widetilde{G}_n$ is surjective. It is clear that the kernel of p is the Galois group $\mathrm{Gal}(L/K)$. Hence we have an exact sequence

$$1 \longrightarrow \mathrm{Gal}(L/K) \longrightarrow \mathcal{G} \xrightarrow{p} \widetilde{G}_n \longrightarrow 1.$$

Let $G_{n-1}(L)$ be the automorphism group of H_{n-1} over L in the generalized sense. By the same way as Lemma 2.4, we have an isomorphism $G_{n-1}(L) \cong \mathrm{Aut}(L) \ltimes S_{n-1}$. Let $A(L/K) \ltimes S_{n-1}$ be the subgroup of $G_{n-1}(L)$. For $(g, \widehat{g}) \in \mathcal{G}$, we have the commutative diagram (2.1). This diagram defines a homomorphism $f\colon\ \mathcal{G} \to A(L/K) \ltimes S_{n-1}$ by $(g, \widehat{g}) \mapsto (\widehat{g}, h(g, \widehat{g}))$.

There are homomorphisms $A(L/K) \to \mathrm{Aut}(K) \to \Gamma$ where the first is restriction and the second is obtained by considering the induced automorphism on the residue field. These homomorphisms are compatible with the action on S_{n-1}. Hence we get a homomorphism $f'\colon\ A(L/K) \ltimes S_{n-1} \to G_{n-1}$. There are homomorphisms $\mathcal{G} \xrightarrow{f' \circ f} G_{n-1} \longrightarrow \Gamma$. By Proposition 2.5, we have a natural isomorphism $\widetilde{G}_n \cong G_n$. We identify $\widetilde{G}_n$ with G_n by this isomorphism. Then we have homomorphisms $\mathcal{G} \xrightarrow{p} G_n \longrightarrow \Gamma$. We verify that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{p} & G_n \\ {\scriptstyle f' \circ f}\downarrow & & \downarrow \\ G_{n-1} & \longrightarrow & \Gamma. \end{array}$$

Then we get a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 \to & \mathrm{Gal}(L/K) & \longrightarrow & \mathcal{G} & \xrightarrow{p} & G_n & \to 1 \\ & \downarrow & & {\scriptstyle f' \circ f}\downarrow & & \downarrow & \\ 1 \to & S_{n-1} & \longrightarrow & G_{n-1} & \longrightarrow & \Gamma & \to 1. \end{array}$$

The left vertical arrow is an isomorphism by Theorem 2.7. Hence we get the following theorem.

THEOREM 2.9. *There are isomorphisms*

$$\mathcal{G} \cong G_n \times_{\Gamma} G_{n-1} \cong \Gamma \ltimes (S_n \times S_{n-1}).$$

The profinite group $\mathcal{G}$ acts on the formal group law (F_n, L) in the generalized sense. Then the natural homomorphism $(F_n, L) \to (F_n, K)$ is compatible with the projection $\mathcal{G} \to G_n$. The inclusion $G_n \subset \mathcal{G}$ gives a lift of the action of G_n on (F_n, K) to (F_n, L) such that the action of the subgroup S_n commutes with the action of Galois group $\mathrm{Gal}(L/K) = S_{n-1}$. Also, the natural homomorphism $(F_n, L) \to (H_{n-1}, \mathbf{F})$ is compatible with the projection $\mathcal{G} \to G_{n-1}$.

3. Continuous cohomology. In this section we study the group cohomology based on continuous cochains. First, we consider inflation maps under some conditions. Then we define quotient groups $\mathcal{G}(i)$ of the profinite group $\mathcal{G}$ and show that inflation maps between the cohomology groups of $\mathcal{G}(i)$ are isomorphisms.

3.1. Inflation maps. Let G be a Hausdorff topological group and let J be a finite normal subgroup. We denote by H the quotient group G/J and π: $G \to H$ the quotient map. In this subsection we assume that there is a continuous section χ: $H \to G$ such that $\chi(e) = e$. Note that χ is not necessarily a group homomorphism. For example, if G is a profinite group, then there is such a section (cf. [20]). Let M be a topological G-module. The fixed submodule M^J is naturally a topological H-module. In this subsection we study the inflation map $H_c^*(H; M^J) \to H_c^*(G; M)$ under some conditions.

A normalized continuous n-cochain for G in M is a continuous function f: $G^n \to M$ such that $f(\gamma_1, \ldots, \gamma_n) = 0$ if γ_i is equal to the identity e for some i $(1 \leq i \leq n)$. We denote by $A^n = A^n(G; M)$ the abelian group of all normalized continuous n-cochains for G in M. The coboundary map d: $A^n \to A^{n+1}$ is given by

$$\begin{aligned} df(\gamma_1, \ldots, \gamma_{n+1}) &= \gamma_1 \cdot f(\gamma_2, \ldots, \gamma_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i f(\gamma_1, \ldots, \gamma_i\gamma_{i+1}, \ldots, \gamma_{n+1}) \\ &\quad + (-1)^{n+1} f(\gamma_1, \ldots, \gamma_n). \end{aligned}$$

It is easy to verify that A^* is a cochain complex. The continuous cohomology of G with coefficients in M based on continuous cochains is defined as the cohomology group of the cochain complex $A^*(G; M)$. We denote by $H_c^*(G; M)$ the continuous cohomology of G with coefficients in M.

We define a filtration on the cochain complex $A^* = A^*(G; M)$. For $j = 0$, we set $F^0A^n = A^n$. For $0 < j \leq n$, F^jA^n is defined as a subgroup of A^n consisting of $f \in A^n$ such that f: $G^n \to M$ factors through a continuous map f': $G^{n-j} \times H^j \to M$. For $j > n$, we set $F^jA^n = 0$. Hence we get a filtration of A^n:

$$A^n = F^0A^n \supset F^1A^n \supset \cdots \supset F^nA^n \supset F^{n+1}A^n = 0.$$

It is easy to verify that $d(F^jA^n) \subset F^jA^{n+1}$. Hence $(F^jA^*)_{j\geq 0}$ is a filtration of the cochain complex A^*:

$$A^* = F^0A^* \supset F^1A^* \supset \cdots \supset F^nA^* \supset \cdots.$$

The normalized continuous n-cochain group $A^n(J;M)$ is naturally isomorphic to a direct product of finite many copies of M since J is finite. We introduce a topology on $A^n(J;M)$ by using this isomorphism and the product topology. Let N be a topological module on which J acts trivially. Then the tensor product $N\otimes\mathbf{Z}[J]$ is naturally a topological J-module. We say that a topological J-module M is a regular representation over N if M is isomorphic to $N\otimes\mathbf{Z}[J]$ as topological J-modules. Then we have a natural isomorphism $A^*(J;M)\cong A^*(J;\mathbf{Z}[J])\otimes N$ as cochain complexes of topological modules. In the following we assume that the topological G-module M is a regular representation of J as topological J-module. Let $A^j(G;A^i(J;M))$ be the abelian group of all normalized continuous j-cochains of G in $A^i(J;M)$. We define a homomorphism $r_j\colon\ F^jA^{i+j} \to A^j(H;A^i(J;M))$ by

$$r_j(f)(\sigma_1,\dots,\sigma_j)(\tau_1,\dots,\tau_i) = f'(\tau_1,\dots,\tau_i,\sigma_1,\dots,\sigma_j)$$

where $f'\colon\ G^i\times H^j \to M$ is a continuous map such that

$$f(\gamma_1,\dots,\gamma_n) = f'(\gamma_1,\dots,\gamma_{n-j},\pi(\gamma_{n-j+1}),\dots,\pi(\gamma_n)).$$

It is easy to see that $r_j(f)=0$ if $f\in F^{j+1}A^{i+j}$. Hence we get a homomorphism

$$\overline{r}_j\colon\ F^jA^{i+j}/F^{j+1}A^{i+j} \longrightarrow A^j(H;A^i(J;M)).$$

We note that $\overline{r}_j\colon\ F^jA^j/F^{j+1}A^j \to A^j(H;M)$ is an isomorphism. Let d be the coboundary operator of $F^jA^*/F^{j+1}A^*$. The coboundary operator of $A^*(J;M)$ induces a homomorphism $d_J\colon\ A^j(H;A^*(J;M)) \to A^j(H;A^{*+1}(J;M))$. Then we obtain that $d_J\circ\overline{r}_j = \overline{r}_j\circ d$.

LEMMA 3.1. *$H(A^j(H;A^*(J;M)),d_J) = A^j(H;M^J)$ for all j.*

Proof. This follows from the fact that the sequence $0\to\mathbf{Z}\xrightarrow{\epsilon}A^0(J;\mathbf{Z}[J])\xrightarrow{d}A^1(J;\mathbf{Z}[J])\xrightarrow{d}\cdots$ is split exact where $\epsilon(1)=\sum_{j\in J} j\in\mathbf{Z}[J]$. □

LEMMA 3.2. *$\overline{r}_j$ induces an isomorphism $H^j(F^jA^*/F^{j+1}A^*)\xrightarrow{\cong}A^j(H;M^J)$.*

Proof. Let $f\in F^jA^j$ such that $df\in F^{j+1}A^{j+1}$. Then $d_J(r_j(f))=0$. By Lemma 3.1, $r_j(f)\in A^j(H;M^J)$. Conversely, let $\widetilde{f}\in A^j(H;M^J)\subset A^j(H;M)$. We define $f\in F^jA^j$ by $f(\gamma_1,\dots,\gamma_j)=\widetilde{f}(\pi(\gamma_1),\dots,\pi(\gamma_j))$. Then for any $\tau\in J$, we easily see that $df(\gamma_1\tau,\gamma_2,\dots) = df(\gamma_1,\gamma_2,\dots)$. □

LEMMA 3.3. *$H^n(F^jA^*/F^{j+1}A^*) = 0$ for all $n > j$.*

Proof. Put $i = n - j - 1 \geq 0$. Let $f \in F^jA^n$ such that $df \in F^{j+1}A^{n+1}$. Since $d_J \circ \bar{r}_j = \bar{r}_j \circ d$, we have $d_J(r_j(f)) = 0$. By Lemma 3.1, there is $u \in A^j(H; A^i(J; M))$ such that $d_J u = r_j(f)$. We define a continuous function g: $J^i \times G^j \to M$ by $g(\sigma_1, \ldots, \sigma_i, \gamma_1, \ldots, \gamma_j) = u(\pi(\gamma_1), \ldots, \pi(\gamma_j))(\sigma_1, \ldots, \sigma_i)$. Set $g_0 = g$. We define a sequence of continuous functions $g_1, \ldots, g_i$ such that g_k is defined on $G^k \times J^{i-k} \times G^j$ with its values in M and g_k is an extension of g_{k-1} for all $1 \leq k \leq i$. We write $\rho_s^t = (\rho_s, \ldots, \rho_t) \in G^{t-s+1}, \gamma_s^t = (\gamma_s, \ldots, \gamma_t) \in G^{t-s+1}$ and $\sigma_s^t = (\sigma_s, \ldots, \sigma_t) \in J^{t-s+1}$ for $1 \leq s \leq t$. We denote by ρ^* the element $\chi(\pi(\rho)) \in G$ and by $\rho^\vee$ the element $\chi(\pi(\rho))^{-1}\rho \in J$. Note that the functions $\rho \mapsto \rho^*$ and $\rho \mapsto \rho^\vee$ are continuous. Let

$$g_1(\rho, \sigma_2^i, \gamma_1^j) = \rho^* \cdot g(\rho^\vee, \sigma_2^i, \gamma_1^j) - f(\rho^*, \rho^\vee, \sigma_2^i, \gamma_1^j).$$

For $k > 1$, we define the g_k's recursively by

$$\begin{aligned} g_k(\rho_1^k, \sigma_{k+1}^i, \gamma_1^j) &= g_{k-1}(\rho_1^{k-2}, \rho_{k-1}\rho_k^*, \rho_k^\vee, \sigma_{k+1}^i, \gamma_1^j) \\ &\quad +(-1)^k f(\rho_1^{k-1}, \rho_k^*, \rho_k^\vee, \sigma_{k+1}^i, \gamma_1^j). \end{aligned}$$

Then we can show that $f - dg_i \in F^{j+1}A^n$ as in the proof of Theorem 2.2.1 of [5]. □

Therefore, we get the E_1-term of the spectral sequence associated with the filtration $(F^jA^*)_{j\geq 0}$ of the cochain complex A^*:

$$E_1^{p,q} \cong \begin{cases} A^p(H; M^J) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

It is easy to verify that the differential d_1 is given by the coboundary map of the normalized continuous cochain complex $A^*(H; M^J)$. Hence we get the E_2-term

$$E_2^{p,q} \cong \begin{cases} H_c^p(H; M^J) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

The spectral sequence collapses from E_2-term and converges to the cohomology group $H^*(A) = H_c^*(G; M)$. It is easy to verify that the edge homomorphism $E_2^{p,0} \to H^p(A)$ is identified with the inflation map $H_c^p(H; M^J) \to H_c^p(G; M)$. Hence we get the following proposition.

PROPOSITION 3.4. *Let G be a Hausdorff topological group, J a finite normal subgroup and $H = G/J$ the quotient group. We assume that there is a continuous section χ: $H \to G$. Let M be a topological G module such that M is a regular representation as topological J-module. Then the inflation map $H_c^*(H; M^J) \to H_c^*(G; M)$*

is an isomorphism

$$H_c^*(H; M^J) \xrightarrow{\cong} H_c^*(G; M).$$

3.2. Cohomology of $\mathcal{G}(i)$. We consider a graded field $L[u^{\pm 1}]$ and its subfield $L_i[u^{\pm 1}]$ where the degree of u is -2. In this subsection we define quotient groups $\mathcal{G}(i)$ of the profinite group $\mathcal{G}$ which act on $L_i[u^{\pm 1}]$. Then we show that the cohomology groups of $\mathcal{G}(i)$ with coefficients in $L_i[u^{\pm 1}]$ are isomorphic to each other through the inflation maps.

We recall that $\mathcal{G}$ is a fibre product $\widetilde{G}_n \times_{\mathrm{Aut}(K)} A(L/K)$ where $\widetilde{G}_n$ is the automorphism group of the universal deformation F_n over E_n, $\mathrm{Aut}(K)$ is the automorphism group of the local field K, and $A(L/K)$ is the subgroup of the automorphism group of L consisting of elements preserving K. By Theorem 2.9, there is an isomorphism $\mathcal{G} \cong \Gamma \ltimes (S_n \times S_{n-1})$ where $\Gamma = \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$. Hence $\mathcal{G}$ is a profinite group. There is an action of $\mathcal{G}$ on L through the projection $\mathcal{G} \to A(L/K)$. In the following we assume that $\mathbf{F}$ is a finite field which contains $\mathbf{F}_{p^n}$ and $\mathbf{F}_{p^{n-1}}$ for simplicity.

We define an action of $\mathcal{G}$ on $L[u^{\pm 1}]$ as automorphisms of graded field which is an extension of the action of $\mathcal{G}$ on the degree 0 part L. An element g of $\widetilde{G}_n$ is identified with a pair $(g, t(g))$ where g is an automorphism of V and $t(g)$ is an isomorphism $t(g)\colon\ F_n \to F_n^g$ over V. The isomorphism $t(g)$ has the form

$$t(g)(X) = \sum_{i \geq 0}{}^{F_n^g} t_i(g) X^{p^i}.$$

For $(g, \widehat{g}) \in \mathcal{G} = \widetilde{G}_n \times_{\mathrm{Aut}(K)} A(L/K)$, we set

$$u^{(g,\widehat{g})} = t_0(g)^{-1} u.$$

This defines a continuous action of $\mathcal{G}$ on $L[u^{\pm 1}]$ as automorphisms of graded field. We note that under the isomorphism $\mathcal{G} \cong \Gamma \ltimes (S_n \times S_{n-1})$, the subgroup $G_{n-1} = \Gamma \ltimes S_{n-1}$ acts on u trivially and on L as the Galois group $\mathrm{Gal}(L/\mathbf{F}_p((u_{n-1})))$.

We recall that there is an open normal subgroup $S_{n-1}^{(i+1)}$ of S_{n-1}. Under the isomorphism $\mathcal{G} \cong \Gamma \ltimes (S_n \times S_{n-1})$, we see that $S_{n-1}^{(i+1)}$ is a normal subgroup of $\mathcal{G}$. We denote by $\mathcal{G}(i)$ the quotient group $\mathcal{G}/S_{n-1}^{(i+1)}$. In particular, $\mathcal{G}(-1) = G_n$. Hence there is an exact sequence of profinite groups

$$1 \longrightarrow S_{n-1}^{(i+1)} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}(i) \longrightarrow 1.$$

By Lemma 2.6, the action of $\mathcal{G}$ on $L[u^{\pm 1}]$ induces an action of $\mathcal{G}$ on the subfield $L_i[u^{\pm 1}]$. Then it is easy to verify that the action of $\mathcal{G}$ on $L_i[u^{\pm 1}]$ factors through the quotient group $\mathcal{G}(i)$.

There is an exact sequence

$$1 \longrightarrow S_{n-1}^{(i+1)}/S_{n-1}^{(i+2)} \longrightarrow \mathcal{G}(i+1) \longrightarrow \mathcal{G}(i) \longrightarrow 1.$$

By Theorem 2.7 and its proof of [4], the kernel $S_{n-1}^{(i+1)}/S_{n-1}^{(i+2)}$ is identified with the Galois group of the extension L_{i+1}/L_i. Hence the invariant subring of the action of $S_{n-1}^{(i+1)}/S_{n-1}^{(i+2)}$ on $L_{i+1}[u^{\pm 1}]$ is $L_i[u^{\pm 1}]$. We consider the inflation map

$$H_c^*(\mathcal{G}(i); L_i[u^{\pm 1}]) \longrightarrow H_c^*(\mathcal{G}(i+1); L_{i+1}[u^{\pm 1}]).$$

For the finite Galois extension L_i/L_{i-1}, the existence of a normal basis implies that the topological $\mathrm{Gal}(L_{i+1}/L_i)$-module L_{i+1} is a regular representation over the discrete valuation field L_i. By Proposition 3.4, we obtain the following theorem.

THEOREM 3.5. *The inflation map*

$$H_c^*(\mathcal{G}(i); L_i[u^{\pm 1}]) \longrightarrow H_c^*(\mathcal{G}(i+1); L_{i+1}[u^{\pm 1}])$$

is an isomorphism for all $i \geq -1$.

3.3. Construction of the ring homomorphism Θ. In §2.4 we showed that there is an isomorphism $\mathcal{G} \cong \Gamma \ltimes (S_n \times S_{n-1})$. In this subsection we construct a ring homomorphism from the cohomology of G_{n-1} with coefficients in $\mathbf{F}[w^{\pm 1}]$ to the cohomology of G_n with coefficients in $K[u^{\pm 1}]$ by using two inflation maps induced by the projections $\mathcal{G} \to G_n$ and $\mathcal{G} \to G_{n-1}$.

Let $\mathbf{F}[w^{\pm 1}]$ be the graded field where the degree of w is -2. The profinite group G_{n-1} acts on $\mathbf{F}[w^{\pm 1}]$ from the right as follows. We recall that we have an expression of $h \in S_{n-1}$ as $h = h_0 + h_1 T + h_2 T^2 + \cdots$ where $h_i \in W(\mathbf{F}_{p^{n-1}})$, $h_i^{p^{n-1}} = h_i$ and $h_0 \neq 0$. The subgroup S_{n-1} of G_{n-1} acts on $\mathbf{F}[w^{\pm 1}]$ as $\mathbf{F}$-algebra automorphisms by

$$w^h = \overline{h}_0^{-1} w, \quad h \in S_{n-1} \tag{3.1}$$

where $\overline{h}_0 \in \mathbf{F}_{p^{n-1}}$ is the reduction of $h_0 \in W(\mathbf{F}_{p^{n-1}})$ to the residue field. The subgroup Γ acts on $\mathbf{F}[w^{\pm 1}]$ by

$$(aw^n)^\sigma = a^\sigma w^n, \qquad \sigma \in \Gamma, a \in \mathbf{F}, n \in \mathbf{Z}. \tag{3.2}$$

Then we obtain an action of G_{n-1} on $\mathbf{F}[w^{\pm 1}]$ compatible with the above actions of the subgroups S_{n-1} and Γ.

We denote by $G_{n-1}(i)$ the quotient group $G_{n-1}/S_{n-1}^{(i+1)}$ for $i \geq -1$. The action of G_{n-1} on $\mathbf{F}[w^{\pm 1}]$ factors through $G_{n-1}(i)$ for all $i \geq 0$. The following lemma is well known on the cohomology of profinite groups.

LEMMA 3.6. (cf. [20]) $H^*_c(G_{n-1};\mathbf{F}[w^{\pm 1}]) \cong \varinjlim_i H^*_c(G_{n-1}(i);\mathbf{F}[w^{\pm 1}])$.

The action of $\mathcal{G}$ on $L[u^{\pm 1}]$ induces the action of the quotient group $\mathcal{G}(i)$ on the subfield $L_i[u^{\pm 1}]$. We identify $\mathbf{F}[w^{\pm 1}]$ as the subfield of $L[u^{\pm 1}]$ by the relation

$$w = \Phi_0^{-1}u.$$

LEMMA 3.7. *$\mathbf{F}[w^{\pm 1}]$ is stable under the action of $\mathcal{G}$. The subgroup S_n of $\mathcal{G}$ acts trivially on $\mathbf{F}[w^{\pm 1}]$. The action of the subgroup G_{n-1} of $\mathcal{G}$ coincides with the action defined in* (3.1) *and* (3.2).

Proof. For $g \in S_n$, we have $\Phi_0^g = t_0(g)^{-1}\Phi_0$ and $u^g = t_0(g)^{-1}u$. Hence S_n acts on w trivially. For $h \in S_{n-1}$, we have $\Phi_0^h = \overline{h}_0\Phi_0$ and $u^h = u$. Hence we obtain $w^h = \overline{h}_0^{-1}w$. Since the action of Γ on Φ_0 and u is trivial, the action on w is also trivial. This shows that $\mathbf{F}[w^{\pm 1}]$ is stable under $\mathcal{G}$ and the action of G_{n-1} is the same as defined in (3.1) and (3.2). □

Remark 3.8. By Lemma 5.10, the invariant ring of $K[u^{\pm 1}]$ under the action of S_n is $\mathbf{F}[v_{n-1}^{\pm 1}]$. Since L is totally ramified over K, the invariant ring of $L[u^{\pm 1}]$ is $\mathbf{F}[w^{\pm 1}]$.

By Lemma 3.7, we see that the inclusion $\mathbf{F}[w^{\pm 1}] \hookrightarrow L_i[u^{\pm 1}]$ is compatible with the projection map $\mathcal{G}(i) \to G_{n-1}(i)$ for all $i \geq 0$. Hence we get an inflation map

$$H^*_c(G_{n-1}(i);\mathbf{F}[w^{\pm 1}]) \longrightarrow H^*_c(\mathcal{G}(i);L_i[u^{\pm 1}]).$$

We consider the following homomorphism of systems

$$\begin{array}{ccccccc} \cdots \to & H^*_c(G_{n-1}(i-1);\mathbf{F}[w^{\pm 1}]) & \to & H^*_c(G_{n-1}(i);\mathbf{F}[w^{\pm 1}]) & \to \cdots \\ & \downarrow & & \downarrow & \\ \cdots \to & H^*_c(\mathcal{G}(i-1);L_{i-1}[u^{\pm 1}]) & \to & H^*_c(\mathcal{G}(i);L_i[u^{\pm 1}]) & \to \cdots . \end{array}$$

By Theorem 3.5, the homomorphisms in the bottom sequence are all isomorphisms and we have a compatible isomorphism

$$H^*_c(G_n;K[u^{\pm 1}]) \xrightarrow{\cong} H^*_c(\mathcal{G}(i);L_i[u^{\pm 1}])$$

for all $i \geq 0$. By passing to the direct limits of the systems, we obtain a ring homomorphism

$$\Theta\colon H^*_c(G_{n-1};\mathbf{F}[w^{\pm 1}]) \longrightarrow H^*_c(G_n;K[u^{\pm 1}]). \tag{3.3}$$

LEMMA 3.9. *Let* $v_{n-1} = u_{n-1}u^{-(p^{n-1}-1)}$. *Then* $w^{-(p^{n-1}-1)} = v_{n-1}$ *and* $H_c^0(G_{n-1}; \mathbf{F}[w^{\pm 1}]) = \mathbf{F}_p[v_{n-1}^{\pm 1}]$. *The homomorphism* Θ *is a morphism of* $\mathbf{F}_p[v_{n-1}^{\pm 1}]$-*algebras.*

Proof. Since Φ is an isomorphism from F_n to H_{n-1}, we have $\Phi([p]^{F_n}(X)) = [p]^{H_{n-1}}(\Phi(X))$. By comparing the leading coefficients, we see that $\Phi_0 u_{n-1} = \Phi_0^{p^{n-1}}$. This implies that $w^{-(p^{n-1}-1)} = u_{n-1}u^{-(p^{n-1}-1)} = v_{n-1}$. Then the lemma follows from Proposition 3.18 (b) of [14]. □

Remark 3.10. In Lemma 5.10 we show that $H_c^0(G_n; K[u^{\pm 1}]) = \mathbf{F}_p[v_{n-1}^{\pm 1}]$ and Θ is the identity map on H^0.

4. Graded modules over $k[X]$.

Let k be a field and let $k[X]$ be a polynomial algebra with one variable X. We define a grading on $k[X]$ by $|X| = 1$. Then $k[X]$ is a discrete valuation ring in the graded sense. For a $\mathbf{Z}$-graded module M, we denote by M^t the homogeneous component of degree t in M. In this section we study $\mathbf{Z}$-graded modules over $k[X]$ satisfying some finiteness condition.

4.1. Torsion modules over $k[X]$.

We say that a $\mathbf{Z}$-graded module over $k[X]$ is of finite type if the dimension over k of M^t is finite for all t. Let M be a $\mathbf{Z}$-graded torsion $k[X]$-module, that is, for every $m \in M$, there is a natural number $n = n(m)$ such that $X^n m = 0$. Let M_n be the kernel of multiplication by X^n for $n \geq 0$:

$$0 \longrightarrow M_n \longrightarrow M \xrightarrow{X^n} M.$$

Note that $M = \cup_n M_n$ since M is a torsion module. In this subsection we study a $\mathbf{Z}$-graded torsion module over $k[X]$ such that M_1 is of finite type. In the following we assume that M_1 is of finite type.

LEMMA 4.1. *The submodule* M_n *is of finite type for all* $n \geq 0$.

Proof. By induction on n, the lemma follows from the exact sequence $0 \longrightarrow M_1 \longrightarrow M_n \xrightarrow{X} M_{n-1}$. □

We say that a torsion module M is divisible if the multiplication by X is surjective, and finite torsion if $\cap_{n>0} X^n M = 0$. First, we study a divisible module M.

LEMMA 4.2. *The multiplication by* X *on a divisible module* M *induces an isomorphism*

$$X\cdot\colon M_{n+1}/M_n \xrightarrow{\cong} M_n/M_{n-1}$$

for all $n \geq 1$.

Proof. This is easy. □

We take a basis B_1 of M_1 over k. Then there is a subset B_2 of M_2 such that the multiplication by X induces a bijection $X\colon\ B_2 \to B_1$. By Lemma 4.2, we see that $B_1 \cup B_2$ is a basis of M_2. By induction, we can take a subset B_n of M_n such that the multiplication by X induces a bijection $X\colon\ B_n \to B_{n-1}$ for all $n > 0$. Then we see that $\cup_n B_n$ is a basis of M over k. Hence we obtain the following proposition.

PROPOSITION 4.3. *A divisible module M is isomorphic to a direct sum of copies of $k[X^{\pm 1}]/k[X]$.*

Remark 4.4. In the proposition it is not necessary to assume that M_1 is of finite type.

Second, we study a finite torsion module M. We define a submodule $M_{n,i}$ of M_n by $M_{n,i} = (M_n \cap X^i M) + M_{n-1}$. Then we get a sequence of submodules of M_n:

$$M_n = M_{n,0} \supset M_{n,1} \supset \cdots \supset M_{n-1}.$$

In the degree t component, the k-module M_n^t is of finite dimension by Lemma 4.1. Since M is finite torsion, there is a non-negative integer i such that $M_n^t \cap X^i M = 0$. Hence we have $M_{n,i}^t = M_{n-1}^t$.

LEMMA 4.5. *The multiplication by X induces an isomorphism*

$$X\cdot\colon\ M_{n+1,i}/M_{n+1,i+1} \xrightarrow{\cong} M_{n,i+1}/M_{n,i+2}$$

for all $n \geq 1$ and $i \geq 0$.

Proof. This is easy. □

For every n, we take a subset $B_{n,0}$ of M_n such that $B_{n,0}$ gives a basis of $M_{n,0}/M_{n,1}$. We define a subset $B_{n,1}$ of M_n to be the image of $B_{n+1,0}$ under the multiplication by X. By Lemma 4.5, we see that $B_{n,1}$ gives a basis of $M_{n,1}/M_{n,2}$. Inductively, we get $B_{n,i}$ for all n and i such that $B_{n,i}$ gives a basis of $M_{n,i}/M_{n,i+1}$. Then we see that $\cup_{n,i} B_{n,i}$ gives a basis of M over k. The $k[X]$-submodule of M generated by $B_{n,0}$ is isomorphic to a finite direct sum of copies of $k[X]/(X^n)$. Hence we obtain the following proposition.

PROPOSITION 4.6. *A finite torsion module M is isomorphic to a direct sum of copies of various $k[X]/(X^n)$.*

Finally, we study a general torsion module M. We recall that we assume that M_1 is of finite type. We define the submodule D by $\cap_n X^n M$ and the quotient T by M/D.

LEMMA 4.7. *The submodule D is divisible and the quotient T is finite torsion.*

Proof. First, we show that D is divisible. We take an element $m \in D$ with a degree $t+1$. Then there is a positive integer n such that $m \in M_{n-1}$. We define a submodule ${M^t_{n,i}}'$ of M^t_n by ${M^t_{n,i}}' = M^t_n \cap X^i M = X^i M^{t-i}_{n+i}$. Then we get a decreasing filtration

$$M^t_n = {M^t_{n,0}}' \supset {M^t_{n,1}}' \supset \cdots \supset {M^t_{n,i}}' \supset \cdots.$$

We define ${M^t_{n,\infty}}' = \cap_i {M^t_{n,i}}'$. Then $D \cap M^t_n = {M^t_{n,\infty}}'$. By Lemma 4.1, the dimension of M^t_n is finite. Hence there is a non-negative integer j such that ${M^t_{n,j}}' = {M^t_{n,\infty}}'$. Let $a \in M^{t-j}$ such that $X^{j+1}a = m$. Define $b = X^j a$. Then we have $Xb = a$ and $b \in {M^t_{n,j}}' = {M^t_{n,\infty}}' \subset D$. This shows that the multiplication by X is surjective on D.

Second, we show that T is finite torsion. We take $a \in \cap_i X^i T$. Then there is $a_i \in T$ such that $X^i a_i = a$ for all i. We take a lift m of a and a lift $m_i \in M$ of a_i for all i. Then we have $m - X^i m_i \in D$. Since D is divisible, there is b_i such that $m - X^i m_i = X^i b_i$. This shows that $m \in \cap_i X^i M = D$. Hence $a = 0$. □

By Lemma 4.7, we obtain a natural exact sequence $0 \longrightarrow D \longrightarrow M \longrightarrow T \longrightarrow 0$ where D is divisible and T is finite torsion. We take a basis B of T. Let $b \in B$ such that $b \in T_n - T_{n-1}$. We take a lift $b' \in M$ of b. Then $X^n b' \in D$. Since D is divisible, there is $d \in D$ such that $X^n b' = X^n d$. Then $b' - d \in M$ is a lift of b and satisfies $X^n(b' - d) = 0$. Hence we obtain a splitting of the above exact sequence.

PROPOSITION 4.8. *Let M be a $\mathbf{Z}$-graded torsion $k[X]$-module such that the kernel M_1 of the multiplication by X is of finite type. Then there is a natural exact sequence*

$$0 \longrightarrow D \longrightarrow M \longrightarrow T \longrightarrow 0$$

where D is divisible and T is finite torsion. There is a (non-canonical) splitting on the exact sequence. Hence M is isomorphic to the direct sum $D \oplus T$.

4.2. Complete modules over $k[X]$. We say that M is complete if the natural map $M \to \varprojlim_i M/X^i M$ is an isomorphism. Let ${}_nM$ be the cokernel of multiplication by X^n on M:

$$M \xrightarrow{X^n} M \longrightarrow {}_nM \longrightarrow 0.$$

In this subsection we study a complete $k[X]$-module M such that ${}_1M$ is of finite type. In the following we assume that ${}_1M$ is of finite type.

LEMMA 4.9. *If ${}_1M$ is of finite type, then ${}_nM$ is also of finite type for all $n > 0$.*

Proof. By induction on n, this follows from the exact sequence ${}_{n-1}M \xrightarrow{X} {}_{n}M \longrightarrow {}_{1}M \longrightarrow 0$. □

We say that a complete $k[X]$-module is torsion free if the multiplication by X is injective. For a submodule N of a complete $k[X]$-module M, we denote by $\overline{N}$ the closure of N in M, that is,

$$\overline{N} = \bigcap_i (N + X^i M) = \lim_{\overleftarrow{i}} (N + X^i M)/X^i M.$$

A submodule N is said to be dense if $\overline{N} = M$. We say that a complete $k[X]$-module M is essentially torsion if there is a dense torsion submodule in M.

Let M be a complete $k[X]$-module such that ${}_1M$ is of finite type. For $s \in \mathbf{Z}$, we denote by $P(s)$ the $\mathbf{Z}$-graded $k[X]$-module given by

$$P(s)^t = \begin{cases} {}_{1+t-s}M^t, & t \geq s, \\ 0, & t < s. \end{cases}$$

Note that $P(s)$ is of finite type and bounded below for all s. Hence $P(s)$ is isomorphic to a direct sum of copies of $k[X]$ and various $k[X]/(X^n)$. We take a basis of $P(s)$ and let B_s be the subset of the basis in degree s. We take a lift B'_s of B_s in M. For $b \in B_s$, we let $b' \in B'_s$ be the lift of b. If $X^n b = 0$ and $X^{n-1} b \neq 0$, then $X^n b' \in X^{1+n} M^{s+n}$. Hence there is $c \in M$ such that $X^n b' = X^{1+n} c$. Then $b'' = b' - Xc$ is a lift of b such that $X^n b'' = 0$. Hence we can take a lift B'_s such that, if $b' \in B'_s$ is a lift of $b \in B_s$ and b generates $k[X]/(X^n)$ in $P(s)$, then $b' \in B_s$ generates $k[X]/(X^n)$ in M. Let $B' = \cup_s B'_s$ and M' the submodule of M generated by B'. Then M' is isomorphic to a direct sum of copies of $k[X]$ and various $k[X]/(X^n)$. Then it is easy to see that ${}_nM' = {}_nM$ for all n. Hence M is the X-adic completion of M'.

Proposition 4.10. *Let M be a complete $k[X]$-module such that ${}_1M$ is of finite type. Then M is isomorphic to a direct product of copies of $k[X]$ and various $k[X]/(X^n)$.*

Remark 4.11. If M itself is of finite type, then we can replace the direct product by the direct sum in the proposition.

We fix an isomorphism between M and a direct product of copies of $k[X]$ and $k[X]/(X^n)$ for $n > 0$. Let T be the submodule of M which is the direct product of all torsion components $k[X]/(X^n)$ of M under the isomorphism. We denote by T' the torsion submodule of M. Then we have $T' \subset T$. Let T'' be the submodule of M which is the direct sum of all torsion components $k[X]/(X^n)$ of M under the isomorphism. Then we see that $T'' \subset T'$ and $\overline{T''} = T$. This implies that $\overline{T'} = T$. Hence T is independent of a choice of isomorphism and essentially torsion.

COROLLARY 4.12. *There is a natural exact sequence of complete* $k[X]$*-modules*

$$0 \longrightarrow T \longrightarrow M \longrightarrow Q \longrightarrow 0$$

where T *is essentially torsion and* Q *is torsion free. There is a (noncanonical) splitting on the exact sequence.*

4.3. Duality between torsion modules and complete modules. In this subsection we study some kind of Pontrjagin duality between torsion modules and complete modules over $k[X]$.

Let D be the graded module $k[X^{\pm 1}]/k[X]$. We denote by $D\{\bar{s}\}$ the shifted graded module such that the degree $s-t$ component of $D\{\bar{s}\}$ is equal to the degree $-t$ component of D. We also denote by $k[X]/(X^n)\{s\}$ (resp. $k[X]/(X^n)\{\bar{s}\}$) the $k[X]$-module isomorphic to $k[X]/(X^n)$ generated by a degree s (resp. $s-n$) element. Hence we have

$$k[X]/(X^n)\{\bar{s}\} = k[X]/(X^n)\{s-n\}, \quad k[X]/(X^n)\{s\} = k[X]/(X^n)\{\overline{s+n}\}.$$

Let $k[X]\{s\}$ be the free $k[X]$-module of rank one generated by a degree s element.

LEMMA 4.13. *There are isomorphisms*

$$\begin{aligned}
\mathrm{Hom}_{k[X]}(k[X]/(X^n)\{s\}, D) &\cong k[X]/(X^n)\{\overline{-s}\}, \\
\mathrm{Hom}_{k[X]}(k[X]/(X^n)\{\bar{s}\}, D) &\cong k[X]/(X^n)\{-s\}, \\
\mathrm{Hom}_{k[X]}(k[X]\{s\}, D) &\cong D\{\overline{-s}\}, \\
\mathrm{Hom}_{k[X]}(D\{\bar{s}\}, D) &\cong K\{-s\}.
\end{aligned}$$

Proof. These are easy. □

Let M be a torsion $k[X]$-module such that M_1 is of finite type. Then we have an isomorphism

$$M \cong \bigoplus_{i \in I} D\{\bar{s}_i\} \oplus \bigoplus_{j \in J} k[X]/(X^{n_j})\{\bar{s}_j\}.$$

The condition that M_1 is of finite type is equivalent to two conditions that the number of i such that $s_i = n$ is finite for all n and the number of j such that $s_j = n$ is finite for all n. We consider $\mathrm{Hom}(M, D)$. Then there is an isomorphism

$$\mathrm{Hom}_{k[X]}(M, D) \cong \prod_{i \in I} k[X]\{-s_i\} \times \prod_{j \in J} k[X]/(X^{n_j})\{-s_j\}.$$

Hence the $k[X]$-module $\mathrm{Hom}(M, D)$ is complete and the cokernel of multiplication by X on $\mathrm{Hom}(M, D)$ is of finite type. Let $\mathcal{T}$ be the category of torsion modules over $k[X]$ such that the kernel of multiplication by X is of finite type. Let $\mathcal{C}$ be the category of complete modules over $k[X]$ such that the cokernel of multiplication by X is of finite type. Then we get a functor

$$\mathrm{Hom}_{k[X]}(-, D)\colon\ \mathcal{T} \longrightarrow \mathcal{C}.$$

Let M be an object of the category of $\mathcal{C}$. We regard M as a topological module by the X-adic topology. Note that a $k[X]$-module homomorphism is continuous with respect to the X-adic topology. Let $\mathrm{Hom}^c(M, D)$ be the set of all continuous homomorphism from M to D. We have an isomorphism

$$M \cong \prod_{i\in I} k[X]\{s_i\} \times \prod_{j\in J} k[X]/(X^{n_j})\{s_j\}.$$

Since the cokernel ${}_1M$ of multiplication by X is of finite type, the number of $i \in I$ (resp. $j \in J$) such that $s_i = n$ (resp. $s_j = n$) is finite for all n. Then we have

$$\mathrm{Hom}^c_{k[X]}(M, D) \cong \bigoplus_{i\in I} D\{-\bar{s}_i\} \oplus \bigoplus_{j\in J} k[X]/(X^{n_j})\{-\bar{s}_j\}.$$

Note that the kernel of multiplication by X on $\mathrm{Hom}^c(M, D)$ is of finite type. Hence we get a functor

$$\mathrm{Hom}^c_{k[X]}(-, D)\colon\ \mathcal{C} \longrightarrow \mathcal{T}.$$

It is easy to see that the functor $\mathrm{Hom}(-, D)$ is an inverse of $\mathrm{Hom}^c(-, D)$ and vice versa.

PROPOSITION 4.14. *The functor* $\mathrm{Hom}(-, D)\colon \mathcal{T} \to \mathcal{C}$ *induces an equivalence of categories. A quasi-inverse is given by* $\mathrm{Hom}^c(-, D)\colon \mathcal{C} \to \mathcal{T}$.

5. $H^*_c(G_n; V[u^{\pm 1}])$ **and cohomology of the comodule** M^1_{n-1}**.** In this section we study the relation between the cohomology $H^*_c(G_n; V[u^{\pm 1}])$ and the Ext of the comodule M^1_{n-1} over the Hopf algebroid $BP_*(BP)$. First, we recall the cohomology groups of comodules over the Hopf algebroid $BP_*(BP)$ and the Morava's change of rings theorem. Then we introduce a filtration on $H^*_c(G_n; \mathbf{F}[w^{\pm 1}])$ and construct a homomorphism of Bockstein type.

5.1. Cohomology of comodules over $BP_*(BP)$**.** Let BP be the Brown-Peterson spectrum at the prime p. The coefficient ring of BP is given by

$$BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \ldots,], \quad |v_i| = 2(p^i - 1).$$

The Hopf algebroid (BP_*, BP_*BP) is an affine groupoid scheme representing the functor which associates to a p-local algebra A the category of p-typical formal group laws over A with strict isomorphisms as morphisms (cf. appendix 2 of [17]). Since the category of comodules over the Hopf algebroid $BP_*(BP)$ is an abelian category with enough injective, we can define the derived functor Ext. For a $BP_*(BP)$-comodule M, we abbreviate $\mathrm{Ext}^{**}_{BP_*BP}(BP_*, M)$ to $H^{**}(M)$. Let M^1_{n-1} be a BP_*-module given by

$$M^1_{n-1} = BP_*/(p, v_1, \ldots, v_{n-2}, v^{\infty}_{n-1})[v_n^{-1}].$$

Then there is a unique $BP_*(BP)$-comodule structure on M^1_{n-1} such that the natural map $BP_* \to M^1_{n-1}$ is a comodule map [13]. Let $M^1_{n-1}(i)$ be the kernel of multiplication by v^i_{n-1} on M^1_{n-1}:

$$0 \longrightarrow M^1_{n-1}(i) \longrightarrow M^1_{n-1} \xrightarrow{v^i_{n-1}} M^1_{n-1}.$$

Since $\eta_R(v_{n-1}) \equiv v_{n-1} \bmod (p, v_1, \ldots, v_{n-2})$, the multiplication by v^i_{n-1} on M^1_{n-1} is a comodule map and the kernel $M^1_{n-1}(i)$ is a subcomodule of M^1_{n-1} for all $i > 0$. We put $M^0_n = M^1_{n-1}(1)$.

5.2. Morava's change of rings theorem. We recall Morava's change of rings theorem [15, 2]. Let M be a comodule over the Hopf algebroid $BP_*(BP)$. We assume that $v_i^{-1}M = 0$ for $0 \le i < n$ where $v_0 = p$. We define the graded ring E_{n*} by

$$E_{n*} = W(\mathbf{F})[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}]$$

where $|u_i| = 0$ for $1 \le i < n$ and $|u| = -2$. There is a ring homomorphism $BP_* \to E_{n*}$ which sends v_i to $u_i u^{-(p^i-1)}$ for $1 \le i < n$, v_n to $u^{-(p^n-1)}$ and v_i to 0 for $i > n$. Morava showed that there is a natural continuous action of G_n on the discrete module $E_{n*} \otimes_{BP_*} M$ and proved the following change of rings theorem.

THEOREM 5.1 (Morava's change of rings theorem). *There is a natural isomorphism*

$$H^*(v_n^{-1}M) \cong H^*_c(G_n; E_{n*} \otimes_{BP_*} M).$$

COROLLARY 5.2. $H^*(M^1_{n-1}(i)) \cong H^*_c(G_n; V/(u^i_{n-1})[u^{\pm 1}])$ *for all* $i > 0$.

5.3. $H^*(M^1_{n-1})$ and $H^*_c(G_n; V[u^{\pm 1}])$. In this subsection we recall the finiteness result on $H^*(M^0_n)$ and study the relation between $H^*(M^1_{n-1})$ and $H^*_c(G_n; V[u^{\pm 1}])$.

We note that M^0_n is a comodule algebra over $BP_*(BP)$. Hence the cohomology $H^*(M^0_n)$ has a ring structure.

LEMMA 5.3. (cf. Theorem 6.2.10 (a) [17]) *The cohomology* $H^*(M^0_n)$ *is a finitely generated algebra over* $\mathbf{F}_p[v_n^{\pm 1}]$. *In particular,* $H^{s,t}(M^0_n)$ *is a finite dimensional vector space over* $\mathbf{F}_p$ *for all* s *and* t.

Since the multiplication by v_{n-1} on M^1_{n-1} is a comodule map, $H^s(M^1_{n-1})$ is a $\mathbf{Z}$-graded torsion module over $\mathbf{F}_p[v_{n-1}]$. There is an exact sequence of comodules

$$0 \longrightarrow M^0_n \longrightarrow M^1_{n-1} \xrightarrow{v_{n-1}} M^1_{n-1} \longrightarrow 0.$$

The short exact sequence induces a long exact sequence

$$\cdots \longrightarrow H^s(M^0_n) \longrightarrow H^s(M^1_{n-1}) \xrightarrow{v_{n-1}} H^s(M^1_{n-1}) \xrightarrow{\delta_s} H^{s+1}(M^0_n) \longrightarrow \cdots.$$

Then we see that the kernel of multiplication by v_{n-1} on $H^s(M^1_{n-1})$ is of finite type. Let D be the divisible $\mathbf{F}_p[v_{n-1}]$-module $\mathbf{F}_p[v^{\pm 1}_{n-1}]/\mathbf{F}_p[v_{n-1}]$. Let $T(l)$ be a finite torsion module $\mathbf{F}_p[v_{n-1}]/(v^l_{n-1})$. We denote by $D\{\overline{a}\}$ $(a \in \mathbf{Z})$ the shifted module of D such that the degree $a-t$ part of $D\{\overline{a}\}$ is the degree $-t$ part of D. Also, $T(l)\{b\}$ $(b \in \mathbf{Z})$ is the shifted module of $T(l)$ generated by a degree b element. By Proposition 4.8, we have an isomorphism

$$H^s(M^1_{n-1}) \cong \bigoplus_{\alpha \in A^s} D\{\overline{a}_\alpha\} \oplus \bigoplus_{\beta \in B^s} T(l_\beta)\{b_\beta\}.$$

We fix an isomorphism and let $D^s = D^s(M^1_{n-1})$ be the divisible part and $T^s = T^s(M^1_{n-1})$ the finite torsion part of $H^s(M^1_{n-1})$.

In order to relate $H^*(M^1_{n-1})$ to $H^*_c(G_n; V[u^{\pm 1}])$, we need the following lemma.

LEMMA 5.4. (cf. [6]) $H^*_c(G_n; V[u^{\pm 1}]) \cong \varprojlim_i H^*_c(G_n; V/(u^i_{n-1})[u^{\pm 1}])$.

From the long exact sequence

$$\cdots \longrightarrow H^s(M^1_{n-1}(i)) \longrightarrow H^s(M^1_{n-1}) \xrightarrow{v^i_{n-1}} H^s(M^1_{n-1}) \longrightarrow \cdots,$$

we get a short exact sequence

$$0 \longrightarrow {}_iT^{s-1} \longrightarrow H^s(M^1_{n-1}(i)) \longrightarrow D^s_i \oplus T^s_i \longrightarrow 0$$

where ${}_iT^{s-1}$ is the cokernel of multiplication by v^i_{n-1} on T^{s-1}, and D^s_i (resp. T^s_i) is the kernel of multiplication by v^i_{n-1} on D^s (resp. T^s). The natural projection $M^1_{n-1}(i) \to M^1_{n-1}(i-1)$ induces an inverse system of short exact sequences

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & {}_iT^{s-1} & \longrightarrow & H^s(M^1_{n-1}(i)) & \longrightarrow & D^s_i \oplus T^s_i & & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & {}_{i-1}T^{s-1} & \longrightarrow & H^s(M^1_{n-1}(i-1)) & \longrightarrow & D^s_{i-1} \oplus T^s_{i-1} & & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & .
\end{array}$$

Note that ${}_1T^s$ is of finite type. Hence ${}_iT^s$ is also of finite type for all $i > 0$. Then $\varprojlim_i^1 {}_iT^{s-1} = 0$. We denote by $Q\{b\}$ a free $\mathbf{F}_p[v_{n-1}]$-module of rank one generated by a degree b element. By passing to the inverse limit of the system, we get a short exact sequence

$$0 \longrightarrow \overline{T}^{s-1} \longrightarrow \varprojlim_i H^s(M^1_{n-1}(i)) \longrightarrow Q^s \longrightarrow 0$$

where

$$\begin{aligned} \overline{T}^{s-1} &\cong \textstyle\prod_{\beta\in B^{s-1}} T(l_\beta)\{b_\beta\}, \\ Q^s &\cong \textstyle\prod_{\alpha\in A^s} Q\{a_\alpha\}. \end{aligned}$$

Then we see that $\varprojlim H^s(M^1_{n-1}(i))$ is a complete $\mathbf{F}_p[v_{n-1}]$-module. The short exact sequence is naturally associated with it and there is a (non-canonical) splitting by Corollary 4.12.

By Morava's change of rings theorem (Theorem 5.1), we obtain the following proposition.

PROPOSITION 5.5. *The cohomology $H^s_c(G_n; V[u^{\pm 1}])$ is a complete module over $\mathbf{F}_p[v_{n-1}]$ for all s. We suppose that there is an isomorphism*

$$H^s(M^1_{n-1}) \cong \bigoplus_{\alpha\in A^s} D\{\overline{a}_\alpha\} \oplus \bigoplus_{\beta\in B^s} T(l_\beta)\{b_\beta\}.$$

Then we have an isomorphism

$$H^s_c(G_n; V[u^{\pm 1}]) \cong \prod_{\beta\in B^{s-1}} T(l_\beta)\{b_\beta\} \times \prod_{\alpha\in A^s} Q\{a_\alpha\}.$$

COROLLARY 5.6. $H^0_c(G_n; V[u^{\pm 1}]) = \mathbf{F}_p[v_{n-1}]$.

Proof. This follows from Proposition 5.5 and Theorem 5.1 of [14]. □

Remark 5.7. If $H^s_c(G_n; V[u^{\pm 1}])$ is of finite type, we can replace the direct product by the direct sum in the right hand side of the isomorphism. It is the case when $n = 2$ by Shimomura [21, 23, 22].

Let r be the projection $H^*_c(G_n; V[u^{\pm 1}]) \to H^*(M^0_n)$. In §6 we need the following lemma.

LEMMA 5.8. $r(\overline{T}^s) = \mathrm{Im}\,\delta_s$.

Proof. We have a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 \to & \overline{T}^s & \longrightarrow & H_c^{s+1}(G_n; V[u^{\pm 1}]) & \longrightarrow & Q^{s+1} & \to 0 \\ & \downarrow & & \downarrow r & & \downarrow & \\ 0 \to & \mathrm{Coker}\ v_{n-1} & \longrightarrow & H^{s+1}(M_n^0) & \longrightarrow & \mathrm{Ker}\ v_{n-1} & \to 0. \end{array}$$

Note that $\mathrm{Coker}\ v_{n-1} = \mathrm{Im}\ \delta_s$. Then the lemma follows from the fact that the left vertical arrow is surjective. □

5.4. Homomorphism of Bockstein type. In this subsection we introduce a filtration on the cohomology group $H_c^*(G_{n-1}; \mathbf{F}[w^{\pm 1}])$ and construct a homomorphism from the associated graded group to some quotient of the cohomology group $H_c^*(G_n; \mathbf{F}[u^{\pm 1}])$.

In §3.3 we constructed a ring homomorphism Θ: $H_c^*(G_{n-1}; \mathbf{F}[w^{\pm 1}]) \to H_c^*(G_n; K[u^{\pm 1}])$ (cf. (3.3)). There are ring homomorphisms

$$\begin{aligned} l\colon\ H_c^*(G_n; V[u^{\pm 1}]) &\longrightarrow H_c^*(G_n; K[u^{\pm 1}]) \\ r\colon\ H_c^*(G_n; V[u^{\pm 1}]) &\longrightarrow H_c^*(G_n; \mathbf{F}[u^{\pm 1}]) \end{aligned}$$

where l is induced by the inclusion $V \to K$ and r is induced by the reduction $V \to \mathbf{F}$. So we get a diagram

$$\begin{array}{ccccc} & & H_c^*(G_n; V[u^{\pm 1}]) & \xrightarrow{r} & H_c^*(G_n; \mathbf{F}[u^{\pm 1}]) \\ & & \downarrow l & & \\ H_c^*(G_{n-1}; \mathbf{F}[w^{\pm 1}]) & \xrightarrow{\Theta} & H_c^*(G_n; K[u^{\pm 1}]). & & \end{array}$$

LEMMA 5.9. *The vertical arrow l in the diagram is the localization inverting the invariant element $v_{n-1} \in H_c^0(G_n; V[u^{\pm 1}])$:*

$$H_c^*(G_n; K[u^{\pm 1}]) = H_c^*(G_n; V[u^{\pm 1}])[v_{n-1}^{-1}].$$

Proof. First we show that $v_{n-1} = u_{n-1}u^{-(p^{n-1}-1)}$ is a G_n-invariant element in $V[u^{\pm 1}]$. It is trivial that v_{n-1} is invariant with respect to the action of Γ. The action of S_n on $V[u^{\pm 1}]$ is given by

$$u_{n-1}^g = u_{n-1}t_0(g)^{-(p^{n-1}-1)}, \quad u^g = t_0(g)^{-1}u.$$

Hence v_{n-1} is invariant. Let $C^*(G_n; V[u^{\pm 1}])$ be the continuous cochain complex for G_n in $V[u^{\pm 1}]$. Then the natural homomorphism $C^*(G_n; V[u^{\pm 1}]) \to$

$C^*(G_n; K[u^{\pm 1}])$ induces an injective homomorphism

$$C^*(G_n; V[u^{\pm 1}])[v_{n-1}^{-1}] \longrightarrow C^*(G_n; K[u^{\pm 1}]).$$

For $f \in C^*(G_n; K[u^{\pm 1}])$, the compactness of G_n implies that f comes from $C^*(G_n; u_{n-1}^{-i} V[u^{\pm 1}])$ for some i. Since $u_{n-1}^{-i} V[u^{\pm 1}] = v_{n-1}^{-i} V[u^{\pm 1}]$, this means that the above natural homomorphism is surjective. Hence $C^*(G_n; V[u^{\pm 1}])[v_{n-1}^{-1}] = C^*(G_n; K[u^{\pm 1}])$. Since the (co)homology of a (co)chain complex commutes with the localization, we get the lemma. □

COROLLARY 5.10. *For $n \geq 2$, $H_c^0(G_n; K[u^{\pm 1}]) = \mathbf{F}_p[v_{n-1}^{\pm 1}]$. The ring homomorphism Θ is the identity on H^0.*

Proof. This follows from Corollary 5.6 and Lemma 5.9. □

We define a filtration on $H_c^*(G_{n-1}; \mathbf{F}[w^{\pm 1}])$ by

$$F^s = F^s H_c^*(G_{n-1}; \mathbf{F}[w^{\pm 1}]) = \{f \mid v_{n-1}^{-s} \cdot \Theta(f) \in \operatorname{Im} l\}.$$

Then we get

$$H_c^*(G_{n-1}; \mathbf{F}[w^{\pm 1}]) \supset \cdots \supset F^s \supset F^{s+1} \supset \cdots.$$

We note that $H_c^*(G_{n-1}; \mathbf{F}[w^{\pm 1}]) = \cup_s F^s$. Let $T \subset H_c^*(G_n; V[u^{\pm 1}])$ be the v_{n-1}-torsion subgroup. There is a natural exact sequence

$$0 \longrightarrow \overline{T} \longrightarrow H_c^*(G_n; V[u^{\pm 1}]) \longrightarrow Q \longrightarrow 0$$

where $\overline{T}$ is the closure of T and Q is torsion free.

LEMMA 5.11. $\cap_s F^s = \Theta^{-1}(l(\overline{T}))$.

Proof. Let $a \in H_c^*(G_{n-1}; \mathbf{F}[w^{\pm 1}])$ such that $\Theta(a) \in l(\overline{T})$. We take $b \in \overline{T}$ such that $\Theta(a) = l(b)$. For $s > 0$, there is $b_s \in \overline{T}$ such that $b \equiv v_{n-1}^s b_s$ mod T. Hence $v_{n-1}^{-s}\Theta(a) = l(b_s)$. This shows that $a \in \cap_s F^s$. Let $a \in \cap_s F^s$. There is $b_s \in H_c^*(G_n\colon V[u^{\pm 1}])$ such that $v_{n-1}^{-s}\Theta(a) = l(b_s)$ for all $s \in \mathbf{Z}$. Then $b_0 \equiv v_{n-1}^s b_s$ mod T. Let c be the image of b_0 in Q. Then we have $c \in \cap_s v_{n-1}^s Q = \{0\}$. This shows that $b_0 \in \overline{T}$ and $\Theta(a) \in l(\overline{T})$. □

LEMMA 5.12. *The multiplication by v_{n-1} induces an isomorphism $F^s/F^{s+1} \xrightarrow{\cong} F^{s+1}/F^{s+2}$. Hence we have*

$$\operatorname{Gr} H_c^*(G_{n-1}; \mathbf{F}[w^{\pm 1}]) = \oplus_s F^s/F^{s+1} \cong F^0/F^1 \otimes \mathbf{F}_p[v_{n-1}^{\pm 1}].$$

Proof. This is easy. □

LEMMA 5.13. $r(T) = r(\overline{T})$.

Proof. It is sufficient to show that $r(\overline{T}) \subset r(T)$. The homomorphism r is the projection to $H^*(M^1_{n-1}(1))$ under the isomorphism $H^*_c(G_n; V[u^{\pm 1}]) \cong \varprojlim H^*(M^1_{n-1}(i))$ (Lemma 5.4). Then $r(\overline{T})$ is contained in the closure of $r(T)$ in $H^*(M^1_{n-1}(1))$. Since $H^*(M^1_{n-1}(1))$ is of finite type, it is discrete. Hence $r(T)$ is closed and $r(\overline{T}) \subset r(T)$. □

For $f \in F^s$, there is $f' \in H^*_c(G_n; V[u^{\pm 1}])$ such that $l(f') = v_{n-1}^{-s}\Theta(f)$. If $f'' \in H^*_c(G_n; V[u^{\pm 1}])$ is another lift of $v_{n-1}^{-s}\Theta(f)$, then $f' - f'' \in T$. Hence we get a homomorphism from F^s to $H^*_c(G_n; V[u^{\pm 1}])/T$. It is clear that this homomorphism induces a homomorphism from F^s/F^{s+1} to $H^*_c(G_n; \mathbf{F}[u^{\pm 1}])/r(T)$. Therefore we get a homomorphism

$$\Xi\colon \text{ Gr } H^*_c(G_{n-1}; \mathbf{F}[w^{\pm 1}]) \longrightarrow H^*_c(G_n; \mathbf{F}[u^{\pm 1}])/r(\overline{T}). \tag{5.1}$$

LEMMA 5.14. *Let S be a subset of $H^*_c(G_n; V[u^{\pm 1}])$ such that $r(S)$ is linearly independent over $\mathbf{F}_p$ in $H^*_c(G_n; \mathbf{F}[u^{\pm 1}])/r(\overline{T})$. Then $l(S)$ is linearly independent over $\mathbf{F}_p[v_{n-1}^{\pm 1}]$ in $H^*_c(G_n; K[u^{\pm 1}])$.*

Proof. Let $s_1, \dots, s_i$ be elements in S such that $\sum a_j s_j = 0$ in $H^*_c(G_n; K[u^{\pm 1}])$ where $a_j \in \mathbf{F}_p[v_{n-1}^{\pm 1}]$. Then we can assume that $a_j \in \mathbf{F}_p[v_{n-1}]$ for all j and $a_1 \in \mathbf{F}_p^{\times}$. This implies that $\sum a_j s_j \in T$ in $H^*_c(G_n; V[u^{\pm 1}])$. Hence we get $\sum \overline{a}_j r(s_j) = 0$ in $H^*_c(G_n; \mathbf{F}[u^{\pm 1}])/r(\overline{T})$ where $\overline{a}_j$ is the image of a_j under the reduction $\mathbf{F}_p[v_{n-1}] \to \mathbf{F}_p$. Since $\overline{a}_1 \neq 0$, this contradicts the assumption that $r(S)$ is linearly independent. □

COROLLARY 5.15. *Let S be a subset of $H^*_c(G_{n-1}; \mathbf{F}[w^{\pm 1}])$ such that $S \cap (\cap_s F^s) = \emptyset$. Let $\overline{S}$ be the subset of* Gr $H^*_c(G_{n-1}; \mathbf{F}[w^{\pm 1}])$ *determined by S. If $\Xi(\overline{S})$ is linearly independent over $\mathbf{F}_p$, then $\Theta(S)$ is linearly independent over $\mathbf{F}_p[v_{n-1}^{\pm 1}]$.*

Proof. We can assume that $S \subset F^0 - F^1$. Then there is a subset S' of $H^*_c(G_n; V[u^{\pm 1}])$ such that l gives a bijection from S' to $\Theta(S)$. Since the image $r(S')$ is linearly independent over $\mathbf{F}_p$ in $H^*_c(G_n; \mathbf{F}[u^{\pm 1}])/r(\overline{T})$, $\Theta(S)$ is linearly independent over $\mathbf{F}_p[v_{n-1}^{\pm 1}]$. □

6. Examples. In this section we study the behavior of the homomorphism Θ constructed in (3.3) on the 1-dimension cohomology groups. We recall that $H^0_c(G_n; \mathbf{F}[u^{\pm 1}]) \cong H^0(M^0_n) \cong \mathbf{F}_p[v_n^{\pm 1}]$. For a ring A of characteristic $p > 0$, let P be the Frobenius operator on A given by $P(x) = x^p$ for $x \in A$. If a group G acts on A as ring automorphisms, then $P\colon A \to A$ is a G-module map. Hence P induces a ring homomorphism on the cohomology ring $H^*(G; A)$. We also denote by P this ring homomorphism. For a homomorphism $f\colon A_1 \to A_2$ of characteristic p rings, we have $P \circ f_* = f_* \circ P$ on cohomology. For a group homomorphism $g\colon G_1 \to G_2$, we also have $P \circ g^* = g^* \circ P$ on cohomology.

Let h_0 be the continuous map from S_n to $\mathbf{F}[u^{\pm 1}]$ given by

$$h_0(g) = \overline{g}_0^{-1}\overline{g}_1 u^{-(p-1)}$$

where $g = g_0 + g_1 S + g_2 S^2 + \cdots \in S_n$ and $\overline{g}_i$ is the reduction of $g_i \in W(\mathbf{F}_{p^n})$ to the residue field $\mathbf{F}_{p^n}$. It is easy to see that $h_0 \in H_c^{1,2(p-1)}(S_n; \mathbf{F}[u^{\pm 1}])$. For $\sigma \in \Gamma$, we have $h_j^{\sigma}(g) = h_j(g)$. Hence we get

$$h_0 \in H_c^{1,2(p-1)}(S_n; \mathbf{F}[u^{\pm 1}])^{\Gamma} = H_c^{1,2(p-1)}(G_n; \mathbf{F}[u^{\pm 1}]).$$

We define

$$h_j = P^j(h_0) \in H_c^{1,2p^j(p-1)}(G_n; \mathbf{F}[u^{\pm 1}]), \qquad j \geq 0.$$

Remark 6.1. $h_{j+n} = v_n^{(p-1)p^j} h_j$ for all $j \geq 0$.

We recall that S_n is isomorphic to the unit group of the maximal order of the central division algebra over the p-adic number field $\mathbf{Q}_p$ with invariant $1/n$. Then we have the reduced norm map $S_n \to \mathbf{Z}_p^{\times}$ (cf. [26]). Note that there is an isomorphism

$$\mathbf{Z}_p^{\times} \cong \begin{cases} (\mathbf{Z}/p)^{\times} \times \mathbf{Z}_p & \text{if } p \text{ is odd,} \\ \mathbf{Z}/2 \times \mathbf{Z}_2 & \text{if } p = 2. \end{cases} \tag{6.1}$$

For p odd, we define a continuous map ζ_n from S_n to $\mathbf{F}$ by

$$\zeta_n\colon\ S_n \longrightarrow \mathbf{Z}_p^{\times} \longrightarrow \mathbf{Z}_p \longrightarrow \mathbf{F}_p \subset \mathbf{F}$$

where the first map is the reduced norm, the second is the projection under the isomorphism (6.1) and the third is the reduction. By properties of the reduced norm [26], we see that

$$\zeta_n \in H_c^{1,0}(S_n; \mathbf{F}[u^{\pm 1}])^{\Gamma} = H_c^{1,0}(G_n; \mathbf{F}[u^{\pm 1}]), \quad p\text{: odd}.$$

Note that $P(\zeta_n) = \zeta_n$ and $v_1 \zeta_1 = h_0$.

For $p = 2$, we define a continuous map ζ_n by

$$\zeta_n\colon\ S_n \longrightarrow \mathbf{Z}_2^{\times} \longrightarrow \mathbf{Z}/2 \subset \mathbf{F}$$

where the first map is the reduced norm and the second is the projection under the isomorphism (6.1). We also define a map ρ_n by

$$\rho_n\colon\ S_n \longrightarrow \mathbf{Z}_2^{\times} \longrightarrow \mathbf{Z}_2 \longrightarrow \mathbf{Z}/2 \subset \mathbf{F}$$

where the first map is the reduced norm, the second is the projection and the third is the reduction. By the same reason for p odd, we see that

$$\zeta_n, \rho_n \in H_c^{1,0}(S_n; \mathbf{F}[u^{\pm 1}])^{\Gamma} = H_c^{1,0}(G_n; \mathbf{F}[u^{\pm 1}]), \qquad p = 2.$$

Note that $P(\zeta_n) = \zeta_n, P(\rho_n) = \rho_n$ and $v_1 \zeta_1 = h_0$.

PROPOSITION 6.2. (cf. Proposition 3.18 (a),(c) [14]) *Over the graded field* $\mathbf{F}_p[v_n^{\pm 1}]$, *we can take as a basis of* $H_c^1(G_n; \mathbf{F}[u^{\pm 1}])$ *the following elements*

$$\begin{array}{ll} \zeta_1, & \text{if } n = 1, p \neq 2, \\ \zeta_1, \rho_1, & \text{if } n = 1, p = 2, \\ h_j \ (0 \leq j < n), \zeta_n, & \text{if } n > 1, p \neq 2, \\ h_j \ (0 \leq j < n), \zeta_n, \rho_n, & \text{if } n > 1, p = 2. \end{array}$$

6.1. Image of h_j under Θ. In this subsection we consider the image of h_j under the homomorphism Θ: $H_c^1(G_{n-1}; \mathbf{F}[w^{\pm 1}]) \to H_c^1(G_n; K[u^{\pm 1}])$. By Corollary 5.10, we have $H_c^0(G_n; K[u^{\pm 1}]) = \mathbf{F}_p[v_{n-1}^{\pm 1}]$. The ring homomorphism Θ is an $\mathbf{F}_p[v_{n-1}^{\pm 1}]$-algebra homomorphism and the identity on H^0. We note that Θ commutes with the Frobenius operator P. This follows from the fact that Θ is the direct limit of the system of two inflation maps $H_c^*(G_{n-1}(i); \mathbf{F}[w^{\pm 1}]) \to H_c^*(\mathcal{G}(i); L_i[u^{\pm 1}]) \xleftarrow{\cong} H_c^*(G_n; K[u^{\pm 1}])$.

We define a continuous map s_0 from S_n to $K[u^{\pm 1}]$ by

$$s_0(g) = t_0(g)^{-1} t_1(g) u^{-(p-1)}.$$

We recall that $t_i(g)$ $(i \geq 0)$ is the coefficient of the isomorphism $t(g)$: $F_n \to F_n^g$ over V given by the following form

$$t(g)(X) = \sum_{i \geq 0}{}^{F_n^g} t_i(g) X^{p^i}.$$

We note that t_i: $S_n \to V$ is Γ-equivariant for all $i \geq 0$. This follows from the fact that $(t(g)(X))^{\sigma}$: $F_n^{\sigma} \to (\alpha(g)^* F_n)^{\sigma}$ is identified with $t(g^{\sigma})(X)$: $F_n \to \alpha(g^{\sigma})^* F_n$ for $\sigma \in \Gamma$.

LEMMA 6.3. *The continuous map s_0 is a 1-cocycle for S_n in $V[u^{\pm 1}]$.*

Proof. We have $t(gg')(X) = t(g)^{g'}(t(g')(X))$. Comparing the coefficients of X and X^p, we get

$$\begin{aligned} t_0(gg') &= t_0(g)^{g'} t_0(g'), \\ t_1(gg') &= t_0(g)^{g'} t_1(g') + t_1(g)^{g'} t_0(g')^p. \end{aligned}$$

Note that $u^g = t_0(g)^{-1}u$. Then

$$\begin{aligned} s_0(gg') &= (t_0(g)^{g'}t_0(g'))^{-1}(t_0(g)^{g'}t_1(g') + t_1(g)^{g'}t_0(g')^p)u^{-(p-1)} \\ &= (t_0(g')^{-1}t_1(g') + (t_0(g)^{-1}t_1(g))^{g'}t_0(g')^{p-1})u^{-(p-1)} \\ &= s_0(g') + s_0(g)^{g'}. \end{aligned}$$

This shows that s_0 is a 1-cocycle. □

Since t_i is Γ-equivariant, $s_0^\sigma(g) = s_0(g)$ for $\sigma \in \Gamma$. Hence we get

$$s_0 \in H_c^{1,2(p-1)}(S_n; V[u^{\pm 1}])^\Gamma = H_c^{1,2(p-1)}(G_n; V[u^{\pm 1}]).$$

We define

$$s_j = P^j(s_0) \in H_c^{1,2p^j(p-1)}(G_n; V[u^{\pm 1}]), \qquad j \geq 0.$$

Proposition 6.4. *$\Theta(h_j) = s_j$ for all $j \geq 0$.*

Proof. For $g' = g'_0 + g'_1T + g'_2T^2 + \cdots \in S_{n-1}$, from the relation $g'(\Phi(X)) = \Phi^{g'}(X)$, we have

$$\begin{aligned} \Phi_0^{g'} &= \overline{g}'_0\Phi_0 \\ \Phi_1^{g'} &= \overline{g}'_0\Phi_1 + \overline{g}'_1\Phi_0^p. \end{aligned}$$

Then we get

$$\begin{aligned} h_0(g') &= \overline{g}'^{\,-1}_0\overline{g}_1 w^{-(p-1)} \\ &= (\Phi_0^{-1}\Phi_1)^{g'}u^{-(p-1)} - (\Phi_0^{-1}\Phi_1)u^{-(p-1)}. \end{aligned}$$

Note that $u^{g'} = u$ for all $g' \in S_{n-1}$. We put $Y = \Phi_0^{-1}\Phi_1 u^{-(p-1)} \in L_1$. Then $h_0(g') = Y^{g'} - Y$ for all $g' \in S_{n-1}$. The cocycle $\Theta(h_0)$ is given by $\Theta(h_0)(g) = Y - Y^g$ ($g \in S_n$). For $g \in S_n$, from the relation $\Phi(X) = \Phi^g(t(g))$, we have

$$\begin{aligned} \Phi_0 &= \Phi_0^g t_0 \\ \Phi_1 &= \Phi_0^g t_1 + \Phi_1^g t_0^p. \end{aligned}$$

Then we obtain

$$\Theta(h_0)(g) = t_0^{-1}t_1u^{-(p-1)} = s_0(g).$$

Since the Frobenius operator P commutes with Θ, we obtain $\Theta(h_j) = s_j$ for all $j \geq 0$. □

In order to show that s_j $(0 \leq j < n-1)$ are linearly independent in $H_c^*(G_n; K[u^{\pm 1}])$, we consider the homomorphism Ξ defined in (5.1). Recall that r is the homomorphism from $H_c^*(G_n; V[u^{\pm 1}])$ to $H_c^*(G_n; \mathbf{F}[u^{\pm 1}])$ induced by the reduction map $V \to \mathbf{F}$.

LEMMA 6.5. *$r(s_j) = h_j$ for all $j \geq 0$.*

Proof. It is easy to show that $r(s_0) = h_0$. Since P commutes with r, we get the lemma. □

Remark 6.6. For $h_{n-1} \in H_c^1(S_{n-1}; \mathbf{F}[w^{\pm 1}])$, we have $h_{n-1} = v_{n-1}^{p-1} h_0$. Then $s_{n-1} = v_{n-1}^{p-1} s_0$ in $H_c^1(G_n; K[u^{\pm 1}])$. This means that $s_{n-1} - v_{n-1}^{p-1} s_0$ is a v_{n-1}-torsion element of $H_c^1(G_n; V[u^{\pm 1}])$. Hence $h_{n-1} = r(s_{n-1}) \in r(\overline{T}) = \mathrm{Im}\ \delta_0$ (cf. (5.9), (5.16) and (5.18) of [14]).

LEMMA 6.7. *$r(h_j)$ $(0 \leq j < n-1)$ are linearly independent over $\mathbf{F}_p$ in $H_c^1(G_n; \mathbf{F}[u^{\pm 1}])/r(\overline{T})$.*

Proof. By Lemma 5.8, we have $H_c^1(G_n; \mathbf{F}[u^{\pm 1}])/r(\overline{T}) = H_c^1(G_n; \mathbf{F}[u^{\pm 1}])/\mathrm{Im}\delta_0$. The lemma follows from (5.9), (5.16) and (5.18) of [14]. □

By Lemma 5.14, we obtain the following proposition.

PROPOSITION 6.8. *s_j $(0 \leq j < n-1)$ are linearly independent over $\mathbf{F}_p[v_{n-1}^{\pm 1}]$ in $H_c^*(G_n; K[u^{\pm 1}])$.*

6.2. The case $n = 2$ and p odd. The 1st Morava stabilizer group S_1 is isomorphic to the unit group of the p-adic integer ring: $S_1 \cong \mathbf{Z}_p{}^{\times}$. If p is an odd prime, then S_1 is isomorphic to $(\mathbf{Z}/p)^{\times} \times \mathbf{Z}_p$. The subgroup isomorphic to $\mathbf{Z}_p$ acts on $\mathbf{F}[w^{\pm 1}]$ trivially. Hence we see that

$$\begin{aligned} H_c^*(S_1; \mathbf{F}[w^{\pm 1}]) &\cong H_c^*(\mathbf{Z}_p; \mathbf{F}) \otimes H_c^*((\mathbf{Z}/p)^{\times}; \mathbf{F}[w^{\pm 1}]) \\ &\cong H_c^*(\mathbf{Z}_p; \mathbf{F}) \otimes \mathbf{F}[v_1^{\pm 1}]. \end{aligned}$$

LEMMA 6.9. *For p odd, $H_c^*(G_1; \mathbf{F}[w^{\pm 1}]) = \Lambda(\zeta_1) \otimes \mathbf{F}_p[v_1^{\pm 1}]$.*

From the results of Shimomura [21, 23] and Proposition 5.5, we have the following lemma.

LEMMA 6.10. *For $p > 2$, $H_c^*(G_2; K[u^{\pm 1}]) = \Lambda(\zeta_2, s_0) \otimes \mathbf{F}_p[v_1^{\pm 1}]$.*

Since $\zeta_1 = v_1^{-1} h_0$, we have $\Theta(\zeta_1) = v_1^{-1} s_0$. Hence we obtain the following proposition.

PROPOSITION 6.11. *The $\mathbf{F}_p[v_1^{\pm 1}]$-algebra homomorphism Θ: $H_c^*(G_1; \mathbf{F}[w^{\pm 1}]) \to H_c^*(G_2; K[u^{\pm 1}])$ is given by $\Theta(\zeta_1) = v_1^{-1} s_0$. The ring homomorphism Θ induces an isomorphism*

$$\Theta \otimes \Lambda(\zeta_2) \colon H_c^*(G_1; \mathbf{F}[w^{\pm 1}]) \otimes \Lambda(\zeta_2) \xrightarrow{\cong} H_c^*(G_2; K[u^{\pm 1}]).$$

6.3. The case $n = 2$ and $p = 2$. If $p = 2$, then S_1 is isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}_2$. We note that the action of S_1 on $\mathbf{F}[w^{\pm 1}]$ is trivial in this case. Hence we obtain that $H_c^*(S_1; \mathbf{F}[w^{\pm 1}])$ is isomorphic to $H_c^*(S_1; \mathbf{F}) \otimes \mathbf{F}[w^{\pm 1}]$.

LEMMA 6.12. *For $p = 2$, $H_c^*(G_1; \mathbf{F}[w^{\pm 1}]) = \mathbf{F}_2[\zeta_1] \otimes \Lambda(\rho_1) \otimes \mathbf{F}_2[v_1^{\pm 1}]$.*

Since $\zeta_1 = v_1^{-1} h_0$, we have $\Theta(\zeta_1) = v_1^{-1} s_0$. For ρ_1, we have the following lemma.

LEMMA 6.13. *We can take $t_0^{-1} t_1 u_1^{-4} + t_0^{-1} t_2 u_1^{-3} + t_0^{-1} t_1 u_1^{-1}$ as a cocycle representing $\Theta(\rho_1)$.*

Proof. For $g' \in S_1$, we have $\rho_1(g') = \overline{g}_1' + \overline{g}_2'$. From the relation $g'(\Phi(X)) = \Phi^{g'}(X)$, we have

$$\begin{aligned} \Phi_0^{g'} &= \Phi_0, \\ \Phi_1^{g'} &= \Phi_1 + \overline{g}_1' u_1^2, \\ \Phi_2^{g'} &= \Phi_2 + \overline{g}_1'(\Phi_1^2 + u_1^2 \Phi_1) + \overline{g}_2' u_1^4. \end{aligned}$$

We define $Y \in L_2$ to be $u_1^{-2}\Phi_1 + u_1^{-4}\Phi_2 + u_1^{-5}\Phi_1$. Then we obtain that $\rho_1(g') = Y^{g'} - Y$ for all $g' \in S_1$.

For $g \in S_2$, the cocycle representing $\Theta(\rho_1)$ is given by $\Theta(\rho_1)(g) = Y - Y^g$. From the relation $\Phi^g(t(g)(X)) = \Phi(X)$, we have

$$\begin{aligned} \Phi_0^g &= t_0^{-1}\Phi_0, \\ \Phi_1^g &= t_0^{-2}(\Phi_1 - t_0^{-1} t_1 u_1), \\ \Phi_2^g &= t_0^{-4}(\Phi_2 - (\Phi_1 - t_0^{-1} t_1 u_1)(t_0^{-2} t_1^2 + t_0^{-1} t_1 u_1) - t_0^{-1} t_2 u_1). \end{aligned}$$

Note that we have a relation $t_1 u_1^2 + t_0 = u_1^g t_1^2 + t_0^4$ from $t(g)([p]^{F_2}(X)) = [p]^{F_2^g}(t(g)(X))$. Then we get $\Theta(\rho_1) = t_0^{-1} t_2 u_1^{-3} + t_0^{-1} t_1 u_1^{-1} + t_0^{-1} t_1 u_1^{-4}$. □

We denote by μ the cocycle representing $\Theta(\rho_1)$ given by Lemma 6.13. It is clear that $v_1^4 \mu$ is a cocycle in $V[u^{\pm 1}]$. Then $v_1^4 \mu \equiv v_2 h_0 \bmod (u_1)$. By (5.16) of [14], $v_2 h_0 = \delta_0(x_{2,1}/v_1^2)$. Hence there is a v_1-torsion element $x \in H_c^1(G_2; V[u^{\pm 1}])$ such that $v_1^4 \Theta(\rho_1) - x$ is divisible by v_1. Then $v_1^3 \Theta(\rho_1) \in \operatorname{Im} l$ where l is the localization map $H_c^1(G_2; V[u^{\pm 1}]) \to H_c^1(G_2; K[u^{\pm 1}])$.

LEMMA 6.14. *$\rho_1 \in F^{-3} - F^{-2}$ and $\Xi(\rho_1) = v_2 \zeta_2 \neq 0$ in $H_c^1(G_2; \mathbf{F}[u^{\pm 1}])/r(\overline{T})$.*

Proof. Let τ be the continuous map $(t_0^6 - 1) u_1^{-2} u^{-4}$ from S_2 to $K[u^{\pm 1}]$. From the relation $t(g)([p]^{F_2}(X)) = [p]^{F_2^g}(t(g)(X))$, we have

$$\begin{aligned} t_0(g) &\equiv \overline{g}_0 + \overline{g}_0^{-1} \overline{g}_1^2 u_1 \mod (u_1^2), \\ t_1(g) &\equiv \overline{g}_1 + \overline{g}_0^{-1} \overline{g}_2^2 u_1 \mod (u_1^2). \end{aligned}$$

Hence we see that τ is a map to $V[u^{\pm1}]$ and $\tau \equiv v_2 h_0 \bmod (u_1)$. By the relation $t_0(gg') = t_0(g)^{g'} t_0(g')$, we see that τ is a 1-cocycle. We consider the continuous map $\nu = (v_1^4 \mu - \tau)/v_1$ from S_2 to $V[u^{\pm1}]$. Then $\nu \equiv v_2(\overline{g}_0^{-1}\overline{g}_2 + \overline{g}_0^{-2}\overline{g}_2^2 + \overline{g}_1^3) \mod (u_1)$. The right hand side represents $v_2\zeta_2 \in H_c^1(G_2; \mathbf{F}[u^{\pm1}])$. Then the lemma follows from (5.16) of [14]. □

Remark 6.15. The cocycle $(t_0^6 - 1)u_1^{-2}u^{-4}v_1$ represents $\partial(v_2^2) \in H_c^1(G_2; V[u^{\pm1}])$ where ∂ is the connecting homomorphism $H_c^0(G_2; \mathbf{F}[u^{\pm1}]) \to H_c^1(G_2; V[u^{\pm1}])$ induced by the short exact sequence $0 \to V[u^{\pm1}] \xrightarrow{v_1} V[u^{\pm1}] \longrightarrow \mathbf{F}[u^{\pm1}] \to 0$.

COROLLARY 6.16. *$\Theta(\zeta_1)$ and $\Theta(\rho_1)$ are linearly independent over $\mathbf{F}_2[v_1^{\pm1}]$ in $H_c^1(G_2; K[u^{\pm1}])$.*

Proof. This follows from Corollary 5.15 and (5.16) of [14]. □

Remark 6.17. From the results of Shimomura [22], we see that

$$H_c^1(G_2; K[u^{\pm1}]) = \mathbf{F}_2[v_1^{\pm1}]\{\zeta_2, \rho_2, \Theta(\zeta_1), \Theta(\rho_1)\}.$$

7. Relation to the chromatic splitting conjecture. In this section we study a relation between the ring homomorphism Θ and the chromatic splitting conjecture (cf. [8]). The conjecture contains the statement that the natural map $L_{n-1}S_p^0 \to L_{n-1}L_{K(n)}S^0$ is a split monomorphism. For a finite spectrum Z of type $n-1$, there are spectral sequences $E_r(1)$ and $E_r(2)$ which converge to $\pi_*(L_{n-1}Z)$ and $\pi_*(L_{n-1}L_{K(n)}Z)$, respectively, and there is a morphism f_r of the spectral sequences which is a lift of the natural map $L_{n-1}Z \to L_{n-1}L_{K(n)}Z$. We show that there are spectral sequences $E_r(3)$ and $E_r(4)$ which converge to the E_2-terms of the spectral sequences $E_r(1)$ and $E_r(2)$, respectively, and the morphism f_2 lifts to a morphism of the spectral sequences $E_r(3) \to E_r(4)$ which is isomorphic to a sum of finite many copies of Θ on the E_1-terms. In particular, if a Toda-Smith spectrum $V(n-2)$ exists, then $E_r(3)$ and $E_r(4)$ collapse, and the morphism f_2 coincides with Θ.

Let L_n and $L_{K(n)}$ be the Bousfield localization functors with respect to $K(0) \vee K(1) \vee \cdots \vee K(n)$ and $K(n)$, respectively, where $K(i)$ is the ith Morava K-theory. Then there is a tower

$$\cdots \to L_n X \to L_{n-1}X \to \cdots \to L_1 X \to L_0 X,$$

which is called the chromatic tower of X. The layers of the tower, that is, the fibres of $L_n X \to L_{n-1}X$ are determined by $L_{K(n)}X$'s and vice versa. There is a natural map of towers from the constant tower $\{X\}$ to the chromatic tower and the chromatic convergence theorem says that the induced map

$$X \longrightarrow \operatorname*{holim}_n L_n X$$

is a homotopy equivalence for all p-local finite spectra X. So we may consider that a finite spectrum X is recovered from the chromatic pieces L_nX through the tower and X is built up from the monochromatic pieces $L_{K(n)}X$'s. The chromatic splitting conjecture says that for understanding a finite spectrum X, it is not necessary to reconstruct X from the chromatic tower and it is sufficient to understand infinite many $L_{K(n)}X$'s. In particular, the chromatic splitting conjecture contains the following assertion.

CONJECTURE 7.1. *The natural map $L_{n-1}S^0_p \to L_{n-1}L_{K(n)}S^0$ is a split monomorphism, where S^0_p is the p-completion of the sphere spectrum S^0.*

We denote by $BP^{\wedge s}$ the s-fold smash product of BP: $BP^{\wedge s} = \overbrace{BP \wedge \cdots \wedge BP}^{s}$. The ring spectrum structure of BP gives a cosimplicial structure on $\{BP^{\wedge s}\}_{s\geq 0}$ where $BP^{\wedge 0} = S^0$ and we obtain the associated cochain complex:

$$* \to S^0 \to BP \xrightarrow{d} BP^{\wedge 2} \xrightarrow{d} BP^{\wedge 3} \xrightarrow{d} \cdots,$$

which is a BP-resolution of S^0 in the sense of [12]. By smashing with a spectrum X, we obtain a BP-resolution of X and a BP-Adams resolution of X, that is, a sequence of exact triangles:

$$\begin{array}{ccccccccc} X = X^0 & \xleftarrow{\;i\;} & & X^1 & \xleftarrow{\;i\;} & & X^2 & \xleftarrow{\;i\;} & X^3 \\ & {\scriptstyle j}\searrow & \nearrow{\scriptstyle k} & & {\scriptstyle j}\searrow & \nearrow{\scriptstyle k} & & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle k} & \quad \cdots, \\ & & BP \wedge X & & & \Sigma^{-1}BP^{\wedge 2} \wedge X & & \Sigma^{-2}BP^{\wedge 3} \wedge X & \end{array} \tag{7.1}$$

where k have degree -1 and $jk = d$. By applying π_* of the diagram, we obtain a spectral sequence E_r^{**} with E_2-term $\mathrm{Ext}^{**}_{BP_*BP}(BP_*, BP_*(X))$. If $X = L_{n-1}Z$ where Z is a finite spectrum of type $n-1$, then $BP_*(X) \cong BP_*(Z)[v_{n-1}^{-1}]$ by Theorem 1 of [18]. Hence we have

$$\begin{aligned} E_2^{**} &\cong \mathrm{Ext}^{**}_{BP_*BP}(BP_*, BP_*(Z)[v_{n-1}^{-1}]) \\ &\cong H_c^{**}(G_{n-1}; E_{n-1*}(Z)) \end{aligned}$$

by the change-of-rings theorem. It is known that the spectral sequence converges to $\pi_*(L_{n-1}Z)$. For a finite spectrum Z of type $n-1$, we denote by

$E_r^{**}(1)(Z)$ the above spectral sequence converging to $\pi_*(L_{n-1}Z)$ with E_2-term $H_c^{**}(G_{n-1}; E_{n-1*}(Z))$.

Let E_n be the Morava E-theory spectrum. Then E_n is complex oriented, the coefficient ring $E_{n*}(pt)$ is $E_{n*} = W(\mathbf{F})[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$, and the associated degree 0 formal group law is the universal deformation F_n. Set $E_n^{\wedge s}$ the $K(n)$-localization of s-fold smash product of E_n: $E_n^{\wedge s} = L_{K(n)}(\overbrace{E_n \wedge \cdots \wedge E_n}^{s})$. As in the case of BP, from the ring spectrum structure of E_n, we obtain a cosimplicial spectrum $\{E_n^{\wedge s}\}_{s \geq 0}$, where $E_n^{\wedge 0} = L_{K(n)}S^0$. The associated cochain complex gives a E_n-resolution of $L_{K(n)}S^0$ in the $K(n)$-local category. By smashing with a finite spectrum X, we obtain an E_n-resolution of $L_{K(n)}X$ and the associated E_n-Adams resolution implies a spectral sequence E_r^{**} with $E_2^{**} \cong H_c^{**}(G_n; E_{n*}(X))$, which strongly converges to $\pi_*(L_{K(n)}X)$. Note that the strong convergence follows from the fact that $i^{(s)}$: $L_{K(n)}(\overbrace{\overline{E}_n \wedge \cdots \wedge \overline{E}_n}^{s}) \to L_{K(n)}S^0$ is null for $s >> 0$, where $\overline{E}_n$ is the fibre of the unit $S^0 \to E_n$ (cf. proof of Corollary 15 of [25]). Applying the functor L_{n-1} to the E_n-resolution of $L_{K(n)}X$, we obtain a cochain complex:

$$* \to L_{n-1}L_{K(n)}X \to L_{n-1}E_n \wedge X \to L_{n-1}E_n^{\wedge 2} \wedge X \to L_{n-1}E_n^{\wedge 3} \wedge X \to \cdots$$

and a sequence of exact triangles:

$$\begin{array}{ccccccc} L_{n-1}L_{K(n)}X = X^0 & \xleftarrow{i} & X^1 & \xleftarrow{i} & X^2 & \xleftarrow{i} & X^3 \\ \searrow^{j} & & \nearrow^{k} \; \searrow^{j} & & \nearrow^{k} \; \searrow^{j} & & \nearrow^{k} \quad \cdots. \\ & L_{n-1}E_n \wedge X & & \Sigma^{-1}L_{n-1}E_n^{\wedge 2} \wedge X & & \Sigma^{-2}L_{n-1}E_n^{\wedge 3} \wedge X & \end{array} \tag{7.2}$$

Hence we obtain a spectral sequence E_r^{**}. Since $i^{(s)} = 0$ for $s >> 0$ and E_r^{**} is obtained by applying L_{n-1} on the E_n-Adams resolution, E_r^{**} strongly converges to $\pi_*(L_{n-1}L_{K(n)}X)$.

If Z is a (p-local) finite spectrum of type $n-1$, then $E_{n*}(Z)$ is a finitely generated module over E_{n*} and $v_i^{-1}BP_*(Z) = 0$ for $0 \leq i < n-1$ by the Landweber filtration theorem. Then $BP_*(L_{n-1}Z) \cong BP_*(Z)[v_{n-1}^{-1}]$ by Theorem 1 of [18]. By Proposition 8.4.(f) of [9], $E_n^{\wedge s}$ is Landweber exact for $s \geq 1$. Hence $E_{n*}^{\wedge s}(L_{n-1}Z)$ is $E_{n*}^{\wedge s}(Z)[v_{n-1}^{-1}]$. Let F_i be the image of $I_n^i E_{n*}(Z) \hookrightarrow E_{n*}(Z) \to E_{n*}(Z)[v_{n-1}^{-1}]$. By taking $\{F_i\}_{i \geq 0}$ as a basis of neighbourhoods of 0, we give a topology on $E_{n*}(L_{n-1}Z)$. By Lemma 14 of [25], we have $E_{n*}^{\wedge s}(Z) \cong C_{G_n}(G_n^{s+1}, E_{n*}(Z))$ for a finite spectrum X. Then E_1-term of the spectral sequence associated with (7.2) is

$$E_1^{**} \cong C_{G_n}^{**}(G_n, E_{n*}(X))[v_{n-1}^{-1}],$$

if X is of type $n-1$. As in the proof of Lemma 5.9, the right hand side is isomorphic to $C^{**}_{G_n}(G_n; E_{n*}(X)[v_{n-1}^{-1}])$. Then the E_2-term of the spectral sequence is identified with

$$E_2^{**} \cong H_c^{**}(G_n; E_{n*}(L_{n-1}X)).$$

We denote by $E_r^{**}(2)(Z)$ the spectral sequence converging to $\pi_*(L_{n-1}L_{K(n)}Z)$ with E_2-term $H_c^{**}(G_n; E_{n*}(L_{n-1}Z))$, where Z is a finite spectrum of type $n-1$.

PROPOSITION 7.2. *There is a morphism of spectral sequences f_r: $E_r^{**}(1)(Z) \to E_r^{**}(2)(Z)$ which is a lift of the natural map $\pi_*(L_{n-1}Z) \to \pi_*(L_{n-1}L_{K(n)}Z)$, where Z is a finite spectrum of type $n-1$.*

Proof. The natural map $BP \to E_n$ induces a map of cochain complexes $\{BP^{\wedge s} \wedge L_{n-1}Z\}_{s\geq 0}$ to $\{L_{n-1}E_n^{\wedge s} \wedge Z\}_{s\geq 0}$ which is a lift of the map $L_{n-1}Z \to L_{n-1}L_{K(n)}Z$. Then we obtain a map of exact triangles from (7.1) for $X = L_{n-1}Z$ to (7.2) for $X = Z$ which gives a morphism of spectral sequences from $E_r^{**}(1)(Z)$ to $E_r^{**}(2)(Z)$. □

Let R be a commutative $\mathbf{Z}_{(p)}$-algebra, and F a p-typical formal group law over R. We suppose that a group Γ acts on (F, R) in generalized sense, and we denote by $f(\gamma)$ the isomorphism from F to F^γ for $\gamma \in \Gamma$. Let $R[u^{\pm 1}]$ be a graded ring such that $|u| = -2$ and $\widetilde{F}$ a degree -2 formal group law given by $uF(u^{-1}X, u^{-1}Y)$. Extend an action of Γ on $R[u^{\pm 1}]$ by $\gamma \cdot u = f(\gamma)_0 u$ where $f(\gamma)_0$ is the leading coefficient of $f(\gamma)(X)$. For $\gamma \in \Gamma$, let $\widetilde{f}(\gamma)(X) = uf(\gamma)(f(\gamma)_0^{-1}u^{-1}X)$. Then $\widetilde{f}(\gamma)$ gives a strict isomorphism from $\widetilde{F}$ to $\widetilde{F}^\gamma$ and we obtain an action of Γ on $(\widetilde{F}, R[u^{\pm 1}])$. Then we obtain a morphism of cosimplicial groups from $BP_*(BP)^{\otimes *}$ to $C_\Gamma(\Gamma^{*+1}; R[u^{\pm 1}])$ where $C_\Gamma(\Gamma^{s+1}; R[u^{\pm 1}])$ is the set of all Γ-equivariant maps from $\Gamma^s = \overbrace{\Gamma \times \cdots \times \Gamma}^{s}$ to $R[u^{\pm 1}]$. For $(\gamma_0, \ldots, \gamma_s) \in \Gamma^{s+1}$, the adjoint

$$\mathrm{ad}(\gamma_0, \ldots, \gamma_s)\colon\ BP_*(BP)^{\otimes s} \longrightarrow R[u^{\pm 1}]$$

is a ring homomorphism represented by p-typical formal group laws and strict isomorphisms

$$\widetilde{F}_0^{\gamma_0} \longrightarrow \widetilde{F}_1^{\gamma_1} \longrightarrow \widetilde{F}_2^{\gamma_2} \longrightarrow \cdots \longrightarrow \widetilde{F}_s^{\gamma_s}$$

over $R[u^{\pm 1}]$.

Let G be another p-typical formal group law over R. We assume that there is an isomorphism Φ between F and G in usual sense. We set

$$\widetilde{G}(X, Y) = \Phi_0^{-1} uG(\Phi_0 u^{-1}X, \Phi_0 u^{-1}Y)$$

and

$$\widetilde{\Phi}(X) = \Phi_0^{-1} u \Phi(u^{-1}X),$$

where Φ_0 is the leading coefficient of $\Phi(X)$. Then $\widetilde{G}$ is a degree -2 p-typical formal group law over $R[u^{\pm 1}]$ and $\widetilde{\Phi}$ is a strict isomorphism between $\widetilde{F}$ and $\widetilde{G}$. Note that Γ also acts on $(\widetilde{G}, R[u^{\pm 1}])$ so that the following diagram is commutative

$$\begin{array}{ccc} \widetilde{F} & \xrightarrow{\widetilde{f}(\gamma)} & \widetilde{F}^{\gamma} \\ {\scriptstyle \widetilde{\Phi}}\downarrow & & \downarrow{\scriptstyle \widetilde{\Phi}^{\gamma}} \\ \widetilde{G} & \xrightarrow[\widetilde{g}(\gamma)]{} & \widetilde{G}^{\gamma} \end{array}$$

for all $\gamma \in \Gamma$. We denote by $\mathbf{F}$ and $\mathbf{G}$ the morphisms of cochain groups from $BP_*(BP)^{\otimes *}$ to $C_\Gamma(\Gamma^{*+1}; R[u^{\pm 1}])$ induced by $\widetilde{F}$ and $\widetilde{G}$ respectively.

LEMMA 7.3. *There is a (co)chain homotopy between* $\mathbf{F}$ *and* $\mathbf{G}$.

Proof. We define a homomorphism h_i: $BP_*(BP)^{\otimes(s+1)} \to C_\Gamma(\Gamma^{s+1}; R[u^{\pm 1}])$ for $(0 \le i \le s)$ as follows. For $\gamma_0, \ldots, \gamma_s \in \Gamma$, the adjoint of h_i is a ring homomorphism $BP_*(BP)^{\otimes(s+1)} \to R[u^{\pm 1}]$ represented by the following string of formal group laws and strict isomorphisms

$$\begin{array}{llll} \widetilde{F}^{\gamma_0} \longrightarrow \widetilde{F}^{\gamma_1} \longrightarrow \cdots \longrightarrow & \widetilde{F}^{\gamma_i} & & \\ & \downarrow{\scriptstyle \widetilde{\Phi}^{\gamma_i}} & & \\ & \widetilde{G}^{\gamma_i} & \longrightarrow \widetilde{G}^{\gamma_{i+1}} \longrightarrow \cdots \longrightarrow \widetilde{G}^{\gamma_s}. \end{array}$$

Then we have the following relations

$$\begin{array}{rcll} h_i d_j & = & d_j h_{i-1} & (0 \le j < i) \\ h_0 d_0 & = & \mathbf{G} & \\ h_i d_i & = & h_{i-1} d_i & (1 \le i \le s) \\ h_s d_{s+1} & = & \mathbf{F} & \\ h_i d_j & = & d_{j-1} h_i & (i+1 < j) \end{array}$$

Set $H = \sum_{i=0}^{s} (-1)^i h_i$. By the above relations, $Hd + dH = \mathbf{G} - \mathbf{F}$. □

If Z is a finite spectrum of type $n-1$, then there is a finite filtration of $BP_*(Z)$ as $BP_*(BP)$-comodules whose associated graded objects are $\Sigma^i BP_*/I_j$ for some i and $j \ge n-1$ by the Landweber filtration theorem. Then $BP_*(L_{n-1}Z) = BP_*(Z)[v_{n-1}^{-1}]$ and $E_{n*}(L_{n-1}Z) = E_{n*} \otimes BP_*(Z)[v_{n-1}^{-1}]$ have induced filtrations whose associated graded objects are $\Sigma^i BP_*/I_{n-1}[v_{n-1}^{-1}]$ and $\Sigma^i E_{n*}/I_{n-1}[v_{n-1}^{-1}]$ for some i, respectively. Note that the natural map $BP_*(L_{n-1}Z) \to E_{n*}(L_{n-1}Z)$ is compatible with the filtrations and the induced map on the associated graded objects is a sum of finite many copies of the natural map $BP_*/I_{n-1}[v_{n-1}^{-1}] \to K[u^{\pm 1}]$.

The filtration of $BP_*(L_{n-1}Z)$ (resp. $E_{n*}(L_{n-1}Z)$) defines a spectral sequence converging to $H_c^{**}(G_{n-1}; E_{n-1*}(Z))$ (resp. $H_c^{**}(G_n; E_{n*}(L_nZ))$). The E_1-term of the spectral sequence is isomorphic to a sum of finite may copies of $H_c^{**}(G_{n-1}; \mathbf{F}[w^{\pm 1}])$ (resp. $H_c^{**}(G_n; K[u^{\pm 1}])$). We denote by $E_*^{**}(3)$ (resp. $E_*^{**}(4)$) the spectral sequence converging to $H_c^{**}(G_{n-1}; E_{n-1*}(Z))$ (resp. $H_c^{**}(G_n; E_{n*}(L_{n-1}Z))$). Since the natural map $BP_*(L_{n-1}Z) \to E_{n*}(L_{n-1}Z)$ is compatible with the filtrations, it induces a morphism of spectral sequences g_r: $E_r^{**}(3) \to E_r^{**}(4)$.

The following theorem gives a relation between the chromatic splitting conjecture and the ring homomorphism Θ.

THEOREM 7.4. *The natural map $L_{n-1}Z \to L_{n-1}L_{K(n)}Z$ lifts to a morphism of spectral sequences g_r: $E_r^{**}(3) \to E_r^{**}(4)$ which coincides with a sum of copies of Θ on E_1-terms.*

Proof. It is sufficient to show that the following diagram is commutative

$$\begin{array}{ccc}
\mathrm{Ext}_{BP_*(BP)}^{**}(BP_*, BP_*/I_{n-1}) & \longrightarrow & H_c^{**}(G_n; E_{n*}/I_{n-1}) \\
\Big\downarrow v_{n-1}^{-1} & & \Big\downarrow v_{n-1}^{-1} \\
\mathrm{Ext}_{BP_*(BP)}^{**}(BP_*, BP_*/I_{n-1}[v_{n-1}^{-1}]) & \longrightarrow & H_c^{**}(G_n; K[u^{\pm 1}]) \\
\Big\downarrow \cong & & \Big\downarrow \cong \\
H_c^{**}(G_{n-1}; \mathbf{F}[w^{\pm 1}]) & \xrightarrow{h} & \varinjlim_i H_c^{**}(\mathcal{G}(i); L_i[u^{\pm 1}]),
\end{array}$$

where the top horizontal arrow is induced by the natural map $BP \to E_n$, the middle horizontal arrow is obtained by inverting v_{n-1} from the top one. Hence the top square is commutative. The bottom horizontal arrow h is an inflation map and Θ is the composition of h with the inverse of the isomorphism $H_c^{**}(G_n; K[u^{\pm 1}]) \xrightarrow{\cong} \varinjlim_i H_c^{**}(\mathcal{G}(i); L_i[u^{\pm 1}])$.

Let $C(1)^*$ be the cochain complex $\{BP_*(BP)^{\otimes *}/I_{n-1}[v_{n-1}^{-1}]\}$, $C(2)^*$ the continuous cochain complex $C_{G_n}(G_n^{*+1}; K[u^{\pm 1}])$ and $C(3)^*$ the direct limit of continuous cochain complexes $\varinjlim_i C_{\mathcal{G}(i)}(\mathcal{G}(i)^{*+1}; L_i[u^{\pm 1}])$. By Theorem 3.5, the natural map $C(2)^* \to C(3)^*$ induces an isomorphism on cohomology groups. The p-typical formal group law (F_n, L) (resp. (H_{n-1}, L)) implies a cochain complex map f (resp. g): $C(1)^* \to C(3)^*$. The isomorphism Φ between F_n and H_{n-1} implies a cochain homotopy between f and g by the same argument in the continuous context of the proof of Lemma 7.3. Note that we may take the cochain homotopy in $C(3)^*$ rather than in $C_{\mathcal{G}}(\mathcal{G}^{*+1}; L[u^{\pm 1}])$ by considering the gradings. This completes the proof. □

A Toda-Smith spectrum $V(n)$ is defined to be a spectrum whose BP-homology $BP_*(V(n))$ is isomorphic to BP_*/I_{n+1}. Then $V(n)$ is a finite spectrum of type $n+1$. It is known that there exists $V(0)$, $V(1)$ for $p \geq 3$, $V(2)$ for $p \geq 5$ and $V(3)$ for $p \geq 7$. But it is not known whether $V(n)$ exists for $n \geq 4$. If $V(n-2)$ exists, then the filtrations of $BP_*(V(n-2))[v_{n-1}^{-1}]$ and $E_{n*}(V(n-2))[v_{n-1}^{-1}]$ are trivial, and the spectral sequences $E_r^{**}(3)$ and $E_r^{**}(4)$ collapse. Hence we obtain the following corollary.

COROLLARY 7.5. *If there exists a Toda-Smith spectrum* $V(n-2)$, *then* f_2: $E_2(1)(V(n-2)) \to E_2(2)(V(n-2))$ *coincides with* Θ.

DEPARTMENT OF APPLIED MATHEMATICS, FUKUOKA UNIVERSITY, FUKUOKA 814–0180, JAPAN
E-mail: torii@bach.sm.fukuoka-u.ac.jp

REFERENCES

[1] M. Ando, J. Morava and H. Sadofsky, Completions of $\mathbf{Z}/(p)$-Tate cohomology of periodic spectra, *Geom. Topol.* **2** (1998), 145–174.
[2] E. S. Devinatz, Morava's change of rings theorem, *The Čech Centennial (Boston, MA, 1993), Contemp. Math.*, vol. 181, Amer. Math. Soc., Providence, RI, 1995, pp. 83–118.
[3] E. S. Devinatz, M. J. Hopkins and J. H. Smith, Nilpotence and stable homotopy theory. I, *Ann. of Math.* (2) **128** (1988), 207–241.
[4] B. H. Gross, Ramification in p-adic Lie extensions, *Journées de Géométrie Algébrique de Rennes (Rennes, 1978), vol. III, Astérisque* **65** (1979), 81–102.
[5] G. Hochschild and J. P. Serre, Cohomology of group extensions, *Trans. Amer. Math. Soc.* **74** (1953), 110–134.
[6] M. J. Hopkins, M. Mahowald and H. Sadofsky, Constructions of elements in Picard groups, *Topology and Representation Theory (Evanston, IL, 1992), Contemp. Math.*, vol. 158, Amer. Math. Soc., Providence, RI, 1994, pp. 89–126.
[7] M. J. Hopkins and J. H. Smith, Nilpotence and stable homotopy theory. II, *Ann. of Math.* (2) **148** (1998), 1–49.
[8] M. Hovey, Bousfield localization functors and Hopkins' chromatic splitting conjecture, *The Čech Centennial (Boston, MA, 1993), Contemp. Math.*, vol. 181, Amer. Math. Soc., Providence, RI, 1995, pp. 225–250.
[9] M. Hovey and N. P. Strickland, Morava K-theories and localisation, *Mem. Amer. Math. Soc.* **139** (1999).
[10] M. Lazard, Sur les groupes de Lie formels à un paramètre, *Bull. Soc. Math. France* **83** (1955), 251–274.
[11] J. Lubin and J. Tate, Formal moduli for one-parameter formal Lie groups, *Bull. Soc. Math. France* **94** (1966), 49–59.
[12] H. R. Miller, On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space, *J. Pure Appl. Algebra* **20** (1981), 287–312.
[13] H. R. Miller and D. C. Ravenel, Morava stabilizer algebras and the localization of Novikov's E_2-term, *Duke Math. J.* **44** (1977), 433–447.
[14] H. R. Miller, D. C. Ravenel and W. S. Wilson, Periodic phenomena in the Adams-Novikov spectral sequence, *Ann. Math.* (2) **106** (1977), 469–516.
[15] J. Morava, Noetherian localisations of categories of cobordism comodules, *Ann. of Math.* (2) **121** (1985), 1–39.

[16] D. C. Ravenel, Localization with respect to certain periodic homology theories, *Amer. J. Math.* **106** (1984), 351–414.
[17] ———, *Complex Cobordism and Stable Homotopy Groups of Spheres, Pure Appl. Math.*, vol. 121, Academic Press, Inc., Orlando, Fla., 1986.
[18] ———, The geometric realization of the chromatic resolution, *Algebraic Topology and Algebraic K-theory (Princeton, NJ, 1983), Ann. of Math. Stud.*, vol. 113, Princeton Univ. Press, Princeton, NJ, 1987, pp. 168–179.
[19] J. P. Serre, *Local Fields, Grad. Texts in Math.*, vol. 67, Springer-Verlag, New York, 1979.
[20] ———, *Galois Cohomology*, Springer-Verlag, Berlin, 1997.
[21] K. Shimomura, On the Adams–Novikov spectral sequence and products of β-elements, *Hiroshima Math. J.* **16** (1986), 209–224.
[22] ———, The Adams-Novikov E_2-term for computing $\pi_*(L_2V(0))$ at the prime 2, *Topology Appl.* **96** (1999), 133–152.
[23] ———, The homotopy groups of the L_2-localized mod 3 Moore spectrum, *J. Math. Soc. Japan* **52** (2000), 65–90.
[24] N. P. Strickland, Finite subgroups of formal groups, *J. Pure Appl. Algebra* **121** (1997), 161–208.
[25] ———, Gross-Hopkins duality, *Topology* **39** (2000), 1021–1033.
[26] A. Weil, *Basic Number Theory*, Springer-Verlag, Berlin, 1995.