3. λ -Rings.

We present the theory of special λ -rings. The algebraic material is mainly taken from the paper [14] by Atiyah and Tall. The reader should consult this paper for additional information. The main theorem to be proven here is an exponential isomorphism for p-adic λ -rings which is an algebraic version of the powerful theorem $J'(X) = J''(X)$ in the work of Adams [2] on fibre homotopy equivalence of vector bundles.

3.1. Definitions.

Let R be a commutative ring with identity. A λ -ring structure on R consists of a sequence $\lambda^n : R \longrightarrow R$, $n \in \mathbb{N}$, of maps such that for all $x, y \in R$

(3.1.1)
$$\lambda^o(x) = 1$$
$$\lambda^1(x) = x$$
$$\lambda^n(x+y) = \sum_{r=o}^{n} \lambda^r(x) \lambda^{n-r}(y).$$

If t is an indeterminate we define

(3.1.2)
$$\lambda_t(x) = \sum_{n \geqslant o} \lambda^n(x) t^n.$$

Then 3.1.1 shows that

(3.1.3)
$$\lambda_t : R \longrightarrow 1 + R[[t]]^+$$

is a homomorphism from the additive group of R into the multiplicative group $1 + R[[t]]^+$ of formal power series over R with constant term 1.

Exterior powers of modules have formal properties like 3.1.1 and we shall see later how exterior powers give λ-ring structures on certain Grothendieck groups.

A ring R together with a λ-ring structure on it is called a λ-<u>ring</u>. A λ-<u>homomorphism</u> is a ring homomorphism commuting with the λ-operations. We have the notions of λ-<u>ideal</u> and λ-<u>subring.</u>

Some further axioms are needed to insure that the λ-operations behave well with respect to ring multiplication and composition.

Let $x_1,\dots,x_p$, $y_1,\dots,y_q$ be indeterminates and let u_i,v_i be the i-th elementary symmetric functions in $x_1,\dots,x_p$ and $y_1,\dots,y_q$ respectively. Define polynomials with integer coefficients:

(3.1.4) $P_n(u_1,\dots,u_n;\ v_1,\dots,v_n)$ is the coefficient of t^n in $\prod_{i,j}(1+x_iy_jt)$.

(3.1.5) $P_{n,m}(u_1,\dots,u_{mn})$ is the coefficient of t^n in $\prod_{i_1<\dots<i_m}(1+x_{i_1}\cdot\ \dots\cdot x_{i_m}t)$.

Then P_n is a polynomial of weight n in the u_i and also in the v_i, and $P_{n,m}$ is of weight nm in the u_i. If we assume $p \geqslant n$, $q \geqslant n$ in 3.1.4 and $p \geqslant mn$ in 3.1.5 then non of the variables u_i,v_i involved are zero and the resulting polynomials are independent of p,q.

A λ-ring R is said to be <u>special</u> if in addition to 3.1.1 the following identities hold for $x,y \in R$

(3.1.6)
$$\lambda_t(1) = 1+t$$
$$\lambda^n(xy) = P_n(\lambda^1x,\dots,\lambda^nx;\ \lambda^1y,\dots,\lambda^ny)$$
$$\lambda^m(\lambda^n(x)) = P_{m,n}(\lambda^1x,\dots,\lambda^{mn}x).$$

One can motivate 3.1.6 as follows. An element x in a λ-ring is called n-dimensional if $\lambda_t(x)$ is a polynomial of degree n. The ring is called finite-dimensional if every element is a difference of finite dimensional elements. If $x = x_1 + \dots + x_p$ and $y = y_1 + \dots + y_q$ in a λ-ring and the x_i, y_i are one-dimensional then

$$\lambda_t(x) = \prod(1+x_i t) = 1 + u_1 t + \dots + u_p t^p$$

(u_i the i-th elementary function of the x_j as above) and we see that the second identity of 3.1.6 is true for such x,y. If moreover the product of one-dimensional elements is again one-dimensional then the third identity of 3.1.6 is true for $x = \sum x_i$. The axioms for a special λ-ring insure that many theorems about λ-rings can be proved by considering just one-dimensional elements. We formalize this remark.

One defines a λ-ring structure on $1+A[[t]]^+$ by:

"addition" is multiplication of power series.

(3.1.7) "multiplication" is given by

$$(1+\sum a_n t^n) \circ (1+\sum b_n t^n) = 1 + P_n(a_1,\dots,a_n;b_1,\dots,b_n)t^n.$$

The "λ-structure" is given by

$$\Lambda^m(1+\sum a_n t^n) = 1 + \sum P_{n,m}(a_1,\dots,a_{mn})t^n.$$

Proposition 3.1.8.

$1 + A[[t]]^+$ is a λ-ring with the structure 3.1.7.

Proof.

Compare Atiyah-Tall [14], p. 258.

Using this structure one sees that A is a special λ-ring if and only if λ_t is a λ-homomorphism. Moreover one has the Theorem of

Grothendieck that $1 + A[[t]]^+$ is a special λ-ring (Atiyah-Tall loc. cit.)

One can use 3.1.8 to show that certain λ-rings are special.

Proposition 3.1.9.

Let R be a λ-ring. Suppose that products of one-dimensional elements in R are again one-dimensional; in particular 1 shall be one-dimensional. Let $R_1 \subset R$ be the subring generated by one-dimensional elements. Then R_1 is a λ-subring which is special.

Proof.

Every element of R_1 has the form x-y where x,y are sums of one-dimensional elements, say $x = x_1 + \ldots + x_p$, $y = y_1 + \ldots + y_q$. Then $\lambda^i(x)$ is the i-th elementary symmetric function in the x_j hence a sum of one-dimensional elements. Moreover $\lambda^i(-y)$ is an integral polynomial in the $\lambda^j(y)$. Hence $\lambda^n(x-y) = \sum_i \lambda^i(x) \lambda^{n-i}(-y) \in R_1$. The remarks before 3.1.7 show that $\lambda_t \mid R_1$ is a ring-homomorphism and $\lambda_t \lambda^i(x) = \lambda^i \lambda_t(x)$ if x is a sum of one-dimensional elements and these two facts imply $\lambda_t \lambda^i(-x) = \lambda^i \lambda_t(-x)$ and then $\lambda_t \lambda^i(x-y) = \lambda^i \lambda_t(x-y)$.

Remark 3.1.10.

One can show (Atiyah-Tall [14]) - and later we shall use this fact - that a λ-ring R is special if and only if for any set $a_1, \ldots, a_n$ of finite-dimensional elements in R there exists a λ-monomorphism $f : R \longrightarrow R'$ such that the fa_i are sums of one-dimensional elements. This is called the splitting principle for special λ-rings.

That a λ-ring structure, even if not special, may be very useful can be seen from the following Proposition due to G. Segal.

Proposition 3.1.11.

Let R be a λ-ring. Then all Z-torsion elements in R are nilpotent.

Proof.

Let a be a p-torsion element, say $p^n a = 0$. Then

$$1 = \lambda_t(o) = \lambda_t(a)^{p^n} = (1+at+\dots)^{p^n} \equiv 1+a^{p^n}t^{p^n}+\dots \mod p\ A$$

and hence $a^{p^n} = p\ b$ for some $b \in A$. Therefore

$$a^{(p^n+1)n} = (p\ a\ b)^n = (p^n a)(a^{n-1}b) = 0 .$$

3.2. Examples.

a) The integers may be given a λ-ring structure by defining $\lambda_t(1) = 1 + \sum m_n t^n$ where $m_1 = 1$. The canonical structure on Z is given by

(3.2.1)

$$\lambda_t(1) = 1 + t$$
$$\lambda_t(m) = (1+t)^m$$
$$\lambda^k(m) = \binom{m}{k} \qquad m \geq 0$$
$$\lambda^k(-m) = (-1)^k \binom{m+k-1}{k}$$

This canonical structure is special by 3.1.9. It can be given the following combinatorial interpretation: Let S be a set with m elements. Let $\Lambda^k S$ be the set of all subsets of cardianlity k. Then $|\Lambda^k S| = \binom{m}{k}$. The theory of special λ-rings may be thought of as an extremely elegant way of handling combinatorial identities for sets, symmetric functions, binomial coefficients, etc.

b) Let E,F be complex G-vector bundles over the (compact) G-space X where G is a compact Lie group. Then exterior powers Λ^i of G-vector bundles satisfy

$$\Lambda^{o}E = 1, \quad \Lambda^{1}E = E, \quad \Lambda^{n}(E \oplus F) = \bigoplus_{i=o}^{n} \Lambda^{i}(E) \otimes \Lambda^{j}(F) .$$

Let $K_G(X)$ be the Grothendieck ring of such G-vector bundles over X (Segal [142]). Then $E \longmapsto 1 + (\Lambda^1 E)t + (\Lambda^2 E)t^2 + \ldots$ is a homomorphism from the additive semi-group of isomorphism classes of G-vector bundles over X into $1 + K_G(X)[[t]]^+$ and extends therefore uniquely to the Grothendieck group giving a map

$$\lambda_t : K_G(X) \longrightarrow 1+K_G(X)[[t]]^+ : x \longmapsto 1+ \lambda^1(x)t+\ldots$$

such that $\lambda^i[E] = [\Lambda^i(E)]$ for E a G-vector bundle. These λ^i yield therefore a λ-ring structure on $K_G(X)$.

<u>Proposition 3.2.2.</u>

$K_G(X)$ <u>with this</u> λ-<u>structure is a special</u> λ-<u>ring</u>.

<u>Proof.</u>

The proof depends on the so called splitting principle which - especially for general G - is highly non-trivial. This splitting principle says: Given vector bundles $E_1,\ldots,E_k$ over X. There exists a compact G-space Y and a G-map $f : Y \longrightarrow X$ such that the induced map $f^* : K_G(X) \longrightarrow K_G(Y)$ is injective and f^*E_i splits into a sum of line bundles. See Atiyah [9] , 2.7.11 or Karoubi [103], p. 193 for the case $G = \{1\}$.

Using the splitting principle 3.2.2 follows essentially from 3.1.9.

For a discussion of λ-operations in K-theory see also Atiyah [9] , ch. III, [7] ; Karoubi [103] IV. 7.

c) Other versions of topological K-theory like real K-Theory or

Real-K-Theory (Atiyah [8]), yield special λ-rings too.

d) A special case of b) is the representation ring R(G) of complex representations. Since representations are detected by restriction to cyclic subgroups and R(C) for a cyclic group C is generated by one-dimensional elements one can directly apply 3.1.9 to show that R(G) is special.

e) The Burnside ring acquires a λ-ring structure if we define $\lambda^i(S)$ for a finite G-set S to be the i-th symmetric power of S. We use the identity $\lambda^n(S+T) = \sum_i \lambda^i(S)\, \lambda^{n-i}(T)$ to extend this to A(G) as under b). This λ-ring structure is in general not special. See Siebeneicher [149] and the exercises to this section.

f) See Atiyah-Tall [14] , I. 2 for the construction of a free λ-ring on one generator.

3.3. γ-operations.

We assume that R is a special λ-ring. Then R contains a subring isomorphic to Z for if $1 \in R$ had finite additive order m, then $1 = \lambda_t(o) = \lambda_t(m\cdot 1) = (1+t)^m$ would give a contradiction (compare coefficients of t^m). A special λ-ring R is called <u>augmented</u> if there is given a λ-homomorphism $e : R \longrightarrow Z$. We call $I = \operatorname{Ker} e$ the <u>augmentation ideal</u>; it is a λ-ideal. Any element $x \in R$ may be written uniquely $x = e(x) + (x-e(x))$ with $e(x) \in Z$ and $x-e(x) \in I$.

Define the γ-<u>operations</u> on a special λ-ring R:

$$\lambda_{t/(1-t)}(x) =: \gamma_t(x) = 1 + \sum_{n\geq 1} \gamma^i(x)t^i. \tag{3.3.1}$$

Then

$$\gamma_t(x+y) = \gamma_t(x)\, \gamma_t(y). \tag{3.3.2}$$

Moreover one has

(3.3.3) $$\gamma^n(x) = \lambda^n(x+n-1).$$

Proof.

Using 3.2.1 we get

$$\lambda_{t/(1-t)}(x) = 1 + \sum_{i\geq 1} \lambda^i(x)\left(\sum_{k\geq o}\binom{i+k-1}{k}t^{k+i}\right)$$

$$= 1 + \sum_{j\geq 1}\left(\sum_{i=1}^{j} \lambda^i(x)\binom{j-1}{j-i}\right)t^j$$

$$= 1 + \sum_{j\geq 1} \lambda^j(x+j-1)t^j .$$

We conclude from 3.3.3 that $\lambda^j(x) = o$ for $j > n$ implies $\gamma^j(x-n)=o$ for $j > n$, i. e. if x is n-dimensional then x-n is of γ -dimension at most n.

Suppose R is an augmented λ -ring with augmentation $e : R \longrightarrow Z$ and augmentation ideal I = ker e. We define the γ-filtration by: $R_n \subset R$ is the additive group generated by monomials $\gamma^{n_1}(a_1)\cdot \ldots \cdot \gamma^{n_r}(a_r)$ where $a_i \in I$ and $\sum n_i \geq n$.

Proposition 3.3.4.

(i) $R_o = R$, $R_1 = I$.

(ii) $R_m R_n \subset R_{m+n}$.

(iii) R_n is a λ -ideal for $n \geq 1$.

Proof.

(i) and (ii) follow directly from the definitions. (iii): $R = Z \oplus R_1$ shows that R_n is an ideal. To show R_n is a λ -ideal, it is sufficient to show $\lambda^r(\gamma^m(x)) \in R_m$ for $x \in I$. First we compute for $i \geq m$

$$\lambda^i(x+m-1) = \gamma^i(x+m-i) = \sum_{s=o}^{i} \gamma^s(x)\ \gamma^{i-s}(m-i)$$

$$= \sum_{s=m}^{i} \gamma^s(x)\ \gamma^{i-s}(m-i) \in R_m$$

because $\gamma^{i-s}(m-i) = \lambda^{i-s}(m-s-1) = o$ for $i \geq m \geq s+1$. We use this in

$$\lambda^r(\gamma^m(x)) = \lambda^r(\lambda^m(x+m-1))$$

$$= P_{r,m}(\lambda^1(x+m-1),\dots,\ \lambda^{rm}(x+m-1))$$

and observe that $P_{r,m}(s_1,\dots,s_{rm})$ is a sum of monomials each containing a term s_i for $i \geq m$ because $P_{r,m}(s_1,\dots,s_{m-1},o,\dots,o) = o$.

Sometimes we want to work only with the augmentation ideal. We define: A ring I without identity is called a <u>special</u> γ-<u>ring</u> if there is an augmented special λ-ring R with I as augmentation ideal. I then carries the induced γ^i-operations. We define the γ-filtration as before, I_n being the ideal generated by monomials $\gamma^{n_1}(a_1)\cdot\dots\cdot\gamma^{n_r}(a_r)$ where $a_i \in I$, $\sum n_i \geq n$. We have

(3.3.5) $\qquad I_1 = I,\ I_m I_n \subset I_{m+n},\quad \gamma^i(I_n) \subset I_n.$

<u>3.4. The Adams operations.</u>

Adams introduced in [1] certain operations derived from the λ^i which are much easier to handle algebraically.

Let R be a special λ-ring. Define maps

$$\psi^n : R \longrightarrow R, \qquad n \geq 1$$

by

$$(3.4.1) \qquad \psi_{-t}(x) = -t \frac{d}{dt}(\lambda_t(x))/\lambda_t(x)$$

$$\psi_t(x) = \sum_{n\geq 1} \psi^n(x)t^n.$$

A more elementary way of defining the ψ^n is: Define the Newton polynomial

$$N_n(s_1,\ldots,s_n) = \sum_{j=1}^{n} x_j^n$$

where s_i is the i-th elementary symmetric function of the x_j. Then put

$$(3.4.2) \qquad \psi^n(x) = N_n(\lambda^1(x),\ldots, \lambda^n(x)).$$

We leave it as an exercise to show that the two definitions are equivalent.

We want to show that the ψ^n are λ-ring homomorphisms. This means we have to verify certain identities between the ψ^n- and λ^j-operations. We use the verification principle which says that it is enough to verify the identities on elements which are sums of one-dimensional elements. A formal proof of this principle is given in Atiyah-Tall [14], I. 3.4, I. 4.5. Since in the applications the λ-rings are finite-dimensional and since we have to prove the splitting principle in order to show that something is a special λ-ring we do not prove the verification principle.

Proposition 3.4.3.

(i) If x is one-dimensional then $\psi^n x = x^n$.

(ii) ψ^n is a λ-homomorphism.

(iii) $\psi^m \psi^n = \psi^n \psi^m = \psi^{mn}$.

(iv) $\psi^{p^r}(x) \equiv x^{p^r} \bmod p$ (p prime).

Proof.

(i) follows directly from 3.4.2.

(ii) Suppose x_i, y_j are one-dimensional. Then $x_i y_j$ is one-dimensional because R is special. From 3.4.1 one obtains that ψ^n is an additive homomorphism. Moreover

$$\psi^n(\Sigma x_i \Sigma y_j) = \psi^n(\Sigma x_i y_j) = \Sigma \psi^n(x_i y_j) = \Sigma (x_i y_j)^n$$

$$= (\Sigma x_i^n)(\Sigma y_j^n) = \psi^n(\Sigma x_i)\psi^n(\Sigma y_j).$$

$$\psi^n(\lambda^m(\Sigma x_i)) = \psi^n(s_m(x_1,\ldots,x_r)) = s_m(x_1^n,\ldots,x_r^n)$$

$$= \lambda^m(\Sigma x_i^n) = \lambda^m(\psi^n(\Sigma x_i)).$$

Now use the verification principle.

(iii) and (iv) are likewise immediate from the verification principle.

As a consequence we have ψ^n on a special γ-ring. Moreover the ψ^n preserve the γ-filtration.

Proposition 3.4.4.

Let I be a special γ-ring. Assume $x \in I_n$. Then the following holds:

(i) $\psi^k(x) - k^n x \in I_{n+1}$

(ii) $\psi^k(x) + (-1)^k k \lambda^k(x) \in I_{n+1}$

(iii) $\lambda^k(x) + (-1)^k k^{n-1} x \in I_{n+1}$.

Proof.

(i) We need only show that $\psi^k(\gamma^m(a)) - k^m \gamma^m(a) \in I_{m+1}$ for $a \in I$, because ψ^k is a γ-homomorphism. If $x_1,\ldots,x_r$ have γ-dimension one, i. e. $\gamma_t(x_i) = 1+x_i t$, then $1+x_i$ has λ-dimension one, hence $\psi^k(x_i) = (1+x_i)^k - 1$ and therefore

$$\psi^k(\gamma^m(x_1+\ldots+x_r)) - k^m \gamma^m(x_1+\ldots+x_r)$$

$$= \psi^k(s_m(x_1,\ldots,x_r)) - k^m s_m(x_1,\ldots,x_r)$$

$$s_m((1+x_1)^k - 1,\ldots,(1+x_r)^k - 1) - k^m s_m(x_1,\ldots,x_r).$$

This is a symmetric polynomial of degree $\geq m+1$, hence (i) is true for $x = \sum x_i$ and, by the verification principle, therefore in general.

(ii) From the Newton polynomials we obtain the well-known identity

$$\psi^k(x) - \psi^{k-1}(x)\lambda^1(x)+\ldots+(-1)^{k-1}\psi^1(x)\lambda^{k-1}(x)+(-1)^k k\lambda^k(x)=0$$

which implies the result, because $\psi^i(x) \in I_n$, $\lambda^i(x) \in I_n$ for $i \geq 1$, and $x \in I_n$.

(iii) From (i) and (ii) we obtain $k\lambda^k(x) + (-1)^k k^n(x) \in I_{n+1}$. Thus the result follows if there is no k-torsion. (One can produce suitable universal situations without torsion, e. g. free λ-rings; thus one gets the result in general. One should note that the assertions are natural with respect to λ-homomorphisms.)

3.5. Adams-operations on representation rings.

Let G be a finite group and R(G;F) be the Grothendieck ring (= representation ring) of finitely generated F[G]-modules where F is a field. We assume for simplicity that F has characteristic zero. Then elements in R(G;F) are determined by their character. We identify R(G;F) with the corresponding character ring. Exterior powers define a special λ-ring structure on R(G;F). We want to compute the associated Adams-operations.

Proposition 3.5.1.

Let $x \in R(G;F)$. Then

$$\psi^k x(g) = x(g^k), \qquad g \in G.$$

In particular

$$\psi^k = \psi^{k+|G|}$$

Proof.

Restrict to the cyclic group C generated by g. Pass to an algebraic closure of F so that $x \mid C = y-z$ where y and z are sums of one-dimensional representations. The result then follows from 3.4.3 taking into account that for a one-dimensional representation x the relation $x^k(g) = x(g^k)$ holds.

Now assume that $F = Q[\zeta_n]$ where ζ_n is a primitive n-th root of unity. Assume that k is prime to the group order $|G|$. The Galois group $Gal(Q[\zeta] : Q)$ is isomorphic to Z/nZ^*, namely so that k mod n corresponds to the field automorphism P^k characterized by $P^k(\zeta_n) = \zeta_n^k$. Since characters of $F[G]$-modules take values in $Q[\zeta_n]$ we can apply P^k to such characters. Let $Q[\zeta_n]$ be a splitting field for G. (By a famous theorem of Brauer it suffices to take for n the exponent of G; see Serre [147], p. 109). Then we show

Proposition 3.5.2.

(i) $\psi^k x = P^k x$ for $x \in R(G;Q[\zeta_n])$ and $(k,|G|) = 1$.

(ii) If x is the character of an irreducibel module then $\psi^k x$ is irreducible too (again k prime to $|G|$).

Proof.

(i) Let x be the character of a matrix representation. Restrict to the cyclic subgroup C generated by $g \in G$. Then the matrix for g is equivalent to a diagonal matrix with roots of unity $u_1,\dots,u_r$ on the diagonal. Then $\psi^k(x)(g) = \sum u_i^k = P^k(\sum u_i) = P^k(x(g))$.

(ii) Apply the Galois automorphism P^k to a matrix representation over $Q[\zeta_n]$.

Remark 3.5.3.

The Adams operation are, of course, independent of the field of definition. Therefore 3.5.2 holds more generally.

3.7. The Bott cannibalistic class θ_k.

Let R be a special λ-ring and let ζ_k be a primitive k-th root of unity. Let $P(R) \subset R$ be the subset of finite-dimensional elements in R. Then P(R) is an additive semi-group. If $x \in P(R)$ we consider the product

$$(3.7.1) \qquad \theta_k(x) := \prod_u \lambda_{-u}(x) \in R \otimes_Z Z[\zeta_k]$$

where the product is taken over all roots of $t^k-1 = o$ except 1. We identify R with its image in $R \otimes Z[\zeta_k]$ under the canonical map $r \longmapsto r \otimes 1$. Then $\theta_k(u)$ is contained in R. [In order to see this consider the following diagram

$$\begin{array}{ccc} R \otimes_Z Z[s_1,\dots,s_{k-1}] & \longrightarrow & R \otimes_Z Z[t_1,\dots,t_{k-1}] \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \otimes_Z Z[\zeta_k] \end{array}$$

where $t_1,\dots,t_{k-1}$ are indeterminates and $s_1,\dots,s_{k-1}$ are the elementary symmetric functions in the t_j. The vertical maps are induced by substituting for $t_1,\dots,t_{k-1}$ the roots of $t^k - 1 = o$ except 1. Then

$\pi_j \lambda_{-t_j}(x)$ is symmetric in the t_j and since $Z[s_1,\ldots,s_{k-1}] \subset Z[t_1,\ldots,t_{k-1}]$ is an inclusion as a direct summand we see that $\pi_j \lambda_{-t_j}(x) \in R \otimes_Z Z[s_1,\ldots,s_{k-1}]$. But the map at the bottom is an injection too because $Z \longrightarrow Z[\zeta_k] : n \longmapsto n$ is a direct injection.] We call it the Bott cannibalistic class θ_k. The following is immediate from the definition.

Proposition 3.7.2.

(i) *If* x *is one-dimensional then*

$$\theta_k(x) = 1 + x + \ldots + x^{k-1}.$$

(ii) *If* $x,y \in P(R)$ *then*

$$\theta_k(x+y) = \theta_k(x)\ \theta_k(y) \ .$$

Since $\theta_k(1) = k$ θ_k is not in general a unit in R so that θ_k cannot be extended to the additive subgroup generated by finite-dimensional elements. In the next section on p-adic γ-rings we find a remedy for this defect.

3.8. p-adic γ-rings.

Let p be a prime number. Let Z_p denote the p-adic integers. One can define Z_p as the inverse limit ring inv lim Z/p^nZ. If A is a finitely generated abelian group then $A \otimes_Z Z_p$ is cannonically isomorphic to the p-adic completion of A

$$A_p := \text{inv lim } A/p^n A.$$

Tensoring with Z_p is an exact functor on the category of finitely generated abelian groups. (See Atiyah-Mac Donald [11] , Ch. 10 for this and other back ground material on completions.) Groups $A_p^{\wedge}$ carry the p-adic topology: a fundamental system of neighbourhoods of zero is given by the subgroups $p^n A_p^{\wedge}$. They are complete and Hausdorff in this topology.

If B is a special γ-ring, then, by definition, there is a special augmented λ-ring R such that B = ker e where e is the augmentation. Then we have the exact sequence (because $e : R \longrightarrow Z$ splits)

$$0 \longrightarrow B \otimes Z_p \longrightarrow R \otimes Z_p \longrightarrow Z_p \longrightarrow 0.$$

We want to define the structure of a special λ-ring on $R \otimes Z_p$ such that $B \otimes Z_p$ is a λ-ideal. We can extend the λ^i by continuity if we have shown

Proposition 3.8.1.

The λ^i are continuous with respect to the p-adic topology.

Proof.

Given i and N chose k_o such that $\binom{p^k}{j}$ is divisible by p^N for $k \geqslant k_o$ and $1 \leqslant j \leqslant i$. Then

$$\lambda^j(p^k x) = P_j(\lambda^1(p^k), \ldots, \lambda^j(p^k); \lambda^1(x), \ldots, \lambda^j(x))$$

is contained in $p^N R$ if $k \geqslant k_o$ and $1 \leqslant j \leqslant i$ because P_j is of weight j in the first j variables. If $x-y = p^k z$ then

$$\lambda^i(y) - \lambda^i(x) = \sum_{j=1}^{i} \lambda^{i-j}(y)\, \lambda^j(p^k z) \in p^N R$$

for $k \geqslant k_o$.

The proof of this Proposition shows that if $a \in Z_p$ is the limit of a sequence (a_n), $a_n \in Z$ then $\lim \lambda^i(a_n x) = \lambda^i(\lim a_n x) = \lambda^i(ax)$ and hence

$$\lambda_t(ax) = \lambda_t(x)^a \qquad a \in Z_p$$

(3.8.2) $$\gamma_t(ax) = \gamma_t(x)^a \qquad x \in R$$

$$\psi^k(ax) = a\,\psi^k(x).$$

After these preliminary remarks we define a p-<u>adic</u> γ-<u>ring</u> A to be a γ-ring which is the completion $A = B \otimes Z_p$ of some γ-ring B which is finitely generated as an abelian group; moreover we require that <u>the</u> γ-<u>topology</u> <u>on</u> B <u>is</u> <u>finer</u> <u>than</u> <u>the</u> p-<u>adic</u> <u>topology</u>.

We now describe some examples of p-adic γ-rings.

<u>Proposition 3.8.3.</u>

<u>Let</u> X <u>be</u> <u>a</u> <u>finite</u> <u>connected</u> CW-<u>complex</u>. <u>Then</u> <u>the</u> n-th γ-<u>filtration</u> <u>on</u> $\tilde{K}(X)$ <u>is</u> <u>contained</u> <u>in</u> <u>the</u> n-th <u>skeleton-filtration</u>. <u>In</u> <u>particular</u> <u>the</u> γ-<u>topology</u> <u>is</u> <u>discrete</u> <u>and</u> $\tilde{K}(X) \otimes Z_p$ <u>is</u> <u>a</u> p-<u>adic</u> γ-<u>ring</u>.

<u>Proof.</u>

Let X^n be the n-skeleton on X. Then the n-th skeleton filtration $S_n\tilde{K}(X)$ is defined to be the kernel of the restriction map $i^* : \tilde{K}(X) \longrightarrow \tilde{K}(X^{n-1})$. Any element of $K(X^{n-1})$ is represented by an element $x = [E] - (n-1)$ where E is an (n-1)-dimensional bundle. Hence $i^* \gamma^n(y) = \gamma^n(i^* y) = \gamma^n(E-n+1) = 0$. The relation $S_n S_m \subset S_{n+m}$ then implies the result.

Let R(G) be the representation ring of the finite group G over the complex numbers. Let $R(G) \longrightarrow Z : x \longmapsto \dim x$ be the augmentation with kernel I(G). Then we can consider three topologies on R(G):

(i) The p-adic topology.

(ii) The I(G)-adic topology.

(iii) The γ-topology, defined by the γ-filtration.

Proposition 3.8.4.

Let G be a p-group. Then the topologies (i), (ii), and (iii) coincide. In particular $I(G) \otimes Z_p$ is a p-adic γ-ring.

We use the next Proposition for the proof of 3.8.4.

Proposition 3.8.5.

Let I be a γ-ring which is generated by a finite number of elements with finite γ-dimension. Then the I-adic topology coincides with the γ-topology.

Proof.

By definition of the γ-filtration we have $I_n \subset I^n$. Let m be the maximal γ-dimension of a given finite set of generators for I. Then γ^{m+1} applied to the monomials in the generators must lie in I^2. Since $\gamma^{m+1}(-x) \equiv -\gamma^{m+1}(x) \bmod I^2$ we obtain $I_{m+1} \subset I^2$. By induction one shows $I_{km+1} \subset I^k$.

Proof of 3.8.4.

Put I = I(G). By 3.8.5 the topologies (ii) and (iii) coincide. Let $m = |G|$. Then

$$(x-e(x))^m \equiv x^m - e(x)^m \mod p\, R(G)$$

because m is a p-power. By 3.5.1 we have $\psi^m x = e(x)$ and by 3.4.3 (iv) we have $\psi^m x \equiv x^m \mod p\, R(G)$. Putting these facts together we obtain

$$(x-e(x))^m \equiv e(x) - e(x)^m \equiv 0 \mod p\, R(G) .$$

This shows $I^m \subset p\,I$, hence the I-adic topology (and therefore the γ-topology) is finer than the p-adic topology. One can show that $mI \subset I^2$ (see Atiyah [6]), so that the p-adic topology is also finer than the I-adic. (This last fact also follows from localization theorems to be proved later in this lecture.)

As a slight generalization of 3.8.4 we mention

Proposition 3.8.6.

Let G *be a* p-*group and* X *a connected finite* G-CW-*complex. Then* $\tilde{K}_G(X) \otimes Z_p$ *is a* p-*adic* γ-*ring*. ($\tilde{K}_G(X)$ = *kernel of* $x \longmapsto \dim x$)

Proof (sketch).

From the fact that X is a finite G-CW-complex one shows by induction over the number of cells that $K_G(X)$ is a finitely generated abelian group. By 3.8.5 the γ-topology coincides with the $\tilde{K}_G(X)$-adic topology. Let X^o be the equivariant zero-skeleton of X. The kernel N of $r : K_G(X) \longrightarrow K_G(X^o)$ is nilpotent (compare Segal [142] , Proposition 5.1). Moreover $K_G(X^o) \cong \prod R(G_x)$, the product taken over the orbits of X^o. Put $I = \tilde{K}_G(X)$. By Atiyah-Mac Donald [11] , Theorem 10.11, the p-adic topology on rI is induced from the p-adic topology on $K_G(X^o)$. Hence from 3.8.4 we see that for some t, $rI^t \subset prI$, or equivalently, $I^t \subset pI + N$. But if $N^k = 0$ then $I^{tk} \subset (pI+N)^k \subset pI$. This shows that the I-adic topology is finer than the p-adic topology.

Now we continue with the general discussion of p-adic γ-rings $A = B \otimes Z_p$. If B_n is the n-th γ-ideal of B we let $A(n) = B_n \otimes Z_p$ be its closure. From 3.8.1 we obtain that the A(n) are γ-ideals. By definition of a p-adic γ-ring the topology defined by the system $A(n)$, $n \geq 1$, is finer than the p-adic topology; in particular this topology is also Hausdorff and one has

$$(3.8.7) \qquad A \cong \text{inv lim } A/A(n).$$

$A(n)$ contains the n-th γ-ideal A_n of A but A_n need not be closed in the p-adic topology. We observe

$$(3.8.8) \qquad A(n)/A(n+1) \cong (B_n/B_{n+1}) \otimes Z_p$$

because $\otimes Z_p$ is exact on finitely generated abelian groups. From 3.4.4 and 3.8.8 we obtain

Proposition 3.8.9.
$A(n)/A(n+1)$ is a p-adic γ-ring. The product of two elements is zero. For $a \in A(n)/A(n+1)$ we have

$$\lambda^k(a) = (-1)^{k-1} k^{n-1} a$$

$$\psi^k(a) = k^n a.$$

We shall show that γ^k acts on $A(n)/A(n+1)$ as multiplication with a certain constant $c(k,n)$ independent of the ring A. From $\gamma^k(x) = \lambda^k(x+k-1)$ one computes

$$(3.8.10) \qquad c(k,n) = \sum_{i=1}^{k} (-1)^{i-1} i^{n-1} \binom{k-1}{k-i} .$$

In order to analyse these numbers we put

$$\gamma_t(x) = 1 + f_n(t)x$$

where

$$f_n(t) = \sum_{j=1}^{\infty} (-1)^{j-1} j^{n-1} \left(\frac{t}{1-t}\right)^j$$

is a certain formal power series in $Z[[t]]$. For $n = 1$ this is a geometric series with sum

$$f_1(t) = t .$$

If we differentiate $f_n(t)$ formally with respect to t we obtain the recursion formula

$$f_{n+1}(t) = t(1-t)\, f_n'(t)$$

so that $f_n(t)$ is actually a polynomial of degree n

$$f_n(t) = \sum_{j=1}^{n} c(j,n) t^j .$$

In particular $\gamma^m = o$ on $A(n)/A(n+1)$ for $m > n$.

3.9. The operation ρ_k.

We describe a variant of the Bott map θ_k for p-adic γ-rings A. A topology shall always be the p-adic topology if not otherwise specified.

A series $\sum_{r \geq 1} a_r$, with $a_r \in A(r)$, converges in the p-adic topology since it converges in the filtration topology $(A(n) \mid n \geq 1)$ which is finer. Therefore the set $1 + A$ of symbols $1 + a$, $a \in A$, with multiplication $(1+a)(1+b) = 1+a+b+ab$ is a group. It is a compact, topological group, with neighbourhood basis of 1 given by $(1+p^nA \mid n \geq 0)$, or equivalently $(1+p^nA+A(n) \mid n \geq 1)$.

Let k be a natural number prime to p. Consider $Z_p[\zeta_k]$ where ζ_k is a primitive k-th root of unity in an algebraic closure of the p-adic numbers. The product $\prod(1-u)$ over all roots u of $t^k - 1 = 0$ except 1 is equal to k, hence a unit in Z_p. Therefore 1-u is a unit in $Z_p[\zeta_k]$ and hence $u/(u-1) \in Z_p[\zeta_k]$. The series

$$\gamma_{u/(u-1)}(a) = 1 + \gamma^1(a)\, u/(u-1) + \gamma^2(a)(u/u-1)^2 + \dots$$

converges in the p-adic topology on $1 + A \otimes_{Z_p} Z_p[\zeta_k]$ hence defines an element $\gamma_{u/(u-1)}(a)$ in this multiplicative group. We define

(3.9.1) $$\rho_k(a) = \prod \gamma_{u/(u-1)}(a) \in 1 + A \otimes Z_p[\zeta_k]$$

where the product is taken over all roots of $t^k - 1 = 0$ except 1. The Z_p-algebra $Z_p[\zeta_k]$ is free as Z_p-module with $Z_p \cdot 1$ as a direct summand; therefore $A = A \otimes_{Z_p} Z_p \subset A \otimes_{Z_p} Z_p[\zeta_k]$ as a subring. (As to the freeness of the module: Let $L \in Q_p[t]$ be an irreducible polynomial with $L(\zeta_k) = 0$. Then L divides the cyclotomic polynomial ϕ_k. Since Z_p is factorial we can choose for L a monic polynomial in $Z_p[t]$, by the Gauß-Lemma. Then $Z_p[\zeta_k] \cong Z_p[t]/L$ and the right-hand side is clearly a free module.) We claim: $\rho_k(a) \in 1 + A$. This follows from the fact that a coefficient of a monomial in the $\gamma^i(a)$ in the expansion of $\rho_k(a)$ according to definition 3.9.1 is symmetric in the roots of $t^k - 1 = 0$ (compare 3.7).

<u>Proposition 3.9.2.</u>

<u>The map</u>

$$\rho_k : A \longrightarrow 1 + A$$

<u>from the additive compact group</u> A <u>into the multiplicative compact group</u> 1 + A <u>is a continuous homomorphism. It commutes with the Adams operations</u>

<u>and maps</u> A(n) <u>into</u> 1 + A(n).

<u>Proof.</u>

ϱ_k is a homomorphism: directly from 3.3.2 and 3.9.1. Since $\varrho_k(p^n a) = (\varrho_k(a))^{p^n}$ and $(1+a)^{p^n} \in 1 + p^N + A(N)$ if $\binom{p^n}{i} \equiv 0 \bmod p^N$ for $1 \leq i \leq N$ we see that ϱ_k is p-adically continuous. Since ψ^j commutes with the γ^i it commutes with ϱ_k. Since A(n) is a γ-ideal $\varrho_k A(n) \subset 1 + A(n)$.

<u>Remark 3.9.3.</u>

If A is a ring without identity we can adjoin an identity in the standard manner: On the additive group $Z \times A$ define a multiplication $(m,a)(n,b) = (mn, mb+na+ab)$. Then $1 + A = \{(1,a) \mid a \in A\} \subset Z \times A$. If $B \subset A$ is an ideal and if 1 + B and 1 + A are groups then $(1+A)/(1+B) \cong 1 + A/B$.

<u>3.10. Oriented γ-rings.</u>

A γ-ring A is said to be <u>oriented</u> if

$$(3.10.1) \qquad \gamma_t(a) = \gamma_{1-t}(a), \qquad a \in A.$$

This terminology has the following reason: Suppose A is the augmentation ideal of the special augmented finite-dimensional λ-ring R. Then

<u>Proposition 3.10.2.</u>

A <u>is oriented if and only if for every finite-dimensional element</u> x, <u>of dimension</u> n <u>say</u>, $\lambda^r(x) = \lambda^{n-r}(x)$ <u>for all</u> r.

<u>Proof.</u>

If 3.10.1 is satisfied for a_1 and a_2 then for $a_1 - a_2$ too. The equation $\lambda^r(x) = \lambda^{n-r}(x)$ implies $\lambda_t(x) = t^n \lambda_{1/t}(x)$ and this yields

$$\gamma_t(x-n) = \lambda_{t/(1-t)}(x-n) = \lambda_{t/(1-t)}(x)(1-t)^n$$

$$= t^n \lambda_{(1-t)/t}(x)$$

$$\gamma_{1-t}(x-n) = \lambda_{(1-t)/t}(x-n) = \lambda_{(1-t)/t}(x)(1+(1-t)/t)^{-n}$$

$$= t^n \lambda_{(1-t)/t}(x) .$$

Note that n must be the augmentation of an n-dimensional element x because $\lambda^n(x) = 1$, so that $x-n \in A$. The same calculation gives $\lambda^r(x) = \lambda^{n-r}(x)$ from 3.10.1.

We call R an <u>oriented</u> λ-<u>ring</u> if $\lambda^r(x) = \lambda^{n-r}(x)$ whenever x is n-dimensional.

<u>Example 3.10.3.</u>

Let $KO_G(X)$ be the Grothendieck ring of real G-vector bundles over the compact G-space X where G is a compact Lie group. An n-dimensional G-vector bundle E is called orientable if the n-th exterior power $\Lambda^n E$ is the G-vector bundle $X \times \mathbb{R} \longrightarrow X$ with trivial G-action on $\mathbb{R}$. If E is orientable then $\Lambda^r E \cong \Lambda^{n-r} E$. Hence

$$KSO_G(X) = \{ E - F \in KO_G(X) \mid E,F \text{ orientable} \}$$

is an oriented λ-ring and the associated augmentation ideal is an oriented γ-ring.

If x is a one-dimensional element in the oriented λ-ring then $\lambda^1(x) = \lambda^o(x) = 1$. Therefore one should think of such a ring as containing essentially only even-dimensional elements.

We now consider a refinement of the operations Θ_k (resp. ς_k) for an oriented λ-ring R (a p-adic oriented γ-ring A).

Let $x \in R$ be an element of dimension 2m. Let k be an odd integer. Let J a set of k-th roots of unity $u \neq 1$ which contains from each pair u, u^{-1} exactly one element. (Since $k \equiv 1(2)$ we have $u \neq u^{-1}$.) The product $k^m \prod_{u \in J}(1-u)^{-2m}$ is an algebraic integer because $\prod_{u \neq 1}(1-u) = k$. Therefore

$$(3.10.4) \qquad k^m \prod_{u \in J} \lambda_{-u}(x)(1-u)^{2m} \in R[\zeta_k]$$

where ζ_k is a primitive k-th root of unity. The fact that R is oriented implies

$$(3.10.5) \qquad \lambda_{-u}(x)(1-u)^{-2m} = \lambda_{-1/u}(x)(1-1/u)^{-2m}.$$

Therefore 3.10.4 is independent of the choice of J. We call this element

$$\Theta_k^{or}(x) .$$

<u>Proposition 3.10.6.</u>

(i) *If x and y are even-dimensional then* $\Theta_k^{or}(x+y) = \Theta_k^{or}(x)\,\Theta_k^{or}(y)$.

(ii) *The square of* $\Theta_k^{or}(x)$ *is* $\Theta_k(x)$.

(iii) $\Theta_k^{or}(x) \in R$.

<u>Proof.</u>

(i) follows directly from the analogous property of λ_t. (ii) follows from the definitions, using 3.10.5. (iii) Using 3.10.5 again one can see that $\Theta_k^{or}(x)$ is formally invariant under the Galois group of $Q(\zeta_k)$ over Q.

If A is an oriented p-adic γ-ring one defines the square root of ρ_k by

$$\rho_k^{or}(x) = \prod_{u \in J} \gamma_{u/u-1}(x) \text{ (3.10.7)}$$

Using $\gamma_t = \gamma_{1-t}$ one shows that the following holds

<u>Proposition 3.10.8.</u>

(i) $\rho_k^{or}(x+y) = \rho_k^{or}(x)\ \rho_k^{or}(y)$.

(ii) <u>The square of</u> $\rho_k^{or}(x)$ <u>is</u> $\rho_k(x)$.

(iii) $\rho_k^{or}(x) \in 1 + A$.

We now compute $\theta_k^{or}(z)$ for a two-dimensional element z. We have $\lambda_{-u}(z) = 1 - uz + u^2$. If we formally write z = x+y with xy = 1 then $\lambda_{-u}(z) = (1-ux)(1-uy)$ and therefore

$$\lambda_{-u}(z)(1-u)^{-2} = y\ \frac{1-ux}{1-u} \cdot \frac{1-u^{-1}x}{1-u^{-1}} \text{ (3.10.9)}$$

If we multiply these expressions according to the definition of $\theta_k^{or}(z)$ we obtain

$$\begin{aligned} \theta_k^{or}(z) &= ky^{(k-1)/2} \prod_u (1-ux) \prod_u (1-u)^{-1} \\ &= y^{(k-1)/2}(1+x+\ldots+x^{k-1}) \\ &= (x^{(k-1)/2} + x^{(k-3)/2} + \ldots + y^{(k-1)/2}). \end{aligned} \text{ (3.10.10)}$$

This last expression may also be written

$$\frac{x^{k/2} - x^{-k/2}}{x^{1/2} - x^{-1/2}} \text{ (3.10.11)}$$

where we use this at this point merely as a suggestive formula without having $x^{1/2}$ defined. Actually $\Theta_k^{or}(z)$ is an integral polynomial in z: The polynomial

$$P_k(t) = \prod_{u \in J} (t-(u+u^{-1}))$$

is contained in $Z[t]$ and has degree $(k-1)/2$, e. g. $P_3(t) = 1+t$, $P_5(t) = -1+1+t^2$. One has for a 2-dimensional z

$$(3.10.12) \qquad \Theta_k^{or}(z) = P_k(z).$$

A proof follows from the identity

$$t^{k-1} P_k(t^2+t^{-2}) = (1+t+...+t^{2k-1})/(1+t)$$

which can be seen by observing that both sides are monic polynomials of degree 2k-2 having the 2k-th roots of unity $\neq \pm 1$ as roots.

From 3.10.10 one obtains for a 2-dimensional z the identiy

$$(3.10.13) \qquad \Theta_k^{or}(z) = 1 + \psi^1 z + \psi^2 z + ... + \psi^{(k-1)/2} z .$$

3.11. The action of ϱ_k on scalar γ-rings.

We consider p-adic γ-rings A with trivial multiplication, like $A(n)/A(n+1)$ in Proposition 3.8.9, on which ψ^k is multiplication by k^n and λ^k multiplication by $(-1)^{k-1}k^{n-1}$. Then we have seen in 3.8. that

$$\gamma_t(x) = 1 + f_n(t)x$$

where $f_n(t)$ in an integral polynomial defined by the recursion formula

$$f_1(t) = t, \qquad f_{n+1}(t) = t(1-t)f_n'(t).$$

Therefore ρ_k is given by

$$\rho_k(x) = \prod_u (1+xf_n(\tfrac{u}{u-1})) = 1 + x \sum_u f_n(\tfrac{u}{u-1})$$

We have to compute the rational number (Galois theory)

$$\sum_u f_n(\tfrac{u}{u-1}) =: b_n(k) \ ,$$

the sum being taken over the k-th roots of unity $u \neq 1$. Put $h_n(t) = f_n(\frac{t}{t-1})$.

<u>Proposition 3.11.1.</u>

<u>We have the following identity between formal power series in</u> x <u>and</u> t <u>over</u> Q

$$\log(1 + \tfrac{t}{1-t}(1-e^x)) = \sum_{n \geq 1} h_n(t) \frac{x^n}{n!} .$$

(The meaning of the left hand side is: Use the power series $\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \ldots$ and replace y with the power series $\frac{t}{1-t}(1-e^x)$ which has no constant term.)

<u>Proof.</u>

We put

$$K(t,x) := \log(1+ \tfrac{t}{1-t}(1-e^x)) = \sum_{n \geq 1} g_n(t) \frac{x^n}{n!}$$

where the $g_n(t)$ are certain power series in t. We differentiate K(t,x) with respect to t and x and obtain

$$\frac{dK}{dt} = \frac{e^x}{te^x - 1} + \frac{1}{1-t} , \quad \frac{dK}{dx} = \frac{te^x}{te^x - 1}$$

hence

$$t \frac{dK}{dt} - \frac{dK}{dx} = \frac{t}{1-t} .$$

We apply this differential equation to $\sum_{n \geqslant 1} g_n(t) \frac{x^n}{n!}$ and compare coefficients, thus obtaining

$$g_1(t) = - \frac{t}{1-t}$$

$$g_n(t) = tg'_{n-1}(t)$$

and these are precisely the recursion formulas for the h_n.

If we replace t in 3.11.1 with a k-th root of unity $u \neq 1$ we obtain an identity between formal power series in x over $Q(\zeta_k)$. We compute the $b_n(k)$ as follows

$$\sum_{n \geqslant 1} b_n(k) \frac{x^n}{n!} = \sum_{u \neq 1} \log \frac{1-ue^x}{1-u}$$

$$= \log \prod_{u-1} \frac{1-ue^x}{1-u} = \log \frac{1}{k} (1+e^x+\ldots+e^{(k-1)x})$$

$$= \log \frac{e^{kx}-1}{kx} - \log \frac{e^x-1}{x}$$

$$= \sum_{n \geqslant 1} (k^n - 1) a_n \frac{x^n}{n!}$$

if we use the expansion $\log \frac{e^x-1}{x} = \sum_{n \geqslant 1} a_n \frac{x^n}{n!}$.

The a_n are easily expressed in terms of <u>Bernoulli numbers</u> B_m which are defined by

$$\frac{t}{e^t - 1} = 1 + \sum_{m \geq 1} B_m \frac{t^m}{m!} .$$

This yields immediately $B_1 = -\frac{1}{2}$, $B_{2m+1} = 0$ for $m \geq 1$. If we differentiate the defining series of the a_n with respect to x we obtain

$$\sum_{n \geq 1} na_n \frac{x^{n-1}}{n!} = 1 - \frac{1}{x} + \sum_{n \geq 0} B_n \frac{x^{n-1}}{n!}$$

and then

$$a_n = \frac{B^n}{n} \quad \text{for} \quad n > 1,\ a_1 = \frac{1}{2} .$$

Collecting these computations we obtain

<u>Proposition 3.11.2.</u>

$\rho_k : A(n)/A(n+1) \longrightarrow 1 + A(n)/A(n+1)$ <u>is the map</u>

$$x \longmapsto 1 + (k^n - 1) \frac{B_n}{n} x .$$

We now come to oriented γ-rings. From the recursion formula for the rational functions $h_n(t)$ one proves by induction

$$h_m(t^{-1}) = (-1)^m h_m(t) \tag{3.11.3}$$

$$f_m(t) = (-1)^m f_m(t) .$$

The previous calculations yield

Proposition 3.11.4.

Let A *be an oriented* p-*adic* γ-*ring. Then*
ρ_k^{or} : A(2n)/A(2n+1) $\longrightarrow$ 1 + A(2n)/A(2n+1) *is the map*

$$x \longmapsto 1 + (k^{2n}-1) \frac{B_{2n}}{4n} x .$$

Remark 3.11.5.

Equating coefficients in $\sum \gamma^r(a)t^r = \sum \gamma^r(a) \ (1-t)^r$ one finds

$$\gamma^k = (-1)^k \gamma^k + (-1)^k(k+1) \gamma^{k+1} + c$$

where c has γ-filtration at least k+2. This gives by induction A(2n-1) = A(2n) for $n \geq 1$.

3.12. The connection between θ_k and ρ_k.

The map θ_k was only defined for finite-dimensional elements x. In order to extend it to negatives of such elements one must have that $\theta_k(x)$ is a unit. This can sometimes be accomplished by passing to the p-adic completion. We describe the formal setting.

Let R be an augmented special λ-ring with augmantation e : R $\longrightarrow$ Z and augmentation ideal B = ker e. Moreover we assume:

(i) R is finitely generated as an abelain group by $x_1 = 1, x_2,\ldots,x_m$ which are finite-dimensional.

(ii) $e(x_r) = \dim x_r$ for r = 1,...,m.

(iii) The γ-topology on B is finer than the p-adic topology.

We then have e(x) = dim x whenever x is finite-dimensional and moreover $\gamma_t(x-e(x))$ is a polynomial in t of degree $\leq$ dim x, hence

$$\gamma\text{-}\dim (x-e(x)) \leq \dim x.$$

Proposition 3.8.5 shows that the B-adic topology coincides with the γ-topology. The ring $A = B \otimes Z_p$ is a p-adic γ-ring, by (iii) above.

Proposition 3.12.1.

Let $i : R \longrightarrow R \otimes Z_p$ be the canonical map and $(k,p) = 1$. Then for finite-dimensional $x \in R$ the element $i\,\theta_k(x)$ is a unit in $R \otimes Z_p$.

Proof.

If dim $x = n$ then $e\theta_k x = \theta_k ex = \theta_k n = k^n$. Put $r = k^n$, then $(r,p) = 1$ and r^{-1} exists in Z_p. Therefore $r^{-1} i\theta_k x = 1+a$, $a \in B \otimes Z_p$. But $1+A \subset B \otimes Z_p$ is a multiplicative subgroup. If $(1+a)(1+b) = 1$ then $r^{-1}(1+b)$ is the inverse of $i\theta_k x$.

We may now extend θ_k to a homomorphism $R \longrightarrow Z_p \otimes R$. If $e' : R \otimes Z_p \longrightarrow Z_p$ is induced by $e : R \longrightarrow Z$ then, for x,y finite-dimensional

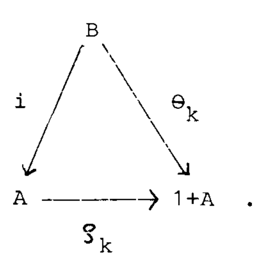
$$e'\theta_k(x-y) = k^{ex-ey} .$$

Therefore θ_k induces a homomorphism

$$\theta_k : B \longrightarrow 1 + A, \qquad A = B \otimes Z_p.$$

Proposition 3.12.2.

The following diagram is commutative:



$(k,p) = 1$

Proof.

Let m = dim x. Then $\gamma_t(x-m)$ is a polynomial of degree $\leq m$. Using

$$\gamma_{t/(t-1)}(x-m)\ \lambda_{-t}(m) = \lambda_{-t}(x)$$

and the definition of Θ_k and ρ_k we obtain

$$\Theta_k(x) = \rho_k\, i(x-m)\ \Theta_k(m)$$

and hence $\Theta_k(x-m) = \rho_k\, i(x-m)$. This suffices for the proof.

3.13. Decomposition of p-adic γ-rings.

Let A be a p-adic γ-ring. A fundamental system of neighbourhoods of zero for the p-adic topology may be taken as $(p^n A + A(n) \mid n \geq 1)$. The natural numbers $\mathbb{N}$ are considered as a (dense) subset of the p-adic numbers.

Proposition 3.13.1.

The map

$$\mathbb{N} \times A \longrightarrow A : (k,a) \longmapsto \psi^k(a)$$

is uniformly continuous.

Proof.

Let M = 2N and suppose p^M divides s. If $x_1,\ldots,x_r$ have γ-dimension one then

$$\psi^{k+s}(\Sigma x_i) - \psi^k(\Sigma x_i) = \Sigma(1+x_i)^k((1+x_i)^s-1)$$

$$= p^N S_1 + S_N$$

where S_j is a symmetric function of weight $\geq j$ in the x_i for $j = 1,N$. Hence given $N \geq 1$ we have shown that there exists $M \geq 0$ such that $p^M | s$ implies

$$\psi^{k+s}(x) - \psi^k(x) \in p^N A + A(N)$$

for all x which are a sum of elements of γ-dimension one. By the verification principle for special γ-rings this holds for all x. Hence our map is uniformly continuous in the first variable. Since it is a homomorphism in the second variable it is uniformly continuous.

We can now extend the map $(k,a) \longmapsto \psi^k(a)$ by continuity to a map $Z_p \times A \longrightarrow A$, denoted with the same symbol. Therefore $\psi^k : A \longrightarrow A$ is defined for all $k \in Z_p$ as a continuous homomorphism. Moreover we still have $\psi^k \psi^l = \psi^{kl}$. If Γ denotes the compact topological group of p-adic units then A becomes a topological Γ-module.

By Hensel's Lemma Z_p contains the roots of $x^{p-1} - 1 = 0$. This is a cyclic group of order p-1 generated by d, say. The additive group A splits into eigenspaces of ψ^d

(3.13.2)
$$A = \bigoplus_{i=0}^{p-2} A_i$$
$$A_i = \{ x \in A \mid \psi^d x = d^i x \} .$$

(This is so because A may be considered as $Z_p[C]$ module, where C is the cyclic group generated by T and T acting as ψ^d; and the group algebra $Z_p[C]$ splits completely because Z_p contains the (p-1)-th roots of unity). Since ψ^d is a ring homomorphism we have

(3.13.3)
$$A_i A_j \subset A_{i+j}$$

so that A becomes a Z/(p-1)-graded ring. Let U be the kernel of the reduction mod p $Z_p^* \longrightarrow Z/pZ$. Then U acts on each group A_i because $u \in U$ commutes with ψ^d. Put

(3.13.4) $$A_i(n) = A_i \cap A(n).$$

Then

Proposition 3.13.5.

$A_i(n) = A_i(n+1)$ *if* $n \not\equiv i \bmod (p-1)$.

Proof. It follows from 3.8.9 that ψ^d acts on $A_i(n)/A_i(n+1)$ as multiplication by d^n. On the other hand, by definition of A_i, it acts as multiplication by d^i. Hence if the quotient is non-zero we must have $n \equiv i \bmod (p-1)$.

3.14. The exponential isomorphism ϱ_k.

We now come to the main result in the theory of p-adic γ-rings which says that ϱ_k is an isomorphism if k generates the p-adic units ($p \neq 2$). This is the algebraic reformulation of Atiyah-Tall [14] of the theorem $J'(X) = J''(X)$ of Adams [2] , which is one essential step in the computation of the group J(X) of stable fibre homotopy classes of vector bundles over X.

Let A be a p-adic γ-ring. The group Z_p^* is topologically cyclic if $p \neq 2$. An integer k is a topological generator if and only if k generates $(Z/p^2)^*$.

Theorem 3.14.1.

Let A be a p-adic γ-ring ($p \neq 2$). Assume that $A(n) = A(n+1)$ for $n \not\equiv 0 \bmod p-1$. Let k generate the p-adic units. Then

$$\varrho_k : A \longrightarrow 1 + A$$

<u>is an isomorphism</u>.

<u>Proof.</u>

We have A = inv lim A/A(n), 3.8.7. We have a commutative diagram with exact rows (see 3.9.2 and 3.9.3)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(n)/A(n+1) & \longrightarrow & A/A(n+1) & \longrightarrow & A/A(n) & \longrightarrow & 0 \\ & & \downarrow \varrho_k & & \downarrow \varrho_k & & \downarrow \varrho_k & & \\ 0 & \longrightarrow & 1 + A(n)/A(n+1) & \longrightarrow & 1 + A/A(n+1) & \longrightarrow & 1 + A/A(n) & \longrightarrow & 0 \end{array}$$

Therefore it suffices to prove the theorem for A(n)/A(n+1). In that case ϱ_k is the map $a \mapsto 1+d(k,n)a$ where $d(k,n) \in Z_p$ is independent of the particular ring, hence is an isomorphism if d(k,n) is a unit. By assumption we only have to consider the case $n \equiv 0$ mod p-1. We have computed the numbers d(k,n) in 3.11.2 and it follows from the Clausen-von Staudt Theorem (Borewicz-Safarevic [30] , p. 410) that d(k,n) is a unit in Z_p if k is a p-adic generator and $n \equiv 0$ (p-1), Actually it has been observed by Atiyah-Tall [14] , p. 283 that the results of 3.11 and the Clausen-von Staudt theorem is not necessary. One only needs to produce a p-adic γ-ring such that $A(n)/A(n+1) \neq 0$ for $n \equiv 0$ (p-1) and ϱ_k is an isomorphism. We shall describe such an example in a moment and thereby completing the proof of Theorem 3.14.1.

<u>Example 3.14.2.</u>

Let R(Z/p;Q) be the Grothendieck ring of Q[Z/p]-modules. There are two irreducible modules: The trivial module 1, and V which splits as

$W + W^2 + \ldots + W^{p-1}$ over the complex numbers. Hence the augmentation ideal I is the free group on a single generator $x = 1 + W + \ldots + W^{p-1} - p$. By 3.5 the Adams operations are given as follows: $\psi^k = \mathrm{id}$ if $(k,p)=1$, $\psi^k = 0$ if p/k. Evaluation of characters at a generator g of Z/p gives an isomorphism $I \longrightarrow pZ : x \longmapsto -p$. We have

$$\gamma_t(x) = \prod_{i=1}^{p-1} \gamma_t(W^i - 1) = \prod_i ((1-t) + W^i t),$$

and evaluating at g maps the right hand polynomial (short calculation) into $(1-t)^p - (-t)^p$. Therefore $\gamma^r(-p) = 0$ for $r \geq p$ and $p \mid \gamma^r(-p)$ for $1 \leq r \leq p-1$. Since ψ^p acts on I_n/I_{n+1} as multiplication by p^n and $\psi^p = 0$ we see that I_n/I_{n+1} is a p-group (cyclic in this case). Moreover I_n/I_{n+1} is non-zero only if $n \equiv 0\ (p-1)$ because ψ^k, $(k,p)=1$, acts as k^n and as identity. Since $\gamma^{p-1}(-p) = (-1)^{p-1} p$ the lowest power of p attainable in I_n is $(\gamma^{p-1}(-p))^v$ where $(v-1)(p-1) < n \leq v(p-1)$. Hence $I_n/I_{n+1} = Z/p$ for $n \equiv 0\ (p-1)$ and the p-adic topology and the γ-topology coincide. We now compute ρ_k on $I_n/I_{n+1} \otimes Z_p \cong I_n/I_{n+1}$ for $n \equiv 0\ (p-1)$. A generator for I_n/I_{n+1} is the image of p^r. Hence

$$\rho_k(p) = \rho_k(-p)^{-1} = \prod_u \left(\left(1 - \frac{u}{u-1}\right)^p - \left(\frac{u}{1-u}\right)^p \right)^{-1}$$

$$= \prod_u \frac{(1-u)^p}{1-u^p} = k^{p-1} = 1 + \frac{k^{p-1}-1}{p} \cdot p$$

Since k generates the p-adic units $m = p^{-1}(k^{p-1}-1)$ is an integer prime to p. We obtain

$$\rho_k(p^r) = \rho_k(p)^{p^{r-1}} = (1+mp)^{p^{r-1}} \equiv 1 + mp^r \bmod p^{r+1}$$

so that ρ_k is on I_n/I_{n+1} the map $\rho_k(a) = 1+ma \in 1 + I_n/I_{n+1}$. Since $I_n/I_{n+1} = Z/p$ this is an isomorphism.

<u>Remark 3.14.3.</u>

We know from 3.11. that for $n = r(p-1)$ ϱ_k in the example above is the map $a \mapsto 1 + (k^n-1)\frac{B_n}{n} a$ and that pB_n is p-integral. We obtain $m \equiv (k^n-1)\frac{B_n}{n} \equiv ((1+mp)^r-1)\frac{B_n}{n} \equiv mrp\frac{B_n}{n} \equiv -m(pB_n) \bmod p$. Hence $pB_n \equiv -1 \bmod p$. This is one of the von Staudt congruences.

We now describe certain instances where the hypothesis of Theorem 3.14.1 is fulfilled.

Let A be any p-adic γ-ring. In 3.13 we have described a splitting of A into eigenspaces A_i of Adams operations $(i = 0,1,\dots,p-2)$. Then ϱ_k induces a map

$$\varrho_k : A_o \longrightarrow 1 + A_o$$

and by 3.13.5 we can apply the Theorem to it:

<u>Proposition 3.14.4.</u>

Let A be a p-adic γ-ring, $p \neq 2$. Let k be a generator of the p-adic units. Then

$$\varrho_k : A_o \longrightarrow 1 + A_o$$

is an isomorphism.

<u>Proposition 3.14.5.</u>

Let A be a p-adic γ-ring. Assume that $\psi^k = id$ for $(k,p) = 1$. Then $A(n)/A(n+1) = 0$ *for* $n \neq 0\ (p-1)$.

<u>Proof.</u>

For $x \in A(n)/A(n+1)$ we have $x = \psi^k x = k^n x$ and $k^n-1 \in Z_p^*$ for $n \not\equiv 0(p-1)$.

Let A be a p-adic γ -ring. Put

(3.14.6) $$A^{\Gamma} = \{a \mid \psi^k a = a, \text{ all } k\}$$

$$A_{\Gamma} = A/N,\ N = \{a - \psi^k a \mid a \in A, \text{ all } k\}.$$

$$(1+A)^{\Gamma} = \{1+a \mid \psi^k a = a, \text{ all } k\}$$

$$(1+A)_{\Gamma} = (1+A)/M,\ M = \{(1+a)/\psi^k(1+a) \mid a \in A, \text{ all } k\}.$$

Since ς_k commutes with the Adams operations we have induced maps

(3.14.7)
$$(\varsigma_k)^{\Gamma} : A^{\Gamma} \longrightarrow (1+A)^{\Gamma}$$

$$(\varsigma_k)_{\Gamma} : A_{\Gamma} \longrightarrow (1+A)_{\Gamma}$$

Theorem 3.14.8.

If $p \neq 2$ and k is a generator of the p-adic units then the maps 3.14.7

$$(\varsigma_k)^{\Gamma} \text{ and } (\varsigma_k)_{\Gamma}$$

are isomorphisms.

Proof.

One first shows: If $0 \to X \to Z \to Y \to 0$ is an exact sequence of p-adic γ -rings and the Theorem is true for X and Y, then it is true for Z. The following diagram with exact rows (ker- coker sequences) is commutative

$$\begin{array}{ccccccccccccccc}
0 \to & X^{\Gamma} & \to & Z^{\Gamma} & \to & Y^{\Gamma} & \to & X_{\Gamma} & \to & Z_{\Gamma} & \to & Y_{\Gamma} & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & (1+X)^{\Gamma} & \to & (1+Z)^{\Gamma} & \to & (1+Y)^{\Gamma} & \to & (1+X)_{\Gamma} & \to & (1+Z)_{\Gamma} & \to & (1+Y)_{\Gamma} & \to 0.
\end{array}$$

One applies the five lemma. (To establish the ker- coker sequence note that

$$0 \longrightarrow X^{\Gamma} \longrightarrow X \xrightarrow{1-\psi^k} X \longrightarrow X_{\Gamma} \longrightarrow 0$$

is exact if k is a generator of the p-adic units). The Theorem is true for A(n)/A(n+1): For $n \not\equiv 0(p-1)$ $A(n)/A(n+1)^{\Gamma} = 0$, $(A(n)/A(n+1))_{\Gamma} = 0$; for $n \equiv 0(p-1)$ φ_k itself is already an isomorphism by 3.14.1. By the first part of the proof the Theorem is true for all A/A(n). From

$$\text{inv lim } (A/A(n)^{\Gamma}) = (\text{inv lim } A/A(n))^{\Gamma}$$

and an analogous equality for (1+A)/(1+A(n)) the Theorem for A follows. (Note that "invlim" is exact on compact groups.)

We now discuss analogous results for p = 2 where oriented γ-rings are needed. The group of 2-adic units $\Gamma = Z_2^*$ is not (topologically) cyclic, but $\Gamma/\{\pm 1\}$ is; e.g. 3 is a generator. Since $-1 \in Z_p$ the operation ψ^{-1} is defined for p-adic γ-rings, see 3.13.

Proposition 3.14.9.

If A is an oriented p-adic γ-ring then $\psi^{-1} = \text{id}$.

Proof.

If x has γ-dimension 1 then 1+x has λ-dimension 1. Therefore

$$1 = \lambda^0(2+2x) = \lambda^2(2+2x) = \lambda^1(1+x)^2 = (1+x)^2$$

so that $\psi^{-1}(x) = \frac{1}{1+x} - 1 = x$. Hence the Proposition is true for a sum of one-dimensional elements. Now apply a "verification principle".

Theorem 3.14.10.

Let A be an oriented p-adic γ-ring (p any prime). Let k be a generator of $\Gamma/\{\pm 1\}$. Then

$$\rho_k^{or} : A \longrightarrow 1 + A$$

induces isomorphisms

$$(\rho_k^{or})^{\Gamma} \quad \text{and} \quad (\rho_k^{or})_{\Gamma} .$$

If p = 2 then ρ_k^{or} is an isomorphism.

Proof.

Let p = 2. We have to show that A(n)/A(n+1) is mapped isomorphically. By 3.11.5 this group is zero if $n \equiv 1 \bmod 2$. So let n = 2m. Then $\rho_k^{or}(a) = 1 + d'(k,n)a$ and $d'(k,n) = (k^n-1)\frac{B_n}{2n} \in Z_2$ by 3.11.4. In this case if $n = 2^r d$, d odd and $r \geq 1$, then $k^n = 1 + 2^{r+2}c$, c odd, because k is a generator of $Z_2^*/\{\pm 1\}$. Hence $(k^n-1)\frac{B_n}{2n} = \frac{c}{d}\, 2B_n$ and by the Clausen-von Staudt theorem $2B_{2m} \equiv -1 \bmod 2$. Therefore $d'(k,n) \in Z_2^*$. If one wants to avoid the Clausen-von Staudt theorem one can compute ρ_k^{or} in a special case as in 3.14.2. For $p \neq 2$ $2d'(k,n) = d(k,n) \in Z_p^*$ hence $d'(k,n) \in Z_p^*$. So one can proceed as in the proof of 3.14.8.

3.15. Thom-isomorphism and the maps θ_k, θ_k^{or}.

Let G be a compact Lie group, $E \longrightarrow X$ a complex G-vector bundle over the compact G-space X. If M(E) is the Thom space of E we have the Thom class $t(E) \in \tilde{K}_G(M(E))$ and $\tilde{K}_G(M(E))$ is a free $K_G(X)$-module with a single generator t(E). Therefore we must have a relation of the type $\psi^k t(E) = \tilde{\theta}_k(E)t(E)$ with a uniquely determined element $\tilde{\theta}_k(E) \in K_G(X)$.

Proposition 3.15.1.

The equality $\Theta_k(E) = \tilde{\Theta}_k(E)$ holds.

Proof. Both Θ_k and $\tilde{\Theta}_k$ are natural for bundle maps and homomorphic from addition to multiplication. By the topological splitting principle it therefore suffices to proof the equality for line bundles E. Let $s^* : \tilde{K}_G(ME) \longrightarrow K_G(X)$ be induced by the zero section. Then $s^* t(E) = 1-E$ and therefore $1-E^k = \psi^k(1-E) = s^* \psi^k t(E) = s^*(\tilde{\Theta}_k(E)t(E)) = \tilde{\Theta}_k(E)(1-E)$. This implies $\Theta_k(E) = 1+E+\ldots+E^{k-1}$ (look e. g. at X a complex projective space). Now use 3.7.2.

For real vector bundles and Θ_k^{or} the situation is analogous but slightly more complicated. We describe the ingredients. Let $E \longrightarrow X$ be a real G-vector bundle of dimension 8n which has a Spin(8n)-structure. With this Spin-structure one defines a Thom-class $t(E) \in \tilde{KO}_G(M(E))$ and the generalized Bott periodicity (Atiyah [10]) says that again $\tilde{KO}_G(M(E))$ is a free $KO_G(X)$-module on t(E). We define $\tilde{\Theta}_k^{or}(E)$ by the equation $\psi^k t(E) = \tilde{\Theta}_k^{or}(E)t(E)$. If k is odd then we also have defined in 3.10 the element $\Theta_k^{or}(E)$ because E, having a Spin-structure, is orientable.

Proposition 3.15.2.

For k odd and E a G-vector bundle with Spin(8n)-structure the equality $\Theta_k^{or}(E) = \tilde{\Theta}_k^{or}(E)$ holds. In particular $\tilde{\Theta}_k^{or}(E)$ is independent of the Spin-structure for odd k.

Proof. Using 3.10.10 a proof is contained in Bott [31] , Proposition 10.3, Theorem B on p. 81 and Theorem C" on p. 89.

3.16. Comments.

This section is based on Atiyah-Tall [14] . That paper axiomatizes certain basic results of Adams [1] , [2] . The reader should also study the relation-ship between λ-rings, formal groups, Witt-vectors, and Hopf-algebras (Hazewinkel [95]). It would be interesting to investigate the topological significance of the number theoretical properties of the Bernoulli numbers. We also mention the exponential isomorphism for λ-rings obtained in Atiyah-Segal [13] ; this is related to ρ_k but gives an isomorphism on the whole ring (under a suitable hypothesis).

3.17. Exercises.

1. Show that the tensor product of special λ-rings A,B is a special λ-ring in a canonical way such that the maps $A \longrightarrow A \otimes B$, $B \longrightarrow A \otimes B$ are λ-homomorphisms.
2. Show that there exists a free special λ-ring U on one generator $u \in U$. This ring is characterized by the following universal property: Given a special λ-ring R and $x \in R$ there is a unique homomorphism $f : U \longrightarrow R$ of λ-rings such that $f(u) = x$.
3. Show that if R is special λ-ring and $x \in R$ n-dimensional then there exists a special λ-ring $S \supset R$ such that $x = x_1 + \ldots + x_n$ where the $x_i \in S$ are one-dimensional (splitting principle).
4. If S is a finite G-set let $\Lambda^i(S)$ be the set of subsets $M \subset S$ with $|M| = i$. The G-action on S induces a G-action on $\Lambda^i(S)$. Show that the $S \longmapsto \Lambda^i(S)$ induce a λ-ring structure on A(G). This structure is in general not special.