8. Equivariant Homotopy Theory

8.1. Generalities.

Let G be a compact Lie group. We consider various categories obtainable from G-spaces:

G-Top : The category of G-spaces and G-maps.

$G\text{-}Top^o$: The category of G-spaces with base point o (always fixed under G) and base-point preserving G-maps.

G-Top (2): Pairs (X,A) of G-spaces and G-maps of pairs.

$G\text{-}Top^o(2)$: Pairs of pointed G-spaces.

All these categories have their associated notion of homotopy. For sets of G-homotopy classes we use the following notation (resprectively):

$$[X,Y]_G\ ,\ [X,Y]_G^o\ ,$$

$$[(X,A),\ (Y,A)]_G\ ,\ [(X,A),\ (Y,A)]_G^o\ .$$

Usually we restrict to suitable subcategories, using notation that should be self-explanatory, e. g. G-CW for the category of G-CW complexes (to be defined later), $G\text{-}CW^o$, G-CW(2), $G\text{-}CW^o(2)$. The standard constructions of homotopy theory using the unit interval , like suspension, mapping cone, path space can be done in G-Top, $G\text{-}Top^o$, etc. using trivial G-action on $I = [0,1]$. There are resulting Barrat-Puppe sequences and their Eckmann-Hilton duals for fibrations. A G-cofibration $i : A \longrightarrow X$ should have the homotopy extension property in G-Top, a G-fibration $p : E \longrightarrow B$ should have the homotopy lifting in G-Top. Of course the problem remains to characterise G-cofibrations etc. in terms of other data, e. g. by considering fixed point sets. This is very important and we return to such questions from time to time (see e. g. the discussion of G-ENR's in I. 5.2). The general theme is to reduce equivariant problems to problems in ordinary topology and the general method will be: induction over the orbit types. For a single orbit type one often has a problem about ordinary bundles (e. g. existence of sections). A basic example of this procedure is the construction and classification of G-maps via sections of an auxiliary map. We describe this transition.

Let X and Y be G-spaces. For a G-map $f : X \longrightarrow Y$ we must have $G_x < G_{fx}$ for all $x \in X$. Therefore we consider the subspace

$$(8.1.1) \qquad I(X,Y) := \{(x,y) \mid G_x \subset G_y\} \subset X \times Y.$$

This is a G-subspace of $X \times Y$ with the diagonal action. Let $(X;Y)$ be the orbit space. The projection $X \times Y$ induces

$$(8.1.2) \qquad q : (X;Y) \longrightarrow X/G.$$

The G-map $f : X \longrightarrow Y$ induces $X \longrightarrow I(X,Y) : x \longmapsto (x,fx)$ and by passing to orbit spaces we obtain a section $s_f : X/G \longrightarrow (X;Y)$ of q.

Proposition 8.1.3. The assignment $f \longmapsto s_f$ _induces a bijection between the set of G-maps_ $X \longrightarrow Y$ _and the set of sections of_ q. _Two G-maps_ $f_1, f_2 : X \longrightarrow Y$ _are G-homotopic if and only if the corresponding sections are homotopic_.

Proof. We claim that

$$(8.1.4) \qquad \begin{array}{ccc} I(X,Y) & \longrightarrow & X \\ \downarrow & & \downarrow p \\ (X;Y) & \underset{q}{\longrightarrow} & X/G \end{array}$$

is a pull-back diagram. Let $Z \longrightarrow (X;Y)$ be the pull-back of p along q. Since $I(X,Y) \longrightarrow X$ is isovariant we obtain from the commutative diagram 8.1.4 a G-map $I(X,Y) \longrightarrow Z$ over $(X;Y)$ which is bijective. In any pull-back diagram

$$\begin{array}{ccc} Z & \longrightarrow & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & X/G \end{array}$$

the map q is canonically homeomorphic to the orbit map $Z \longrightarrow Z/G$. Since X and Y are assumed to be Hausdorff spaces the spaces $I(X,Y)$, Z and their orbit spaces are Hausdorff and the orbit maps are proper (Bourbaki [32], III § 4.1. Prop. 2). By Bourbaki [32], I § 10.1. Prop. 5 the map $I(X,Y) \longrightarrow Z$ is proper and therefore, being bijective, a homeomorphism.

Now given a section $s : X/G \longrightarrow (X;Y)$ we have in the pull-back 8.1.4 the induced section $t : X \longrightarrow I(X,Y)$ which composed with the projection $I(X,Y) \longrightarrow Y$ yields a G-map $f_s : X \longrightarrow Y$. (Verify that t is a G-map.) The correspondences $s \longmapsto f_s$, $f \longmapsto s_f$ are seen to be mutually inverse. A G-homotopy $X \times I \longrightarrow Y$ induces a section $(X\times I)/G \longrightarrow (X\times I;Y)$ which, via canonical homeomorphisms $(X\times I)/G \cong X/G \times I$ and $(X\times I;Y) \cong (X;Y) \times I$ corresponds to a homotopy of sections (and vice versa).

We now explain the principle of constructing G-maps via induction over orbit-types. Suppose that Or is a finite set of conjugacy classes of subgroups of G. We can choose an admissible indexing $Or = \{(H_1), (H_2), \dots, (H_k)\}$, this meaning that $(H_j) < (H_i)$ implies $i < j$. If the G-space X has finite orbit type we always choose an admissible indexing of its set of orbit types $Or(X)$. Let $f : X \longrightarrow Y$ be a G-map between spaces of finite orbit-type. Let

$$\mathrm{Or}(X) \cup \mathrm{Or}(Y) = \{(H_1),\dots,(H_k)\}$$

be an admissible ordering. Define a filtration of X by closed G-subspaces

$$X_1 \subset X_2 \subset \dots \subset X_k = X$$

$$X_i = \{x \in X \mid \text{for some } j \leq i \quad (G_x) = (H_j)\}.$$

Then $X_i \setminus X_{i-1}$ is the orbit bundle $X_{(H)}$, $H = H_i$. The G-map f induces G-maps $f_i : X_i \longrightarrow Y_i$. If a G-map $k : X_{i-1} \longrightarrow Y_{i-1}$ is given we are interested in its extensions $K : X_i \longrightarrow Y_i$.

<u>Proposition 8.1.5. The extensions</u> K <u>of</u> k <u>are in bijective correspondence with the</u> NH/H-<u>extensions</u> $e : X_i^H \longrightarrow Y_i^H$ <u>of</u> $k^H : X_{i-1}^H \longrightarrow Y_{i-1}^H$ $(H = H_i)$.

<u>Proof.</u> Given K we have $e = K^H$ and since $GX_H = X_i \setminus X_{i-1}$ the G-map K is uniquely determined by K^H. Now suppose we are given an NH/H-map $e : X_i^H \longrightarrow Y_i^H$ extending k^H. We define a map

$$\begin{aligned} &E : X_i \longrightarrow Y_i && \text{by} \\ &E(x) = K(x) && \text{if } x \in X_{i-1} \\ &E(x) = g\,e(y) && \text{if } x = gy,\ y \in X_i^H . \end{aligned}$$

We have to show that E is well-defined and continuous. If $x = g_1y_1 = g_2y_2$ and $y_1 \in X_{i-1}^H$ then $y_2 \in X_{i-1}^H$ and $g_1e(y_1) = g_1K(y_1) = K(g_1y_1) = K(x)$ $= g_2e(y_2)$ because K is a G-map. If $x = g_1y_1 = g_2y_2$ and $y_1, y_2 \in X_i^H \setminus X_{i-1}^H$ then $g_1 = g_2n$ with $n \in NH$ and therefore

$$g_1e(y_1) = g_2ne(y_1) = g_2e(ny_1) = g_2e(y_2)$$

because e is an NH-map. Hence E is well-defined. E is continuous on the closed subsets X_{i-1} and GX_1^H, hence continuous.

We combine 8.1.3 and 8.1.5 in the following manner: The action of NH/H on $X_i^H \setminus X_{i-1}^H$ is free. Hence we are in the following situation: Let (X,A) and (Y,B) be pairs of G-spaces (A and B closed subspaces). The action of G on $X \setminus A$ and $Y \setminus B$ shall be free. We want to extend G-maps $f : A \longrightarrow B$ to G-maps $F : X \longrightarrow Y$. By 8.1.3 we have to extend a partial section of $(X;Y) \longrightarrow X/G$ given over A/G (a closed subspace of X/G) to a section. But over $(X \setminus A)/G$ we have an ordinary fibre bundle with fibre Y (locally trivial by the slice theorem). (See Bredon [37], II. 2 for the special case of free actions.) So one usually encounters a sequence of fibre bundle problems and moreover one has to deal with the singular behaviour of $(X;Y) \longrightarrow X/G$ over A and near A.

8.2. Homotopy equivalences.

We show that under suitable hypotheses a G-map $f : X \longrightarrow Y$ is a G-homotopy equivalence if and only if the fixed point mappings f^H are ordinary homotopy equivalences. This holds in particular if X and Y are G-ENR's.

An assertion as above should be true if X and Y are free G-spaces. This is a fibre bundle problem. A free G-space X is called <u>numerable</u> if $X \longrightarrow X/G$ is a numerable principal G-bundle in the sense of Dold [71] , i. e. locally trivially over an open cover which has a subordinate locally finite partition of unity.

<u>Proposition 8.2.1.</u> *Let $f : X \longrightarrow Y$ be a G-map from a G-space to a numerable free G-space Y. Then f is a G-homotopy equivalence if and only if f is an ordinary homotopy equivalence.*

<u>Proof.</u> Certainly X must be a free G-space. Since X maps into a locally

trivial space if is itself locally trivial (Bredon [37] , II. 3.2). Moreover $X \longrightarrow X/G$ is numerable, by pulling back a numeration of $Y \longrightarrow Y/G$. Let $EG \longrightarrow BG$ be the universal principal G-bundle (this is numerable, Dold [71] , 8). Consider the following diagram of G-maps

$$\begin{array}{ccc} EG \times X & \xrightarrow{\;id \times f\;} & EG \times Y \\ \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle pr} \\ X & \xrightarrow{\;f\;} & Y \end{array}$$

We show that pr and id x f are G-homotopy equivalences. The map (idxf)/G

$$\begin{array}{ccc} (EG \times X)/G & \longrightarrow & (EG \times Y)/G \\ & \searrow \quad \swarrow & \\ & BG & \end{array}$$

is a fibre-wise map over BG between fibrations. The induced map on each fibre is an ordinary homotopy equivalence because f is. By Dold [71] , 6.3. and 8. (id x f)/G is a fibre homotopy equivalence and by the covering homotopy theorem for bundle maps Dold [71] , 7.8, the map id x f is a bundle equivalence hence a G-homotopy equivalence. A similar argument applies to pr: The map $(EG \times X)/G \longrightarrow X/G$ is a fibration with contractible fibre EG hence a homotopy equivalence (actually shrinkable, Dold [71] , 3.2). Now apply the covering homotopy theorem for bundle maps again.

Proposition 8.2.2. *Given a diagram of G-spaces and G-maps*

$$\begin{array}{ccc} Y & \xrightarrow{\quad p \quad} & Z \\ {\scriptstyle f_A}\uparrow & & \uparrow{\scriptstyle h} \\ A & \subset & X \end{array}$$

and a G-homotopy $H_A : h|A \simeq pf_A$. *Assume that* $A \subset X$ *is a G-cofibration. Then there exists a G-map* $f : X \longrightarrow Y$ *extending* f_A *and a G-homotopy* $H : h \simeq pf$ *extending* H_A *provided*

(a) *p is an equivariant homotopy equivalence*

or (b) *p is an ordinary homotopy equivalence and* $X \setminus A$ *is a numerable free G-space.*

Proof. Replace p by the equivariantly homotopy equivalent G-fibration $q : E \longrightarrow Z$, where E is the path-space

$$E = \{(w,y) \in Z^I \times Y \mid w(1) = p(y)\} \quad , \; q(w,y) = w(o).$$

The G-action on E is given by $g(w,y) = (g\cdot w, gy)$, where $(g\cdot w)(t) = gw(t)$. Let $r : F \longrightarrow X$ be the G-fibration over X induced by, i. e.

$$F = \{(x,w,y) \in X \times Z^I \times Y \mid w(o) = h(x),\; w(1) = p(y)\}$$
$$r(x,w,y) = x.$$

Define $k : A \longrightarrow F$ by $k(a) = (a, w_a, f_A(o))$ with

$$w_a(t) = \begin{cases} h(a) & o \leq t \leq 1/2 \\ H_A(a, 2t-1) & \frac{1}{2} \leq t \leq 1 . \end{cases}$$

Then k is an equivariant section of r over A. From the description of F above we see that the theorem is proved if we can extend k to an equivariant section of r over X.

Since $A \subset X$ is a G-cofibration, there is an equivariant map $u : X \longrightarrow I$ and a G-homotopy $K : X \times I \longrightarrow X$ such that $A \subset u^{-1}(o)$, $K(x,o) = x$, $K(a,t) = a$ for all $a \in A$ and $t \in I$, and $K(x,1) \in A$ for $x \in u^{-1}[0,1[$ (this is the equivariant analogue of Strøm [156]; see also tom Dieck-Kamps-Puppe [70], § 3). Put $U = u^{-1}[0,1[$. Extend k to an equivariant section r over U by $k(x) = (x, w_x, f_A K(x,1))$ with

$$w_x(t) = \begin{cases} hK(x,2t) & o \leq t \leq 1/2 \\ H_A(K(x,1),2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases} \qquad x \in U$$

The restriction $r_{X \setminus A} : F_{X \setminus A} = r^{-1}(X \setminus A) \longrightarrow X \setminus A$ is G-shrinkable: Since p is a homotopy equivalence and a G-fibration it is shrinkable (Dold [71], 6.2), hence the induced r is shrinkable (Dold [71], 3.1). Hence $r_{X \setminus A}$ is a homotopy equivalence and by 8.2.1 (in case (b)) G-homotopy equivalence, and, being a G-fibration, $r_{X \setminus A}$ is shrinkable. (In case (a) $r_{X \setminus A}$ is induced from the G-shrinkable q). G-Shrinkable means: There exists an equivariant section t of $r_{X \setminus A}$ and a G-homotopy over $X \setminus A$ L from the identity to $tr_{X \setminus A}$. The required equivariant section s of r over X is now given by

$$s(x) = \begin{cases} t(x) & x \in X \setminus U \\ L(k(x), \max[2u(x)-1,o]) & x \in U \setminus A \\ k(x) & x \in A \end{cases}$$

<u>Proposition 8.2.3.</u> <u>Let</u> $p : (X,A) \longrightarrow (Y,B)$ <u>be a G-map such that</u> $p_A = p|A : A \longrightarrow B$ <u>is a G-homotopy equivalence and</u> p <u>is an ordinary homotopy equivalence. Suppose that</u> $X \setminus A$ <u>and</u> $Y \setminus B$ <u>are numerable free</u>

G-spaces and $A \subset X$, $B \subset Y$ *are G-cofibrations. Then any G-homotopy inverse* q_B *of* p_A *can be extended to a G-homotopy inverse* q *of* p *and any G-homotopy* $H_B : id_B \simeq p_A\, q_B$ *to a G-homotopy* $H : id_Y \simeq pq$.

Proof. We apply 8.2.2 (b) to the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad p \quad} & Y \\ \uparrow{\scriptstyle q_B} & & \uparrow{\scriptstyle id} \\ B & \underset{i}{\subset} & Y \end{array}$$

and obtain a G-extension $q : Y \longrightarrow X$ of q_B and $H : Y \times I \longrightarrow Y$ of iH_B such that $H : id_Y \simeq pq$. Hence $(p,p_A)(q,q_B) \simeq id$ as maps between G-pairs. Since p was an ordinary homotopy equivalence q must be an ordinary homotopy equivalence. Hence we can apply 8.2.2 (b) once more to find an extension $\bar{p} : X \longrightarrow Y$ of p_A such that $(q,q_B)(\bar{p},p_A) \simeq id$ as maps of G-pairs. Hence (q,q_B) is a G-homotopy equivalence of G-pairs with G-homotopy inverse (p,p_A).

Proposition 8.2.4. Let $f : X \longrightarrow Y$ *be a G-map such that for all* $H < G$ *the map* f^H *is an ordinary homotopy equivalence. Suppose that for all* $H < G$ X_H, Y_H *are numerable free NH/H-spaces and* $G(X^H \setminus X_H) \subset GX^H$, $G(Y^H \setminus Y_H) \subset GY^H$ *are G-cofibrations. Suppose moreover that* X *and* Y *have finite orbit-type. Then* f *is a G-homotopy equivalence.*

Proof. Choose an admissible indexing of $Or(X) \cup Or(Y)$ as explained in 8.1. We have the associated filtration (X_n) and (Y_n) of X and Y and we show by induction over n that $f_n : X_n \longrightarrow Y_n$ is a G-homotopy equivalence. The induction starts, using 8.2.1. Suppose f_{n-1} is a G-homotopy equivalence with inverse h_{n-1}. Using 8.2.3 we see that h^H_{n-1} can be extended to an NH/H-homotopy inverse of f_n^H if $X_n \setminus X_{n-1} = X_{(H)}$. By 8.1.5 we find the required of f_n.

Remark 8.2.5. The hypotheses of 8.2.4 are satisfied if X and Y are G-ENR's. This follows from the theorem of Jaworowski 5.2.6 and the fact that an inclusion of G-ENR's is a G-cofibration.

We also mention a theorem of Segal-James [101], Theorem 1.1, giving another variant of 8.2.4.

Proposition 8.2.6. Let X and Y be G-ANR's. Then a G-map $f : X \longrightarrow Y$ is a G-homotopy equivalence if the map $f^H : X^H \longrightarrow Y^H$ is a homotopy equivalence for all closed subgroups H of G.

8.3. Obstruction theory.

According to 8.1.5 the basic extension problem in equivariant homotopy theory may be formulated as follows:

Extension problem: Given G-spaces $A \subset X$, A closed in X, and Y and a G-map $f : A \longrightarrow Y$. Suppose G acts freely on $X \setminus A$. Can f be extended to a G-map $F : X \longrightarrow Y$? If F exists, how can one classify G-homotopy classes of such extensions?

We want to reduce these problems to problems in classical obstruction theory, as presented in the books by Steenrod [154] or Baues [17] . By 8.1.3 we have to consider $q : (X;Y) \longrightarrow X/G$ with given partial section $s : A/G \longrightarrow (X\ Y)$ corresponding to f and we have to extend this section over X/G. This looks like a problem in obstruction theory, but the additional technical problem that arises comes from the fact that q is not, in general, a fibration. Over $(X \setminus A)/G$, q is the fibre bundle $((X \setminus A) \times Y)/G \longrightarrow (X \setminus A)/G$ with fibre Y, but when we approach

A/G the fibre change (the fibration has "singularities"). One possibility to circumvent this problem is to assume that the section s has an extension to a neighbourhood, i. e. the G-map f may be extended to a neighbourhood. This is the case when $A \subset X$ is a G-retract of a neighbourhood an in particular when $A \subset X$ is a G-cofibration, or when Y is a G-ANR and X is normal. (This extension property is the definition of a G ANR in Palais [124] , 1.6. In particular a G-ENR is a G-ANR.)

Proposition 8.3.1. *Let* (X,A) *be a relative* G-CW-*complex of dimension* $\leq n$ *with free* G-*action on* $X \setminus A$. *Let* Y *be a* G-*space which is* n-*connected and* n-*simple* ($n \geq 1$). *Then any* G-*map* $f : A \longrightarrow Y$ *has an extension* $F: X \rightarrow Y$. *The* G-*homotopy classes* rel. A *of such extensions correspond bijectively to elements of* $H^n(X/G, A/G; \pi_n Y)$ (where singular cohomology with suitable local coefficients is used).

Remarks. The assumption about (X,A) means that X is obtained from A by attaching cells $G \times D^i$ for $i \leq n$. Then (X/G,A/G) is an ordinary relative CW-complex of dimension $\leq n$. The inclusion $A \subset X$ is a G-cofibration, in fact a strong neighbourhood deformation retract (in G-Top): There exists a G-neighbourhood U of A in X such that $A \subset U$ is a G-homotopy equivalence rel. A. Over $X \setminus A$ we have the local coefficient system $((X \setminus A) \times \pi_n Y)/G \longrightarrow (X \setminus A)/G$ where the G-action on Y induces an action on $\pi_n Y$. By excision
$H^n(X/G, A/G; \pi_n Y) \cong H^n(X \setminus A/G, U \setminus A/G; \pi_n Y)$ and in the latter group we use the local coefficient system just defined.

Proof. Using 8.1 the problem is translated into a section extension problem and then classical obstruction theory is applied.

One of the immediate applications of obstruction theory is a proof of H. Hopf's theorem which determines the homotopy classes of maps from an n-manifold into an n-sphere. We generalize this to the equivariant situation in the next section.

8.4. The equivariant Hopf theorem.

A classical theorem of H. Hopf asserts that the homotopy classes of a closed connected orientable n-manifold M into the n-sphere are characterized by their degree and every integer occurs as degree of a suitable map. If M and S^n carry free actions of a finite group G then the equivariant homotopy classes are still determined by their degree, but no longer does every integer occur as a degree (e. g. if $G = Z/pZ$ and $M = S^n$ as G-spaces then the degree must be congruent one modulo p). We shall describe in this section the straighforward generalization to transformation groups, using the obstruction theory of 8.3.

We give the data needed to state the results. Let X be a G-CW-complex of finite orbit type. Then X^H is a WH-complex ($WH := NH/H$). We assume that all X^H are finite-dimensional. If H is an isotropy group of X we let n(H)be the dimension of X^H. For simplicity we assume that $n(H) \geq 1$. If $H \lneq K$ then we should have $n(H) > n(K)$, for $H,K \in Iso(X)$ of course. We assume that $H^{n(H)}(X^H;Z) \cong Z$. The action of WH on X^H then induces a homomorphism $e_{H,X} : WH \longrightarrow Z^* = \{\pm 1\} = Aut\ Z$ which is called the orientation behaviour of X at H. We put $\bar{X}^H = \cup X^K$, $K \gneq H$; this is a WH-subspace of X^H. The map $e_{H,X}$ defines a WH-module $Z_{H,X}$ which we use for local coefficients in order to define the group $H^{n(H)}(X^H/WH, \bar{X}^H/WH; Z_{H,X})$. We assume that this cohomology group is isomorphic to Z if WH is finite. But be have the

<u>Lemma 8.4.1. If under the assumption above</u> $n(H) \geq n(K)+2$ <u>for all</u> $K > H$, $K \neq H$, $K \in Iso(X)$ <u>then</u> $H^{n(H)}(X^H/WH, \bar{X}^H/WH; Z_{H,X}) \cong Z$.

<u>Proof.</u> Using the exact cohomology sequence of the pair we see that it

suffices to show that $H^{n(H)}(X^H/WH; Z_{H,X}) \cong Z$. We look at cellular cochains $\mathrm{Hom}_{WH}(C_{n(H)}(X^H), Z_{H,X})$. If $n(H) \geq n(K) + 1$ for $K > H$, $K \neq H$, $K \in \mathrm{Iso}(X)$ then $C_{n(H)}(X^H)$ is a free WH-module (for WH finite) hence the trace map which makes cochains WH-equivariant is surjective, hence the transfer $H^{n(H)}(X^H; Z) \longrightarrow H^{n(H)}(X^H/WH; Z_{H,X})$ is surjective. The composition of this map with the map in the other direction induced by $X^H \longrightarrow X^H/WH$ is multiplication by $|WH|$. So we only show that the group in question is torsion free. But one shows easily, using the trace operator that

$$\mathrm{Hom}_W(Z_H, Z_H) \longleftarrow \mathrm{Hom}_W(C_n, Z_H) \longleftarrow \mathrm{Hom}_W(C_{n-1}, Z_H)$$

is exact.

We now continue to describe data. Let Y be another G-space. We assume that Y^H is n(H)-connected and $\pi_{n(H)} Y^H \cong Z$ for $H \in \mathrm{Iso}(X)$. Then $H^{n(H)}(Y^H; Z) \cong Z$ and we obtain the orientation behaviour $e_{H,Y}: WH \longrightarrow Z^*$ of Y at H. We assume that $e_{H,X} = e_{H,Y}$ for all $H \in \mathrm{Iso}(X)$. We orient X be choosing a generator of $H^{n(H)}(X^H; Z)$ for every H and similarly for Y. We assume that X and Y have been oriented. Then given a G-map $f: X \longrightarrow Y$ the fixed point mapping $f^H : X^H \longrightarrow Y^H$ has a well-defined degree $d(f^H) \in Z$.

<u>Theorem 8.4.1.</u> <u>Under the assumption above the equivariant homotopy set</u> $[X,Y]_G$ <u>is not empty. Elements</u> $[f] \in [X,Y]_G$ <u>are determined by the set of degrees</u> $d(f^H)$, $H \in \mathrm{Iso}\ X$, WH <u>finite</u>. <u>The degree</u> $d(f^H)$ <u>is modulo</u> $|WH|$ <u>determined by the</u> $d(f^K)$, $K > H$, $K \neq H$ <u>and fixing these</u> $d(f^K)$ <u>the possible</u> $d(f^H)$ <u>fill the whole residue class</u> mod $|WH|$.

<u>Proof.</u> We order the isotropy types $(H_1), \dots, (H_r)$ of X such that $(H_i) < (H_j)$ implies $i > j$. Let $(H) = (H_i)$ and suppose that we already

have a G-map $f : \bigcup_{j<i} GX^{H_j} =: X_{i-1} \longrightarrow Y$. We want to extend this G-map to X_i. As we have explained in 8.1 the homotopy classes rel X_{i-1} of such extension correspond to WH-extensions of $f|\bar{X}^H$ to X^H. The obstructions to such extensions lie in $H^i(X^H/WH, \bar{X}^H/WH; \pi_{i-1}(Y^H))$ and these groups are all zero by our assumptions. Hence there exists at least one extension.

Given two WH-maps $f,g : X^H \longrightarrow Y^H$ with $f|\bar{X}^H = g|\bar{X}^H$ the obstructions against a homotopy between them lie in the groups $H^i(X^H/WH, \bar{X}^H/WH; \pi_i(Y^H))$ and these groups are all zero except for $n(H) = i$ and WH finite where $\pi_{n(H)}(Y^H) = Z_{H,Y} = Z_{H,X}$ and the group is Z by assumption. Hence we get a single integer $d(f,g)$ as an obstruction. We claim that $d(f,g)$ is divisible by $|WH|$ and moreover $d(f,g) = d(f) - d(g)$. We look at the natural map

$$p^* : H^{n(H)}(X^H/WH, \bar{X}^H/WH; Z_{H,X}) \longrightarrow H^{n(H)}(X^H, \bar{X}^H; Z).$$

By naturality of the obstruction class $d(f,g)$ is mapped onto the obstruction against a non-equivariant homotopy between f and g and this is by the classical Hopf theorem just the difference of the degrees. We have already seen above that image $p^* \subset |WH|\, Z$. Together with 8.3.1, applied to this induction step, this finishes the proof of 8.4.1.

8.5. Geometric modules over the Burnside ring.

We shall prove in this section that the Burnside ring A(G) is isomorphic to stable cohomotopy of spheres in dimension zero via the Lefschetz-Dold index, see 7.6.7. The proof will be computational but gives at the same time information about certain other modules over A(G). We recall

Theorem 8.5.1. If we assign to a compact G-ENR X _the Lefschetz-Dold index_ I(X) _we obtain a well-defined map_ $I_G : A(G) \longrightarrow \omega_G^0$. _This map is an isomorphism of rings._

Proof. If H is a closed subgroup of G we define a ring homomorphism $d_H : \omega_G^0 \longrightarrow Z$ by assigning to $x \in \omega_G^0$, represented by $f : V^c \longrightarrow V^c$, the degree of the H-fixed point map f^H. Recall that we introduced in section 5 a homomorphism $\varphi_H : A(G) \longrightarrow Z : [X] \longmapsto \chi(X^H)$, where χ denotes the Euler characteristic.

We show: Let X be a compact G-ENR. Then $d_H I(X) = \chi(X^H)$. By 7.6.8 we have $d_H I(X) = I(X^H) \in \omega_{\{1\}}^0 \cong Z$. The fixed point index $I(X^H)$ of $id(X^H)$ is the Euler characteristic of X^H (compare Dold [75] , XII 6.6 and [76]). This proves $d_H I(X) = \chi(X^H)$. By 8.4.1 the elements of ω_G^0 are detected by the maps d_H. From the definition of the Burnside ring we now obtain that I_G is a well-defined injective ring homomorphisms. That this map is also surjective will follow if we show that the $d_H(x)$ satisfy congruences analogous to 5.8.5. (See 8.5.9) We shall prove this in a moment for a slightly more general situation.

Remark 8.5.2. If $f : X \longrightarrow X$ is an endomorphism of the compact G-ENR X then the Lefschetz-Dold index of (X,f) is an element of $\omega_G^0 = A(G)$. By 5.5.1 this index element is a linear combination of homogeneous spaces. It is a non-trivial exercise for the reader to figure out which linear combination this is.

The isomorphism of Theorem 8.5.1 is natural, i. e. commutes with the various restriction and induction processes. If $f : G \longrightarrow K$ is a continuous homomorphism then we obtain by pull-back along f homomorphisms $f^* : \omega_K^0 \longrightarrow \omega_G^0$ and $f^* : A(K) \longrightarrow A(G)$ and we have

$$I_G \, f^* = f^* I_K.$$

The adjointness $[G^+ \wedge_H X, Y]^o_G \cong [X,Y]^o_H$ for a pointed H-space X and a pointed G-space Y together with the G-homeomorphism

$$G^+ \wedge_H X \longrightarrow G/H^+ \wedge X \ : \ (g,x) \longmapsto (g,gx)$$

for a G-space X induces an isomorphism

$$i^G_H \ : \quad \omega^o_H \cong \omega^o_G(G/H).$$

If we compose this with the transfer induced by G/H $\longrightarrow$ Point we get the induction

$$\mathrm{ind}^G_H \ : \quad \omega^o_H \longrightarrow \omega^o_G.$$

Note that we also have a map

$$I_{[G/H]} \ : A\,[G/H] \longrightarrow \omega^o_G(G/H)$$

which assigns to a submersion $f : M \longrightarrow G/H$ the Lefschetz-Dold index I_f. In 5.12 we constructed an isomorphism $i^G_H : A(H) \longrightarrow A\,[G/H]$.

<u>Proposition 8.5.3.</u> $I_{[G/H]}\, i^G_H = i^G_H\, I_H$

$$\mathrm{ind}^G_H\, i^G_H = i^G_H\, \mathrm{ind}^G_H.$$

<u>Proof.</u> This follows from properties 7.6.8 of the transfer.

Finally we mention that the maps I_H are compatible with the multiplicative induction. If H has finite index in G we showed in 5.12

that the multiplicative induction $X \longmapsto \mathrm{Hom}_H(G,X)$ induced a map $A(H) \longrightarrow A(G)$. This map is transformed under the isomorphisms I_H, I_G into a map $\omega_H^o \longrightarrow \omega_G^o$ which has the following description on representatives: Note that $\mathrm{Hom}_H(G,X)$ as a space is just $\prod(gH \times_H X)$, the product taken over the cosets G/H; but this formulation also indicates the G-action. If now X is a pointed H-space then we can similarly form the smashed product $\bigwedge(gH \times_H X)$ with G-action defined similarly. This gives a functor from pointed H-spaces to pointed G-spaces which maps H-homotopies to G-homotopies. If V is an H-module then $\bigwedge(gH \times_H V^c)$ is the one-point-compactification of the induced representation $\mathrm{Hom}_H(G,V)$. The map in question is now induced by $[V^c, V^c]_H^o \longrightarrow [\bigwedge(gH \times_H V^c), \bigwedge(gH \times_H V^c)]_G^o \longrightarrow [(\mathrm{Hom}_H(G,V))^c, (\mathrm{Hom}_H(G,V))^c]_G^o$. More generally, multiplicative induction is a map $\omega_H^o(X) \longrightarrow \omega_G^o(\mathrm{Hom}_H(G,X))$. The reader may check that multiplicative induction is compatible with the Lefschetz index.

Suppose now that we given complex representations V and W such that

(8.5.4) $$\dim V^H = \dim W^H \quad \text{for all } H < G.$$

We call $\omega_\alpha = \omega_o^G(V^c, W^c)$ the ω_G^o-module for $\alpha = V-W$. For each $H < G$ we have a degree map

(8.5.5) $$d_{\alpha,H} : \omega_\alpha \longrightarrow Z : [f] \longmapsto \text{degree } f^H.$$

The degree is computed with respect to the canonical orientations of $(V^H)^c$, $(W^H)^c$ which are induced by the complex structure. By 8.4.1 the maps $d_{\alpha,H}$ detect the elements of ω_α. So we ask: What are the relations between the possible degrees $d_{\alpha,H}(x)$? The assignment $(H) \longmapsto d_{\alpha,H}(x)$ is a continuous function. Therefore we obtain an injective map

(8.5.6) $$d_{\alpha} : \omega_{\alpha} \longrightarrow C(\phi, Z).$$

We want to describe the image by congruence relations.

Theorem 8.5.7. There exists a collection of integers $n_{H,K}(\alpha)$, *depending on* α, $(H) \in \phi(G)$, *and* (K) *with* H *normal in* K *and* K/H *cyclic, such that* $n_{\alpha}(H,H) = 1$ *and such that the following holds*: $x \in C(\phi, Z)$ *is contained in the image of* d_{α} *if and only if*:

$$\Sigma_{(K)} n_{H,K}(\alpha)\ x(K) \equiv 0 \bmod |NH/H|.$$

The sum is taken over the conjugacy classes (K) *such that* H *is normal in* K *and* K/H *is cyclic.*

Proof. We first show that any set of congruence relations of the type considered in 8.5.7 suffices to the describe the module ω_{α}. Later we derive specific congruences as indicated, using K-theory.

Suppose we are given for each $(H) \in \phi$ a map $r_H : C(\phi, Z) \longrightarrow Z/|WH|$ of the form

(8.5.8) $$r_H(z) = z(H) + \sum n_{H,K}\ z(K) \bmod |WH|$$

where the $n_{H,K}$ are integers and the sum is taken over the conjugacy classes (K) such that H is normal in K and K/H is a non-trivial cyclic group. Suppose that for $\alpha = E - F$ with $\dim E^H = \dim F^H$ the image of d_{α} is contained in

$$C_{\alpha} = \{ z \in C(\phi, Z) \mid (H) \in \phi \Rightarrow r_H(z) = 0 \}$$

Then we claim $d_{\alpha}\omega_{\alpha} = C_{\alpha}$.

Given $z \in C_\alpha$. We have to show that for a suitable U there exists a map $f : S(E \oplus U) \longrightarrow S(F \oplus U)$ such that for each $(H) \in \phi$ degree $f^H = z(H)$. To begin with we choose U large enough so as to satisfy the following conditions:

i) $Iso(E \oplus U) = Iso(F \oplus U)$

ii) $(1), (G) \in Iso(E \oplus U)$

iii) $(K), (L) \in Iso(E \oplus U) \Rightarrow (K \cap L) \in Iso(E \oplus U)$

iv) Choose an integer $n \neq o$ such that $x = nz$ is contained in C_o. Then there shall exist a representative $S(E \oplus U) \longrightarrow S(F \oplus U)$ for $x \in \omega_o$.

Once (iv) is satisfied for U it is also satisfied for any U' containing U as a direct summand. Hence by enlarging U we can also satisfy (i) - (iii).

We set $X = S(E \oplus U)$ and $Y = S(F \oplus U)$. Let $Iso(X) = \{(H_1), \dots, (H_r)\}$ where $(H_i) > (H_j)$ implies $i < j$. If $X_i = \{x \in X \mid (G_x) = (H_j) \text{ for some } j \leq i\}$ we construct inductively G-maps $f_r : X_r \longrightarrow Y$ such that

v) degree $f_r^L = z(L)$ if $(L) \in \phi$, $(L) \geq (H_i)$, $i \leq r$
or if $(L) > (H_{r+1})$, $(L) \in \phi$.

Note that $X_r^L = X^L$ for such L. Put $H = H_{r+1}$. The G-extensions $f_{r+1} : X_{r+1} \longrightarrow Y$ of f_r correspond via restriction bijectively to the WH-extensions $h : X^H \longrightarrow Y^H$ of $f_r' = f_r \mid X_H : X_H \to Y^H$ where $X_H = X^H \cap X_r$. The obstructions to the existence of h lie in $H^*(X^H/HN, X_H/NH; \pi_{*-1}(Y^H))$, as in 8.4. These groups are zero by our assumptions. Let f_{r+1}' be a WH-extension of f_r'. Let $f_1 : X \longrightarrow Y$ be a map with $f_1^H = f_{r+1}'$ which exists by the same obstruction argument. Then, if $(H) \in \phi$, we have for the fixed point degrees

$$d(f_1^H) + \sum n_{H,K}\, d(f_1^K) \equiv 0 \bmod |WH| .$$

By induction $d(f_1^K) = z(K)$ so that in this case $d(f'_{r+1}) \equiv z(H) \bmod |WH|$. Since WH acts freely on $X^H \setminus X_H$ we can alter f'_{r+1} rel X_H to an NH-map f''_{r+1} so that $d(f''_{r+1}) = z(H)$. Let f_{r+1} be the map with $f_{r+1}| X^H = f''_{r+1}$ if $(H) \in \phi$ and $f_{r+1} | X^H = f'_{r+1}$ if $(H) \notin \phi$. Then $d(f^L_{r+1}) = z(L)$ whenever $(L) \geq (H_i)$, $i \leq r+1$. Suppose $(L) > (H_{r+1})$, $(L) \in \phi$. Since $\mathrm{Iso}(X) = \mathrm{Iso}(Y)$ is closed under intersections there exists a unique isotropy group $(P) = (H_s)$ such that $(P) \geq (L)$ and $(P) \in \phi$, $X^L = X^P$, $Y^L = Y^P$, degree f^L_{r+1} = degree $f^P_{r+1} = z(P)$. We have to show $z(L) = z(P)$. But by (iv) above nz is represented by a map $g : X \longrightarrow Y$ hence $g^P = g^L$ implies $nz(L) = nz(P)$. This finishes the proof of $d_\alpha \omega_\alpha = c_\alpha$.

We now derive specific functions of the type 8.5.8. Let $f : E \longrightarrow F$ be a proper G-map between complex G-modules. Let $C < G$ be a topologically cyclic group with generator h. Put $E = E^C \oplus E_C$, $j_E : E_C \subset E$. We apply equivariant K-theory with compact support and obtain for $f^* : K_C(F) \longrightarrow K_C(E)$ and $(f^C)^*$ the equality $j_E^* f^* = j_F^* (f^C)^*$. Let $\lambda(E) \in K_G(E)$ be the Bott class, a free R(G)-generator of $K_G(E)$. Then we define $a \in R(G)$ by $f^* \lambda(E) = a\, \lambda(F)$ and obtain $(a|C)\, \lambda_{-1}(E_C) = \lambda_{-1}(F_C)$ degree f^C. We evaluate characters at h and use $\lambda_{-1}(E_C)(h) \neq 0$. If G is finite then $\sum_{g \in G} a(g) \equiv 0 \bmod |G|$. If $C < G$ is cyclic and C^* its set of generators we put $a^*(C) = \sum_{g \in C^*} a(g)$. With $n(E-F,C) = \sum_{g \in C^*} \lambda_{-1}(F_C)(g)/\lambda_{-1}(E_C)(g)$ we obtain

$$a^*(C) = n(E-F,C) \text{ degree } f^C$$

$$a(g) = \sum_{(C)} |G|\, |NC|^{-1}\, a^*(C) \equiv 0 \bmod |G|.$$

By elementary Galois theory $n(E-F,C)$ is an integer. We apply these considerations to f^H considered as WH-map and obtain

$$\Sigma_K \; |NH/NH \cap NK| \; n(E^H - F^H, K/H) \; d(f^K) \equiv 0 \bmod |WH|$$

where the sum is taken over the NH-conjugacy classes (K) with $H \triangleleft K$ and K/H cyclic. This yields the desired functions 8.5.8.

Remark 8.5.9. Comparing the case E = F of the above congruences with 5.8.5 we see that the map I_G of 8.5.1 is surjective.

8.6. Prime ideals of equivariant cohomotopy rings.

Let X be a compact G-ENR, G finite. We are going to determine the prime ideal spectrum of the ring $\omega^0_G(X)$.

The orbit category O(X) of X shall have as objects the G-homotopy classes of maps $G/H \longrightarrow X$ and as morphisms from $u : G/H \longrightarrow X$ to $v : G/K \longrightarrow X$ the G-homotopy classes $t : G/H \longrightarrow G/K$ such that $vt = u$.

If $u : G/H \longrightarrow X$ is given we have the induced ring homomorphism $u^* : \omega^0_G(X) \longrightarrow \omega^0_G(G/H)$ and the maps u^* combine to a ring homomorphism

$$(8.6.1) \qquad \nu : \omega^0_G(X) \longrightarrow \lim \omega^0_G(G/H)$$

where the limit (= inverse limit) is taken over the category O(X). Let Spec ν be the induced map of prime ideal spectra.

Theorem 8.6.2. The kernel of ν is the nilradical of $\omega^0_G(X)$. For each $x \in \lim \omega^0_G(G/H)$ there exists an $n \in \mathbb{N}$ with $x^n \in$ image ν. The map ν induces a homeomorphism Spec ν of prime ideal spectra.

Next we show that taking prime ideal spectra commutes with taking limits over the category $O(X)$. The canonical maps $\lim \omega^{0}_{G}(G/H) \to \omega^{0}_{G}(G/H)$ induce a continuous map μ: colim Spec $\omega^{0}_{G}(G/H) \longrightarrow$ Spec lim $\omega^{0}_{G}(G/H)$.

<u>Theorem 8.6.3. The map μ is a homeomorphism.</u>

We now enter the proofs of these Theorems.

Recall that one has Bredon cohomology **[36]** $H^{*}(X;\omega)$ of X with coefficient system ω given by $\omega : G/H \longrightarrow \omega^{0}_{G}(G/H)$ on objects and induced maps (see also Bröcker **[38]** or Illman for an exposition of this cohomology theory). Let

$$e : \omega^{0}_{G}(X) \longrightarrow H^{0}(X;\omega)$$

be the edge-homomorphism associated to the Atiyah-Hirzebruch spectral sequence of $\omega^{0}_{G}(-)$. More directly: $H^{0}(X;\omega)$ is canonically isomorphic to lim $\omega^{0}_{G}(G/H)$ and under this isomorphism e corresponds to ν.

<u>Proposition 8.6.4.</u> (i) <u>The map $e \otimes Q$ is an isomorphism.</u>
(ii) <u>The torsion subgroup of $\omega^{0}_{G}(X)$ as abelian group is equal to the nilradical of the ring $\omega^{0}_{G}(X)$.</u>

<u>Proof.</u> (i) If $e \otimes Q$ is an isomorphism for a space X then also for any G-retract of X. Since any G-ENR is a retract of a finite G-CW-complex (dominated by a finite G-CW-complex suffices and this is easier to see) it is enough to consider finite G-CW-complexes. But e is a natural transformation of half-exact homotopy functors, so by a standard comparison theorem (see e. g. Dold **[72]**) it suffices to show that $e \otimes Q$ is an isomorphism on cells. This is true for zero-cells by the very definition of $H^{0}(X;\omega)$. If $i > 0$ then $H^{0}(G/H \times (D^{i},S^{i-1});\omega) = 0$ by the dimension axiom of this equivariant cohomology theory. On the other

hand

$$\omega_G^{o}(G/H \times (D^i, S^{i-1})) \cong \omega_H^{o}(D^i, S^{i-1}) \cong \omega_i^H$$

and by the splitting theorem of Segal [145] , (see also tom Dieck [63] , Satz 2) we have

$$\omega_i^H \cong \oplus_{(K)} \; \omega_i \; (BWK^+)$$

(the product is over conjugacy classes (K) of subgroups of H; WK=NK/K, NK normalizer of K in H). But $\omega_i(BWK^+)$ is for $i > o$ a torsion group.

(ii) The kernel of e is the nilradical of $\omega_G^{o}(X)$. The nilradical is certainly contained in this kernel because $H^{o}(X;\omega)$ is contained in in product of rings of the type $\omega_G^{o}(G/H)$ and these rings have no (non-zero) nilpotent elements (being isomorphic to the Burnside ring A(H).) On the other hand the kernel consists precisely of elements of skeleton filtration one hence consists of nilpotent elements. (See Segal [142] for an analogous statement.) Since $H^{o}(X;\omega)$ is torsion-free we have Torsion $\omega_G^{o}(X) \subset$ Nil $\omega_G^{o}(X)$. Tensoring the exact sequence

$$0 \longrightarrow \text{Nil}\; \omega_G^{o}(X) \longrightarrow \omega_G^{o}(X) \longrightarrow H^{o}(X;\omega)$$

with Q and using (i) we obtain (ii).

Note that Proposition 8.6.4 proves the first statement of Theorem 8.6.2. We now come to the second statement.

<u>Proposition 8.6.5.</u> <u>The map</u> e : $\omega_G^{o}(X) \longrightarrow H^{o}(X;\omega)$ <u>has "nilpotent cokernel", i. e. a suitable power of every element of</u> $H^{o}(X;\omega)$ <u>is contained in the image of</u> e.

<u>Proof.</u> (Compare Quillen [127]). If the assertion of the Proposition is true for X then also for any G-retract of X. Since X is a compact G-ENR it is a retract of a compact differentiable G-manifold with boundary. So we need only prove the Proposition for those X which are locally contratible (i. e. each orbit of X is a G-deformation retract of a neighbourhood). If X is G-homotopy equivalent to an orbit then the map e is an isomorphism. Now assume that $X = U_1 \cup \ldots \cup U_n$, the U_i being compact G-ENR's which are G-homotopy equivalent to an orbit. Assume that the Proposition is true for $X_1 = U_1 \cup \ldots \cup U_{n-1}$. We consider the following diagram of Mayer-Vietoris sequences where $H^o(X) = H^o(X;\omega)$ and e_i are instances of the transformation e.

$$\begin{array}{ccccccc}
 & & \omega_G^o(X) & \xrightarrow{\;t\;} & \omega_G^o(X_1) \oplus \omega_G^o(U_n) & \xrightarrow{\;s\;} & \omega_G^o(X_1 \cap U_n) \\
 & & \Big\downarrow e & & \Big\downarrow e_1 \oplus e_2 & & \Big\downarrow e_3 \\
o & \longrightarrow & H^o(X) & \xrightarrow{\;t'\;} & H^o(X_1) \oplus H^o(U_n) & \xrightarrow{\;s'\;} & H^o(X_1 \cap U_n)
\end{array}$$

Given $x \in H^o(X)$ we put $t'(x) = (x_1, x_2)$. By induction hypothesis there exists k such that

$$t'x^k = (x_1^k, x_2^k) = (e_1u_1, e_2u_2)$$

for suitable u_i. By exactness $s'x_1^k = s'x_2^k$ hence $su_2 = su_1 + n$, where n is a suitable nilpotent element by Proposition 8.6.4. Suppose $n^1 = 0$. Then for $p > t$, with $z = su_1$,

$$(z+n)^p = z^p + \binom{p}{1} z^{p-1}n + \ldots + \binom{p}{t-1} z^{p-t+1}n^{t-1}.$$

By Proposition 8.6.4 the elements $n, n^2, \ldots, n^{t-1}$ are torsion elements. Choose $q \in \mathbb{N}$ such that $qn^i = 0$ for $1 \leq i \leq t-1$. Choose p such that q divides $\binom{p}{1}, \ldots, \binom{p}{t-1}$, e. g. $p = (t-1)!q$. Then we obtain

$$(z+n)^p = z^p ,$$

i. e.

$$(su_1)^p = s(u_1^p) = s(u_2^p)$$

and we can find y with $ty = (u_1^p, u_2^p)$, so that finally $fy = x^{pk}$. This proves the induction step.

The final assertion of Theorem 8.6.2 comes from commutative algebra. We have the following situation: $A \xrightarrow[f]{} A/\mathrm{Nil}\, A \xrightarrow[g]{} B$ where f is the canonical quotient map and g is an injection with nilpotent cokernel. Then Spec f is a homeomorphism. Since g has nilpotent cokernel it is easy to see that Spec g is injective. On the other hand g is an integral extension; by the going up theorem Spec g is a closed surjective mapping. Hence also Spec g is a homeomorphism in our case. This finishes the proof of Theorem 2.

Theorem 8.6.4 is contained in Quillen [127], Corollary B.7 in the Appendix B.

We are going to give more explicit statements for some of the results above. Let $x \in X$ and let $H < G_x$ be a subgroup of the isotropy group at x. We define a ring homomorphism $\varphi_{x,H} : \omega_G^0(X) \longrightarrow Z$ as the composition

$$\omega_G^0(X) \longrightarrow \omega_H^0(X) \longrightarrow \omega_H^0(\{x\}) \cong A(H) \longrightarrow Z$$

where the first two maps are restrictions and the last one takes the degree or Euler characteristic of the H-fixed point object.

<u>Proposition 8.6.6. Every ring homomorphism</u> $\varphi : \omega_G^0(X) \longrightarrow Z$ <u>is of the</u>

form $\varphi_{x,H}$ for suitable $x \in X$ and $H < G_x$. We have $\varphi_{x,H} = \varphi_{y,K}$ if and only if $(H) = (K)$ and x and y are in the same orbit under WH of the path-components of X^H. The prime ideals of $\omega_G^0(X)$ have the form $\varphi_{x,H}^{-1}(p)$, $(p) \subset Z$ a prime ideal.

Proof. Let q be the kernel of φ. This is a prime ideal which by Theorem 8.6.2 and 8.6.3 is equal to the kernel of some $\varphi_{x,H}$. Therefore we must have $\varphi = \varphi_{x,H}$.

The different homomorphisms $\varphi : \omega_G^0(X) \longrightarrow Z$ correspond bijectively to the minimal prime ideals of $\omega_G^0(X)$ and bijectively to the homomorphism $\omega_G^0(X) \otimes Q \longrightarrow Q$ of Q-algebras. But by the results of section 7 we have a natural ring isomorphism

$$\omega_G^0(X) \otimes Q \cong \bigoplus_{(H)} \omega^0(X^H)^{WH} \otimes Q$$

where the sum is over the conjugacy classes (H) of subgroups $H < G$. From this fact one easily deduces the second statement of the Proposition. The third one is again a restatement of the Theorems above.

8.7. Comments.

This section is rather rudimentary. We give some references to further developments. A detaild discussion of the Hopf theorem 8.4.1 for maps between spheres can be found in Hauschild [93] . A more conceptual proof of 8.5.1 uses splitting theorem of tom Dieck [63] , Satz 2. Other splitting theorems may be found in Segal [145] , Rubinsztein [136] , Kosniowski [105] , Hauschild [90] , [93] ; relevant is also Wirthmüller [168] and Schultz [138] . 8.5.7 has been generalized to unstable and real modules by Tornehave [160] . 8.2 is based on Hauschild [94] and Vogt [23] , Appendix. For the use of obstruction theory as in 8.3 to equivariant versions of the Blakers-Massey theorem and the suspension theorem see Hauschild [92] . 8.6 was presented in lectures by the author in Newcastle-upon-Tyne, April 1975; also the double coset formula for the equivariant transfer (see exercises).

8.8. Exercises.

1. Show that the double coset formula of 5.12 holds in equivariant cohomotopy and hence in any stable equivariant cohomology of homology theory. (This genralizes various results in Feshbach [82] , Brumfiel-Madsen [43] etc.) More specifically: Let $x_M \in \omega_o^G(M)$ be the transfer element corresponding to $M \longrightarrow$ Point. Let $M = \Sigma\, n_{(H),b}\, M_{(H),b}$ with $n_{(H),b} = \chi_c(S_{(H),b}/G)$ be the decomposition in the Burnside ring as in 5.12. Let $x_{(H),b} \in \omega_o^G(M_{(H),b})$ be the transfer element corresponding to $M_{(H),b} \longrightarrow$ Point. Let $i_{(H),b} : \omega_o^G(M_{(H),b}) \longrightarrow \omega_o^G(M)$ be induced by the inclusion. Then show

$$x_M = \Sigma\, n_{(H),b}\, i_{(H),b}(x_{(H),b}).$$

2. Let $H < G$ and let L be the tangent space of G/NH at 1. Show that there exists a natural isomorphism

$$\omega_n^{NH}(L^c \wedge EW^+ \wedge X) \longrightarrow \omega_n^G((G \times_N EW)^+ \wedge X),$$

$n \in Z$.

3. (tom Dieck [63]) Show that there exists a natural isomorphism

$$\oplus_{(H)}\, \omega_n^{WH}(EWH^+ \wedge X^H) \longrightarrow \omega_n^G(X),$$

$n \in Z$, G compact Lie group, the sum over conjugacy classes of subgroups.