7. Equivariant homology and cohomology.

We describe localization and splitting theorems for equivariant homology and cohomology theories. In particular we use the fact that such theories are modules over the Burnside ring. We compute localizations at prime ideals of the Burnside ring. Our treatment in this chapter is axiomatic.

7.1. A general localization theorem.

Let G be a compact Lie group. A G-equivariant cohomology theory consists of a contravariant, G-homotopy invariant functor $h_G^*(?,?)$ from a suitable category of pairs of G-spaces (e. g. compact spaces, or G-CW-complexes) into graded abelian groups. The grading is by an abelian group A which may be the integers, the real representation ring or some subquotient of it, etc. It assumed that A is equipped with a homomorphism $i : Z \longrightarrow A$ so that expressions like $a+i(n) = a+n$, $a \in A$, $n \in Z$, make sense. We require the long exact cohomology sequence to hold (at least for closed G-cofibrations $A \subset X$) and the suspension isomorphism $\tilde{h}_G^*(X) \cong \tilde{h}_G^{*+1}(SX)$. In the following we gradually add more and more axioms, like suspension isomorphisms for representations, product structure, continuity etc.

If H is a subgroup of G we write

(7.1.1) $$h_H^*(X,Y) = h_G^*(G \times_H X,\ G \times_H Y)$$

for a pair (X,Y) of H-spaces and consider $h_H(?,?)$ as H-equivariant cohomology theory.

Let now $k_G^*(?,?)$ be another equivariant cohomology theory with the same grading as h_H^* and which is multiplicative. In particular $k_G^*(X)$ is a graded-commutative ring with unit. We assume given a pairing

$$k_G^*(X,Y) \times h_G^*(X,Y) \longrightarrow h_G^*(X,Y)$$

of cohomology theories which makes $h_G^*(X,Y)$ a $k_G^*(X,Y)$-module. In particular $h_G^*(G\times_H X)$ is via the projection $p : G\times_H X \longrightarrow G/H$ and $k_G^*(G/H) \longrightarrow h_G^*(G/H)$ an $k_G^*(G/H)$-module. Moreover it is also via $k_G^* \longrightarrow k_G^*(G\times_H X) \longrightarrow h_G^*(G\times_H X)$ an $k_G^* = k_G^*(\text{Point})$ module and this module structure "factors" over the ring homomorphism $k_G^* \longrightarrow k_G^*(G/H) = k_H^*$, called restriction homomorphism.

Let $S \subset k_G^*$ be a multiplicatively closed subset which (for simplicity) lies in the center of k_G^*, and is in particular commutative in the ungraded sense (center also in the ungraded sense). Let X be a G-space and put

(7.1.2) $$X^S = \{x \in X \mid S \cap \text{Kernel}(k_G^* \longrightarrow k_G^*(G/G_x)) = \emptyset \quad .$$

<u>Proposition 7.1.3.</u> *Let* X *be a compact* G-*space with* $X^S = \emptyset$. *Then the localization*

$$S^{-1}\, h_G^*\,(X) = 0.$$

(Graded localization. Elements of S are made invertible.)

<u>Proof.</u> Given $x \in X$ we can find by the slice theorem (Bredon [37], II 5.4) a G-neighbourhood U of the orbit Gx and a G-map $r : U \longrightarrow G/G_x$. If $U_o = r^{-1}(G_x)$ then canonically $U = G \times_{G_x} U_o$ and r is the G-extension of $U_o \longrightarrow \text{Point}$. Since x does not lie in X^S we can find $s \in S$ which is contained in the kernel of $k_G^* \longrightarrow k_G^*(G/G_x)$. Since the k_G^*-module structure of $h_G^*(U)$ factors over $k_G^* \longrightarrow k_G^*(G/G_x)$ we see that $s h_G^*(U) = 0$,

hence $S^{-1}h_G^*(U) = 0$. Covering X by a finite number of such U, using the Mayer - Vietoris sequence and the exactness of the localization functor we conclude that $S^{-1}h_G^*(X) = 0$.

We now consider compact G-spaces X in general. If V is a compact G-neighbourhood of X^S in X then by excision and 7.1.3 we have $S^{-1}h_G^*(X,V)=0$. Now assuming either the continuity

$$\text{(7.1.4)} \qquad \operatorname*{colim}_V h_G^*(X,V) = h_G^*(X,X^S)$$

of the cohomology theory or local properties of the pair X,X^S which imply 7.1.4 (e. g. neighbourhood retract). We obtain

<u>Proposition 7.1.5.</u> <u>Let</u> X <u>be</u> <u>a</u> <u>compact</u> G-<u>space</u> <u>such</u> <u>that</u> 7.1.4 <u>holds</u>. <u>Then</u> <u>the</u> <u>inclusion</u> $X^S \longrightarrow X$ <u>induces</u> <u>an</u> <u>isomorphism</u>

$$S^{-1}h_G^*(X) \cong S^{-1}h_G^*(X^S) .$$

There are many variants of 7.1.3 and 7.1.5 according to the different technical (axiomatic) assumptions about theories and spaces involved. We mention some of them. First of all the treatment of homology is quite analogous. Compactness of the space in 7.1.3 may be replaced by finite dimensionality, working with the spectral sequence of a covering and an additive theory.

We now describe a particular of the localization process. We assume that our cohomology theory h_G has suspension isomorphisms for a suitable set of representations, i. e.: Given a family $(V_j \mid j \in J)$ of complex representations and to each j a natural isomorphism $s_j : \tilde{h}_G^*(X) \cong \tilde{h}_G^{*+|j|}(V_j^c \wedge X)$ where $V_j^c = V_j \cup \infty$ is the one-point-compactification and $|j|$ is a suitable index depending additively on V_j (e. g.

the dimension or V_j itself). We assume for simplicity that the representations are complex in order to avoid sign problems. The s_j are assumed to commute. We define the <u>multiplication with the Euler class</u> of V_j to be the composite map

$$\tilde{h}_G^*(X) \xrightarrow[s_j]{} \tilde{h}_G^{*+|j|}(V_j^c \wedge X) \longrightarrow \tilde{h}_G^{*+|j|}(X)$$

where the second map is induced by inclusion of the zero in V_j. Actually this is a special case of the previously discussed module structure, coming from a natural transformation of stable equivariant cohomotopy into h_G. Let S be the multiplicatively closed subset generated by all such Euler-classes. Then $X \setminus X^S$ is the set of all orbits which can be mapped into $V \setminus \{o\}$, where V is any finite direct sum of V_j's. If in particular V_j consists of all non-trivial irreducible representations then X^S is the fixed point set of X. (See tom Dieck [56] for further information on this construction.)

<u>7.2. Classifying spaces for families of isotropy groups.</u>

Let G be a compact Lie group. A set F of closed subgroups is called a <u>familiy</u> if it is closed under conjugation and taking subgroups. (For some of the following investigations it suffices: closed under conjugation and intersection).

Let F be a family. A G-space X is called F-<u>trivial</u> if there exists a G-map $X \longrightarrow G/H$ for some $H \in F$. The G-space X is called F-<u>numerable</u>, if there exists a numerable covering $(U_j \mid j \in J)$ of X by F-trivial G-subsets. See Dold [71] for the notion of numerable covering. Partitions of unity in our context should consist of G-invariant functions.

Let F be a family. We denote by T(G,F) the category of F-numerable G-spaces. The isotropy groups of such spaces lie in F. Let T(G,F)h be the corresponding homotopy category.

Proposition 7.2.1. The category $T(G,F)h$ contains a terminal object $E(F)$, i. e. an object $E(F)$ such that each F-numerable G-space X admits a G-map $X \longrightarrow E(F)$ unique up to G-homotopy.

Proof. We immitate the Milnor construction [115] of a universal bundle. There exists a countable system $(H_j | j \in J)$ of groups $H_j \in F$ such that every group in F is conjugate to an H_j (Palais [124], 1.7.27) Let $E_j = G/H_j * G/H_j * \ldots$ be the join of a countibly infinite number of copies G/H_j. Let $E(F) = *_{j \in J} E_j$ be the join of the E_j (always carrying the Milnor topology).

Let X be an object of $T(G,F)$. We choose a numerable covering $(U_a | a \in A)$ by G-sets $U_a \subset X$ and G-maps $f_a : U_a \longrightarrow G/H_a$ with $H_a \in \{H_j | j \in J\}$. One can assume that A is countable (compare tom Dieck [49], Hilfssatz 2). From $(U_a | a \in A)$ and a subordinate G-invariant partition of unity one constructs a G-map $X \longrightarrow E(F)$ and shows as in tom Dieck [49] that any two G-maps are G-homotopic. The space $E(F)$ is contained in $T(G,F)$ (see Dold [71], 8. for numerability). Hence $E(F)$ is the desired terminal object.

Remark 7.2.2. A terminal object of $T(G,F)$ is uniquely determined up to G-homotopy equivalence. If F_∞ is the family of all subgroups of G then $E(F_\infty)$ is G-contractible because a point is a terminal object in $T(G,F_\infty)h$.

Proposition 7.2.3. Let X be an object in $T(G,F)$. Then $X * E(F)$ is G-homotopy-equivalent to $E(F)$.

Proof. By the methods of tom Dieck [49] one proves that any two G-maps $Z \longrightarrow Y$ for

$$Y = E(F) * X * X * \ldots .$$

are G-homotopic. If $X \in T(G,F)$ then $Y \in T(G,F)$, so that Y is a terminal object in $T(G,F)h$. This yields the G-homotopy equivalences

$$E(F) * X \simeq Y * X \simeq Y \simeq E(F) \ .$$

Let H be a subgroup of G. For a G-space X let $res_H X$ be the H-space obtained by restricting the group action. If F is a family of subgroups of G let $F/H = \{L \cap H \mid L \in F\}$ be the induced family of subgroups of H. With these notations we have

Proposition 7.2.4. $res_H\, E(F) = E(F/H)$.

Proof. By adjointness

$$[Y, res_H\, E(F)]_H = [G \times_H Y, E(F)]_G.$$

If $Y \in T(H,F/H)$ then $G \times_H Y \in T/G,F)$. Hence the H-equivariant homotopy set $[Y, res_H\, E(F)]_H$ contains a single element which means that $res_H E(F)$ is a terminal object. Note that $res_H E(F) \in T(H,F/H)$.

7.3. Adjacent families.

Families of isotropy groups have been used successfully in bordism theory by Conner and Floyd [47] and later by Stong [155] , Kosniowski [106] and others. The classifying spaces E(F) of 7.2 allow to extend some of these methods to arbitrary equivariant homology and cohomology theories. We give some indications of how this can be done.

Let $F_1 \subset F_2$ be two families of subgroups of G and let E_i by a terminal object of $T(G,F_i)h$. Then we have a G-map $f : E_1 \longrightarrow E_2$ unique up to G-homotopy. In the following we assume f to be a closed G-cofibration (replace, if necessary, E_2 with the mapping cylinder of f). If $f' : E_1' \longrightarrow E_2'$ is another such G-cofibration then the pair (E_2,E_1) is G-homotopy-equivalent to the pair (E_2',E_1'); compare tom Dieck-Kamps-Puppe [70] , Satz 2.32. The G-homotopy equivalence moreover is unique in the category of pairs (use terminality).

Suppose an equivariant homology theory h_* is given. We define a new homology theory by

$$(7.3.1) \qquad h_*[F_2,F_1](X,Y) := h_*(E_2 \times X,\ E_1 \times X \cup E_2 \times Y).$$

The exact homology sequence of a pair follows without trouble if Y is closed in X (or use mapping cylinders). Another choice of (E_2,E_1) yields, by the remarks above, a functor which is canonically isomorphic to $h_*[F_2,F_1]$. We put $h_*[F_2,\emptyset] = h_*[F_2]$ if F_1 is empty, i. e.

$$h_*[F_2](X,Y) := h_*(E_2 \times X,\ E_1 \times X).$$

The exact homology sequence of the triple

$$(E_2 \times X,\ E_1 \times X \cup E_2 \times Y,\ E_2 \times Y)$$

gives via the excision isomorphism

$$h_*(E_1 \times X,\ E_1 \times Y) \cong h_*(E_1 \times X \cup E_2 \times Y,\ E_2 \times Y)$$

the long exact sequence of homology theories

$$(7.3.2) \quad \cdots \longrightarrow h_{n+1}[F_2,F_1](X,Y) \longrightarrow h_n[F_1](X,Y) \longrightarrow$$
$$\longrightarrow h_n[F_2](X,Y) \longrightarrow \cdots$$

where n again is taken from a suitable index set.

The relation of the homology theories to the exact sequences of Conner and Floyd is as follows. (We use the notations of Stong [155] .) Let $\mathfrak{N}_*^G(F_2,F_1)$ be the unoriented G-bordism theory of manifolds in $T(G,F_2)$ with boundary in $T(G,F_1)$. Then

Proposition 7.3.3. There exists a natural isomorphism

$$\mathfrak{N}_*^G(F_2,F_1) \cong \mathfrak{N}_*^G[F_2,F_1] .$$

Proof. Exercise. (See tom Dieck [57] .)

Proposition 7.3.3 tells us that bordism with families is unrestricted bordism of suitable spaces.

One of the main uses of families is the induction over orbit types using adjacent families. Two families $F_2 \supset F_1$ are called adjacent if their difference $F_2 \setminus F_1$ is just a single conjugacy class. We are going to analyze this situation.

Let $F_2 \supset F_1$ be adjacent, differing by the conjugacy class of H. Let CZ denote the cone over the space Z. Then we have

Proposition 7.3.4. There exists a canonical natural isomorphism

$$h_*[F_2,F_1](X,A) \cong h_*(G\times_{NH}E(NH/H) \times (CEF_2,EF_2) \times (X,A)).$$

Proof. In the statement of the proposition E(NH/H) is of course the free numerable NH/H-space. One shows that

$$(G\times_{NH} E(NH/H)) * EF_1$$

is a terminal object of $T(G,F_2)h$, hence can be taken as space EF_2. To prove this one recondires the proof of 7.2.1. The above claim then follows from the following considerations: If A and B are G-spaces and P is a point, then we have a G-homeomorphism

$$A * B \cong (A * P) \times B \cup A \times (B * P).$$

Using excision this yields

$$\begin{aligned} & h_*(A * B, B) \cong \\ & h_*((A * P) \times B \cup A \times (B * P), B) \cong \\ & h_*((A * P) \times B \cup A \times (B * P), (A * P) \times B) \cong \\ & h\ (A \times (B * P), A \times B). \end{aligned}$$

Moreover the pair $(B * P, B)$ is G-homotopy-equivalent to the pair (CB, B).

7.4. Localization and orbit families.

We assume given an additive G-homology theory h_* which is stable in the following sense: Let V be a complex G-module. Then we are given suspension isomorphism as in 7.1

$$s_V : \tilde{h}_*(X) \cong \tilde{h}_{*+|V|}(V^c \wedge X)$$

which are compatible $s_W s_V = s_{W \oplus V}$. We assume that the theory is multiplicative with unit $1 \in \tilde{h}_o(S^o)$. The image of 1 under

$$\tilde{h}_o(S^o) \xrightarrow[n_*]{} \tilde{h}_o(V^c) \xleftarrow[\cong]{} \tilde{h}_{-|V|}(S^o)$$

is called Euler class $e(V)$ of V (n is the zero section $S^o = \{o, \infty\} \to V^c$). Let M be a set of G-modules which is closed under direct sums. Let

$$S = S(M) = \{e(V) \mid V \in M\}.$$

We formally invert the elements of S and obtain a new homology theory

$$S^{-1} h_*(X,A).$$

Theories of this type where investigated e. g. in tom Dieck [56], [53], [58], [59].

Let F_∞ be the family of all subgroups of G. Let F_S be the family of isotropy groups appearing on unit spheres $S(V)$, $V \in M$. Then we have

Proposition 7.4.1. *There exists a natural isomorphism of homology theories*

$$S^{-1} h_*(X,A) \cong h_*[F_\infty, F_S](X,A) .$$

Proof. As in tom Dieck [56] one sees that $S^{-1} h_*(X,A)$ is a direct limit over groups $h_*((DV,SV) \times (X,A))$ where V runs through the G-modules in M. Since an additive homology theory is compatible with direct limits we have to show essentially the following: Let V_∞ be the direct sum of a countable number of all irreducible representations which appear as direct summands in modules of M. Then the unit sphere $S(V_\infty)$ is a terminal object in $T(G,F_S)h$. Obviously $S(V_\infty) \in F(G,F_S)$. Any two G-maps $S(V_\infty) \longrightarrow S(V_\infty)$ are G-homotopic (Husemoller [99], 3.6 page 31 - 32). The existence of a G-map $E(F_S) \longrightarrow S(V_\infty)$ is seen as follows: If

$a : G/H \longrightarrow S(V)$ is a G-map, then

$$(u_1 g_1 H,\ u_2 g_2 H, \ldots) \longmapsto \sum_{j=1}^{\infty} u_j \cdot a(g_j H)$$

is a G-map from $G/H * G/H * \ldots$ into $\sum_{j=1}^{\infty} V - \{o\}$.

We have seen in 7.1 that localization allows to cut out suitable pieces of G-spaces. This is also true in the context of families. Let F be a family and X a G-space. Put

$$X_F = \{x \in X \mid G_x \in F\} \quad , \quad X^F = X \setminus X_F \ .$$

We assume that X, X_F etc. are numerable and that the pairs $(X, X_F), (A, A_F)$ have suitable excision properties.

<u>Proposition 7.4.2.</u> <u>The</u> <u>inclusion</u> $(X^F, A^F) \longrightarrow (X, A)$ <u>induces</u> <u>an</u> <u>iso-</u>morphism

$$h_* [F_\infty, F]\ (X^F, A^F) \cong h_* [F_\infty, F]\ (X, A).$$

<u>Proof.</u> 7.2.3 gives $h\ (E(F) * X_F, X_F) = 0$. Since $E(F_\infty) = CE(F)$ we have as in the proof of 7.3.4 $h_* (E(F) * X_F, X_F) \cong h_*(E(F_\infty) \times X_F,\ E(F) \times X_F)$ and the latter group is by excision isomorphic to $h_*((E(F_\infty),\ E(F) x (X, X^F))$. (One has to assume that this excision is actually possible.) The exact homology sequence of $h_* [F_\infty, F]$ for the pair (X, X^F) now yields the asserted isomorphism.

We have to discuss the excision problem. To begin with we have $h_* [F_\infty, F]\ (K) = 0$ for G-subsets K of K_F. If X is completely regular then X^F is closed in X (Palais [124] , 1.7.22). If $K \subset X_F$ is closed in X, then we have ordinary excision $h_*[F_\infty, F](X\ K) = h_*[F_\infty, F]\ (X)$. In order to pass from $(X \setminus K)$ to $X \setminus X_F$ we must investigate the natural map

$$l : h_*[F_\infty, F]\,(X^F) \longrightarrow \text{inv lim}\; h_*[F_\infty, F]\,(U),$$

where the inverse limit is taken over the open G-neighbourhoods U of X^F, and see under which conditions l is an isomorphism.

Now one can use continuity conditions of the theory h_*. But for many spaces X one does not use this continuity. One notes that the inverse limit is taken over isomorphisms. Therefore l is injective if X^F is a G-retract of a neighbourhood U and l is surjective if a retraction $r : U \longrightarrow X^F$ is G-homotopic to the inclusion $U \subset X$.

We now discuss localization of equivariant homology at prime ideals of the Burnside ring and its relation to families of isotropy groups. Again we adopt an axiomatic approach.

We are given the G-equivariant theory $t_*^G(X,Y)$. We assume that $t_*^G(X,Y)$ is naturally a module over A(G). We put $t_*^U(X,Y) = t_*^G(G/U \times X, G/U \times Y)$ and assume that t_*^U is an A(U)-module. The restriction

$$\text{res} = r : t_*^G(X) \longrightarrow t_*^U(X)$$

shall be compatible with the restriction $s : A(G) \longrightarrow A(U)$ i. e. $r(x \cdot y) = s(x) \cdot r(y)$, $x \in A(G)$, $y \in t_*^G(X)$. Moreover we have natural transformations (induction) $\text{ind} : t_*^U(U/K \times X) \longrightarrow t_*^U(X)$ such that the composion

$$t_*^U(X) \xrightarrow[\text{res}]{} t_*^U(U/K \times X) \xrightarrow[\text{ind}]{} t_*^U(X)$$

is multiplication with $U/K \in A(U)$.

Consider the prime ideal $q = q(H,p)$ of $A(G)$ (see 5. 7.) where $H < G$, NH/H is finite of order prime to p if $p \neq 0$. Assume that we have families $F_1 \supset F_2$ such that for $K \in F_1 \setminus F_2$ $q(K,p) = q(H,p)$. Let an index (p) or q denote localization at the prime ideal $(p) \subset Z$ or $q \subset A(G)$. Then we have

Proposition 7.4.3. Multiplication with $y \in q(H,p)$, e. g. $y = [G/H]$, is an automorphism of the homology theory $t_*^G [F_1, F_2]_{(p)}$. The canonical map $t_*^G [F_1, F_2]_{(p)} \longrightarrow t_*^G [F_1, F_2]_q$ is an isomorphism.

Proof. Using exact sequences 7.3.2 and the exactness of localization we see that it suffices to consider adjacent $F_1 \supset F_2$, say with $F_1 \setminus F_2 = (K)$ and $q(K,p) = q(H,p)$. We then use the isomorphism of 7.3.4. We abbreviate $NK = N$. The space $E(N/K)$ is the classifying space (in the sense of Segal [144]) of the category with objects the elements of N/K and exactly one morphism between any two objects. This category defines a simplicial space and its geometric realisation is $E(N/K)$. The skeleton filtration of this simplicial space gives a spectral sequence which has as E_2-term the homology of the following chain complex

$$\ldots \longleftarrow t_*(G \times_N (N/K^i \times Z) \underset{d_i}{\longleftarrow} t_*(G \times_N (N/K)^{i+1} \times Z) \longleftarrow \ldots$$

with $Z = (CEF_2, EF_2) \times (X,A)$ and $d_i = \sum_{j=o}^{i} (-1)^j (pr_j)_*$ where pr_j omits the $(1+j)$-th factor. Multiplication by y, being a natural transformation of homology theories, induces an endomorphism of this spectral sequences. Hence it suffices to show that multiplication with y is an isomorphism on $t_*^G(G \times_N (N/K)^i \times (CEF_2, EF_2) \times (X,A))_{(p)}$ for $i \geq 1$. The group in question is isomorphic to $t_*^G(G/K \times (N/K)^{i-1} \times (CEF_2, EF_2) \times (X,A))_{(p)}$ and therefore the action of $y \in A(G)$ only depends on its restriction $y' \in A(K)$. By 5. 5. this restriction has the form

$y' = \varphi_K y\,[K/K] + \sum a_i\,[K/K_i]$ with $a_i \in Z$ and $(K_i) < (K)$, $(K_i) \neq (K)$. But $\varphi_K(y) \equiv \varphi_H(y) \not\equiv 0 \bmod p$, because $y \in q(H,p)$. Since we localized at (p) multiplication with $\varphi_K(y)\,[K/K]$ is an isomorphism. The proof of 7.3.4 will be finished if we can show that multiplication with $[K/K_i]$ is zero. But by the axiomatic assumption this multiplication factors over $t_*^G(G/K \times K/K_i \times (N/K)^{i-1} \times (CEF_2, EF_2) \times (X,A))_{(p)}$ and this group is zero by 7.4.2.

7.5. Localization and splitting of equivariant homology.

Again we are given an equivariant homology theory t_*^G which is a module over A(G) such that the axioms of the previous section are satisfied. If we localize at (p) the theory becomes a module over $A(G)_{(p)}$. The idempotents of $A(G)_{(p)}$ split off direct factors we are going to describe these direct factors.

Let $q = q(H,p)$ a prime ideal of A(G) where H is the defining group of q (i. e. G/H q). We consider two chain complexes

$$t_*^G \xleftarrow{d_o} t_*^G(G/H) \xleftarrow{d_1} t_*^G((G/H)^2) \xleftarrow{d_2} \cdots$$

$$t_*^G \xrightarrow{d^o} t_*^G(G/H) \xrightarrow{d^1} t_*^G((G/H)^2) \xrightarrow{d^2} \cdots$$

with $d_i = \sum_{j=o}^{i} (-1)^j (pr_j)_*$ and $d^i = \sum_{j=o}^{i} (-1)^j (pr_j)^*$

(Here $(pr_j)^*$ is the induction (alias transfer) which is assumed to exist with suitable properties.)

<u>Proposition 7.5.1. The homology of these chain complexes is zero when localized at</u> q.

<u>Proof.</u> We define a contracting homotopy s for the first chain complex

by the formula

$$s = [G/H]^{-1}(pr_o)^* : t^G_*((G/H)^i)_q \longrightarrow t^G_*((G/H^{i+1})_q.$$

One verifies $ds + sd = id$ using that $pr_* \; pr^*$ is multiplication by $[G/H]$. A similar proof works for the second chain complex. (Compare also section 6.)

We apply the foregoing in the following situation. We put $t^G_*(G/H \times X) = t^H_*(X)$ for G-spaces X. The restriction $t^G_*(X) \longrightarrow t^H_*(X)$ becomes injective when localized at $q(H,p)$ and the image is equal to the kernel of

$$pr_o^* - pr_1^* : t^G_*(G/H \times X)_q \longrightarrow t^G_*(G/H^2 \times X)_q.$$

We denote this kernel by $t^H_*(X)^{inv}_q$, the invariant elements.

Let FH be the family of all subgroups subconjugate to H and let F'H be the family of those $K \in FH$ with $q(K,p) \neq q(H,p)$. Then we have a natural transformation of homology theories

(7.5.2) $$r_H : t^G_*(X)_{(p)} \longrightarrow t^H_*(X)^{inv}_{(p)} \longrightarrow t^H_*[FH,F'H](X)^{inv}_{(p)}$$

where the first map is restriction and the second comes from the exact homology sequence of the pair FH,F'H. (Note that EFH is H-contractible by 7.2.4)

Theorem 7.5.3. (a) $(r_H)_q$ is an isomorphism.

(b) r_H is split surjective.

(c) The product of the maps r_H

$$r = (r_H) : t^G_*(X)_{(p)} \longrightarrow \prod_{(H) \in \phi(p)} t^H_*[FH,F'H](X)^{inv}_{(p)}$$

<u>is injective and an isomorphism if only a finite number of factors on the right are non-zero</u>.

<u>Proof.</u> (a) From 7.4.3 we know that

$$t^H_* [FH, F'H] (X)^{inv}_{(p)} \cong t^H_* [FH, F'H] (X)^{inv}_{(p)}$$

because the isomorphism holds without "inv" and localization is exact. We have for any space X the isomorphism $t^G_*(X)_q = t^H_*(X)^{inv}_q$. What remains to be shown is that $t^H_*(X)_q \longrightarrow t^H_*[FH, F'H](X)_q$ is an isomorphism or, equivalently, that $t^H_*[F'H](X)_q$ is zero. Because of the additivity of the theory it is enough to show that $t^H_*(G/K \times X)_q = 0$ for $K \in F'H$. This follows from the homology version of 7.1.3 because $A(K)_{q(H,p)} = 0$.

(b) In view of (a) r_H is up to isomorphism obtained from tensoring the canonical map $A(G)_{(p)} \longrightarrow A(G)_q$ with $t^G_*(X)$. This canonical map is split surjective, because q has an associated idempotent $e(q) \in A(G)_{(p)}$ and $e(q)\, A(G)_{(p)} = A(G)_q$.

(c) The analogous assertion is true if we localize at maximal ideals of A(G).

<u>Remark 7.5.3.</u> Let G be a finite group. Let p be a prime number or 0. Write $|G| = p^k m$ with m prime to p if $p \neq 0$. Write $|G| = m$ is case p=0. If we can divide by m in the groups $t^G_*(X,A)$ then the map r in 7.5.3 is an isomorphism without localization at (p). In particular if we invert the order of the group, then the homology theory splits into summands

$$t^H_* [FH, F'H] (X)^{NH/H}$$

where FH (resp. F'H) is the family of all (resp. all proper) subgroups of G and the NH/H means the ordinary invariants under the NH/H-action.

Remark 7.5.4. We have seen that A(G) may contain many idempotents even without localization. Such idempotents split off direct factors from equivariant homology theories and these direct factors may be described using families. This is quite analogous to the considerations above. For details see tom Dieck [66] .

7.6. Transfer and Mackey structure.

We have to describe examples of homology theories which satisfy the axioms of 7.4. We use some homotopy theory which is developed in the next chapter which should be consulted for notation and some details. The application of the Burnside ring to equivariant (co-)homology and (co-) homotopy makes use of the Lefschetz fixed point index and fixed point transfer developed by Dold [76] , [77] in the non-equivariant case. We refer to these papers for details and further information. We recall the results that we need in a slightly different set up.

Let G be a compact Lie group. A G-map $p : E \longrightarrow B$ is called G-ENR_B (= euclidean G-neighbourhood retract over B) if there exists a real G-module V with G-invariant inner product, an open G-subset $U \subset B \times V$, and G-maps $i : E \longrightarrow U$, $r : U \longrightarrow E$ over B with $ri = id(E)$. Let $(B \times V)^c$ be the Thom space of the trivial bundle $B \times V \longrightarrow B$. Note that $(B \times V)^c$ is canonically G-homeomorphic to the smashed product $B^+ \wedge V^c$ where B^+ is B with a separate base point added.

If p, i, and r are as above, if p is a proper map and B locally compact and paracompact there exists a G-invariant continous function $\varrho : B \longrightarrow]0, \infty[$ such that for all $b \in B$ we have $\varrho(b) < d(ip^{-1})b)$, $\{b\} \times V \setminus U)$, where d denotes the metric derived from the inner product on V.

For such maps we call <u>transfer map</u> associated to the data p,i, and r any pointed G-map

$$h : (B \times V)^c \longrightarrow (E \times V)^c$$

with the following properties

(7.6.1) The inverse image of $E \times \{o\}$ under h is iE.

(7.6.2) For $u = (b,v) \in U$ and $2d(v, pr_2 iru) < \rho(b)$ the map h has the form
$h(u) = (ru, v - pr_2\, iru)$.

If X and Y are pointed G-spaces we let $\omega_G^o(X;Y)$ denote the direct limit over pointed G-homotopy sets $[V^c \wedge X, V^c \wedge Y]_G^o$ using suspensions over all (complex) G-modules; see chapter 8. Using suspension isomorphisms we extend this functor to functors $\omega_G^{\alpha}(X;Y)$, graded over α in the real representation ring RO(G) of G. We get a cohomology theory in the variable X and a homology theory in the variable Y.

<u>Proposition 7.6.3. Let</u> $p : E \longrightarrow B$ <u>be</u> G-ENR_B <u>with retract representation</u> i,r <u>as above</u>. <u>Let</u> p <u>be proper and</u> B <u>locally compact and paracompact</u>. <u>Then transfer maps</u> h <u>exist and their pointed</u> G-<u>homotopy class is uniquely determined by</u> 7.6.1 <u>and</u> 7.6.2. <u>The stable</u> $\tilde{p} \in \omega_G^o(B^+;E^+)$ <u>of</u> h <u>is independent of the retract representation</u> i,r.

<u>Proof.</u> A proof may be extracted from Dold [77] . (Note that we consider a somewhat simpler situation.)

<u>Example 7.6.4.</u> Let $p : E \longrightarrow B$ be a submersion between compact differentiable G-manifolds. Let $j : E \longrightarrow V$ be an equivariant embedding

into a G-module V. Then $i = (p,j) : E \longrightarrow B \times V$ is an embedding over B. A retract representation may be obtained from a tubular neighbourhood U of i. Hence p is G-ENR_B.

If $t_G^*(-)$ is a cohomology theory for G-spaces which has suspension isomorphisms for all G-modules (or all complex G-modules, etc.) then a transfer map h or $\tilde{p}$ as is in 7.6.3 induces a homomorphism

$$(7.6.5) \qquad p_! : t_G^*(E) \longrightarrow t_G^*(B)$$

called transfer. Similarly for homology theories t_*^G we get a transfer

$$(7.6.6) \qquad p^! : t_*^G(B) \longrightarrow t_*^G(E) .$$

The composition $p_! \, p^*$ is in the case of a multiplicative cohomology theory multiplication with the Lefschetz-Dold index $I_p \in t_G^0(B)$ (see Dold [76]). In particular we have the index $I(X) \in \omega_G^0$ for the map $X \longrightarrow$ Point, where X is a compact G-ENR and $\omega_G^0 = \operatorname{colim} [V^c, V^c]_G^0$ are the coefficients of equivariant stable cohomology in dimension zero. As usual ω_G^0 is a commutative ring with unit. In the next chapter we shall prove the following basic result.

Theorem 7.6.7. The assignment induces a map $I_G : A(G) \longrightarrow \omega_G^0$. This map is an isomorphism of rings.

We now collect the formal properties of the transfer which are used to establish the axioms used in the localization theorems in 7.4 and 7.5. We call a G-ENR_B $p : E \longrightarrow B$ with p proper and B locally compact and paracompact a transfer situation. If P is a point we abbreviate $\omega_G^0(B;P^+) = \omega_G^0(B)$; this is a commutative ring, with unit if $B = C^+$. The cohomology group $t_G^*(B^+ \wedge X)$ carries a $\omega_G^0(B^+)$-module structure which is natural in X. The definition runs as follows: If $a \in \omega^0_G(B^+)$ is represented by $a : V^c \wedge B^+ \longrightarrow V^c$ let $a_1 : V^c \wedge B^+ \longrightarrow V^c \wedge B^+$ be given as $(v,b) \longmapsto (a(v,b),b)$. Then the action of a is the map

$$t^*_G(B^+ \wedge X) \cong t^*_G(V^c \wedge B^+ \wedge X) \xrightarrow[(a_1 \wedge id)^*]{} t^*_G(V^c \wedge B^+ \wedge X) \cong t^*_G(B^+ \quad X)$$

where the isomorphisms are suspensions. Similarly for homology. The next proposition collects what we need about the transfer and this module structure.

<u>Proposition 7.6.8.</u> Let $h : E' \longrightarrow E$ and $f : E \longrightarrow B$ be transfer situations.

(a) fh is a transfer situation and $h^! f^! = (fh)^!$, $f_! h_! = (fh)_!$.

(b) Let

$$\begin{array}{ccc} E_1 & \xrightarrow{\ \phi\ } & E \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ B_1 & \xrightarrow{\ \varphi\ } & B \end{array}$$

be a pull-back and B_1 locally compact and paracompact. Then f_1 is a transfer situation and

$$f^! \varphi_* = \phi_* f_1^!, \quad (f_1)_! \varphi^* = \phi^* f_! .$$

(c) For $f_* : t^G_*(E^+ \wedge X) \longrightarrow t^G_*(B^+ \wedge X)$ and $a \in \omega^0_G(B^+)$ we have

$$f_*(f^* a \cdot s) = a \cdot f_*(s) .$$

(d) For $f^* : t^*_G(B^+ \wedge X) \longrightarrow t_G(E^+ \wedge X)$ and $b \in \omega^0_G(B^+)$ we have

$$f^*(b \cdot x) = f^*(b) \cdot f^*(x) .$$

(e) For $f^! : t^G_*(B^+ \wedge X) \longrightarrow t^G(E^+ \wedge X)$ and $a \in \omega^0_G(B^+)$ we have

$$f^!(a\cdot x) = f^*(a)\cdot f^!(x).$$

(f) For $f_! : t_G^*(E^+ \wedge X) \longrightarrow t_G^*(B^+ \wedge X)$ and $a \in \omega_G^o(B^+)$ we have

$$f_!(f^*a\cdot b) = a\cdot f_!(b).$$

(g) If $p : E \longrightarrow B$ is a transfer situation and $H < G$ a closed subgroup then the H-fixed point map $p^H : E^H \longrightarrow B^H$ is again a transfer situation (for the group NH/H) and $(p^H)^\sim = r\tilde{p}$, where $r : \omega_G^o(B^+;E^+) \longrightarrow \omega^o_{NH/H}(B^{H+};E^{H+})$ is induced by restriction to H-fixed points.

(h) If $p : E \longrightarrow B$ is a transfer situation for the subgroup H of G then $G \times_H p : G \times_H E \longrightarrow G \times_H B$ is a transfer situation for the group G and $j(\tilde{p}) = (G \times_H p)^\sim$ where

$$j : \omega_H^o(B^+;E^+) \longrightarrow \omega_G^o(G \times_H B^+; G \times_H E^+)$$

is induced by the functor $X \longmapsto G \times_H X$.

For the proof of (a) and (b) we refer to the above mentioned work of Dold. Using this and our description of transfer maps, (c) to (h) become fairly routine verifications.

The applications to the axiomatic treatment in 7.4 is as follows: $res : t_*^G(X) \longrightarrow t_*^G(G/H \times X)$ is the transfer for $f : G/H \longrightarrow$ Point and $ind : t_*^G(G/H \times X) \longrightarrow t_*^G(X)$ is induced by f. The relevant properties follow from 7.6.7 and 7.6.8.

For finite groups there exist important equivariant homology theories which are not stable in the sense that they admit suspension isomorphisms for enough G-modules. Examples are the bordism theories of

Conner and Floyd. Nevertheless the methods of 7.4 and 7.5 are applicable. The relevant axioms can be established by direct geometric methods, without using transfer and stable homotopy as above. For bordism theories "restriction" is just the usual restriction to a subgroup and "induction" is induced by the functor $X \longmapsto G \times_H X$ from H-spaces to G-spaces. For an axiomatic treatment along these lines see tom Dieck [60] . The Bredon equivariant homology and cohomology (Bredon [36] , Bröcker [38] , Illman) have canonical restriction and induction if the coefficient system is a Mackey functor.

7.7. Localization of equivariant K-theory.

In order to add some meat to the vegetable soup 7.1 - 7.6 we consider equivariant K-theory as an example of the previous general theory. Of course, one can treat K-theory more directly, using representation theoretic methods. We let $K_G(X)$ be the Grothendieck ring of complex G-vector bundles over the (compact) G-space X (see Segal [142]).

Let G be a compact Lie group. As in Segal [143] we use the

Definition 7.7.1. A closed subgroup S of G is called Cartan subgroup of G if NS/S is finite and S is topologically cyclic (i. e. powers of a suitable elements are dense $\Leftrightarrow$ S is the product of a torus and a finite cyclic group). A Cartan subgroup is p-regular if the group of components has order prime to p, for a prime number p.

Let C be the set of conjugacy classes of Cartan subgroups of G and C(p) the subset of p-regular groups. We refer to Segal [143] for the proof of

Proposition 7.7.2. The set C is finite.

If $(S) \in C(p)$, $P < NS/S$ a p-Sylow subgroup and $Q < NS$ the pre-image of P then $|NQ/Q| \not\equiv 0$ mod p. Hence $Q = Q_S$ is the defining group of the prime ideal $q(S,p)$.

By the equivariant Bott-isomorphism the cohomology theory $K_G(-)$ has suspension isomorphism for complex G-modules. Thus $K_G(-)$ becomes and A(G)-module and $K_G(\text{Point}) = R(G)$ becomes an A(G)-algebra. Actually the map $A(G) \longrightarrow R(G)$ which comes from the homotopy considerations of 7.6. coincides with the equivariant Euler characteristic of chapter 5.

If $H < G$ let H_p be the smallest normal subgroup such that H/H_p is a p-group.

<u>Proposition 7.7.3.</u> $R(G)_{q(H,p)} = 0$ <u>if and only if</u> H_p <u>is a p-regular Cartan subgroup</u>.

<u>Proof.</u> Let $S < G$ be a topologically cyclic subgroup with generator G. The diagram

$$\begin{array}{ccc} A(G) & \xrightarrow{\chi_G} & R(G) \\ {\scriptstyle \varphi_S}\downarrow & & \downarrow{\scriptstyle e_g} \\ Z & \xrightarrow{\subset} & \mathbb{C} \end{array}$$

is a commutative diagram of ring homomorphisms (χ_G equivariant Euler characteristic 5.5.6 ; e_g evaluation of characters at g). We view everything as A(G)-module and localize at $q = q(H,p)$. Since elements of R(G) are detected by the various e_g we can find an S with $\mathbb{C}_q \neq 0$ if $R(G)_q \neq 0$. But then $Z_q \neq 0$ and this implies $q(S,p) = q(H,p)$. Since S is cyclic there exists a Cartan subgroup T with $S \triangleleft T$ such that T/S is torus, by Segal [143] , 1.2 and 1.5. Hence $q(T,p) = q(S,p)$. One can take a p-regular subgroup T' of T with $q(T,p) = q(T',p)$. The assertion then follows from 5.   . An analogous argument shows that $R(G)_{q(S,p)} \neq 0$ for a p-regular Cartan group p.

From 7.7.3 and 7.6 we obtain natural isomorphisms

(7.7.4) $$K_G(X)_{(p)} \cong \bigoplus_{(S) \in C(p)} K_G(X)_{q(S,p)} \qquad p \neq o$$

(7.7.5) $$K_G(X)_{(o)} \cong \bigoplus_{(S) \in C} K_G(X)_{q(S,o)}$$

(7.7.6) $$K_G(X)_{q(S,p)} \cong K_{QS}(X)^{inv}_{q(S,p)}$$

where $QS < NS$ is the pre-image of a p-Sylow subgroup of NS/S. Moreover in 7.7.6 X can be replaced by $X(S) = \{ x \mid q(G_{x,p}) = q(S,p)\}$.

We are going to study the case of finite groups G more closely. Then S is a cyclic group of order prime to p and we have $1 \longrightarrow S \longrightarrow QS = H \longrightarrow P \longrightarrow 1$ with a p-group P, hence H is a semi-direct product and a p-hyperelementary group. Moreover

$$K_H(X)_{q(H,p)} = K_H(X^S)_{q(H,p)} \ .$$

One can describe H-equivariant vector bundles over X^S. The fibre consists of S-modules and these have to be grouped together according to the conjugation action of P.

We specialize further to the case $H = S \times P$. Then naturally $K_H(X^S) = R(S) \otimes K_p(X^S)$. Moreover $A(H) = A(S) \otimes A(P)$ and the following diagram of equivariant Euler characteristics is commutative

$$\begin{array}{ccccc} A(S) & \longrightarrow & A(H) & \longleftarrow & A(P) \\ \downarrow \chi_S & & \downarrow \chi_H & & \downarrow \chi_P \\ R(S) & \longrightarrow & R(H) & \longleftarrow & R(P) \end{array}$$

Let S be the cyclic group of order m and generator g. Suppose $(m,p) = 1$. Let x denote the irreducible standard representation of G. Then $R(S) \cong Z[x]/(x^m-1)$. Let $E = \{1-x^i \mid 1 \leq i \leq m-1\}$ be the set of Euler classes of non-trivial irreducible S-modules. Let $e : R(S) \longrightarrow Z[u_m]$ be evaluation of characters at g; here u_m is a primitive m-th root of unity.

<u>Proposition 7.7.7. The map</u> e <u>induces an isomorphism of rings</u>

$$\tilde{e} : R(S)[E^{-1}] \cong Z[m^{-1}, u_m].$$

<u>Proof.</u> We have to invert the $1 - u_m^i$, $1 \leq i \leq m-1$. If $m = p_1^{a(1)} \ldots p_r^{a(r)}$ is the factorization into prime powers and if $u(i)$ is a primitive $p_i^{a(i)}$-th root of unity then $1-u(i)$ has norm p_i hence is invertible in $Z[m^{-1}, u_m]$. Moreover we see that m^{-1} and u_m are in the image of e. Therefore e is surjective. The map e factorizes .

$$Z[x]/(x^m-1) \xrightarrow[e_1]{} Z[x]/\Phi_m(x) \xrightarrow[e_2]{} Z[u_m]$$

where Φ_m is the m-th cyclotomic polynomial. The map e_2 is an isomorphism. If we put $x^m-1 = \Phi_m(x)\, P_m(x)$ then Φ_m and P_m are relatively prime and the canonical map

$$Z\,[x]/(x^m-1) \longrightarrow Z\,[x]/\Phi_m \oplus Z\,[x]/P_m$$

is injective. The prime factors of P_m divide certain $1-x^i$, $1 \leq i \leq m-1$, and since these elements are to be inverted the P_m have to be inverted too. This can only happen if the localization E^{-1} trivialises the factor $Z\,[x]/P_m$, so that

$$Z\,[x]/(x^m-1)\,[E^{-1}] \longrightarrow Z\,[x]/\Phi_m \quad E^{-1}$$

must be injective and hence $\check{e}$ is injective too.

<u>Proposition 7.7.8. The map</u> e <u>induces an isomorphism of rings</u>

$$e' : R(S)_{q(S,p)} \longrightarrow Z_{(p)}\,[u_m]\,.$$

<u>Proof.</u> We have to invert the image of $A(S) \setminus q(S,p)$ under $\chi_S : A(S) \longrightarrow R(S)$. If $y \notin q(S,p)$ then $e\,\chi_S(y) = |y^g| = |y^S| \not\equiv 0(p)$. Hence e induces a surjective map e'. The product of the Euler classes $\prod_{i=1}^{m-1} (1-x^i)$ is a rational representation and therefore equal to $\chi_S(y)$ for a suitable $y \in A(S)$. One has $|y^S| = m$, so $y \notin q(S,p)$. Hence the map in question is a localization of e in 7.7.7 and therefore injective.

We now come back to $H = S \times P$. We note that $A(P)_{q(P,p)} = A(P)_{(p)}$ is a local ring and

$$A(H)_{q(H,p)} \cong A(S)_{q(S,p)} \otimes A(P)_{q(P,p)}$$

and more generally therefore

$$(7.7.9) \qquad K_H(X^S)_{q(H,p)} \cong R(S)_{q(S,p)} \otimes K_p(X^S)_{(p)}\;.$$

Corollary 7.7.10. Let $m = |G|$. Then we have a canonical isomorphism of rings

$$K_G(X)[m^{-1}] \cong \bigoplus_{(C)} (R(C)[E_C^{-1}] \otimes K(X^C))^{NC/C}$$

where (C) runs through the conjugacy classes of cyclic subgroups of G, and $E_C \subset R(C)$ is the set of Euler classes of non-trivial irreducible C-modules.

7.8. Localization of the Burnside ring.

Let $F_1 \supset F_2$ be families of subgroups of G. We denote by $A(G;F_1)$ the ideal of A(G) generated by sets (or spaces) X with isotropy groups in F_1 and by $A(G;F_1,F_2)$ the ideal $A(G;F_1)$ modulo the subideal $A(G;F_2)$.

For simplicity let G be a finite group. If $(H) \in \phi(p)$, i. e. $|NH/H| \not\equiv 0 \bmod p$ let H_p be the smallest normal subgroup such that H/H_p is a p-group. Then $\{K | q(K,p) = q(H,p)\} = \{K | (H_p) \lesssim (K) \leq (H)\}$. Call this set $F_o(H)$. We put $F(H) = \{K | (K) \leq (H)\}$ and $F'(H) = F(H) \setminus F_o(H)$.

The ring $A(G)_{(p)}$ splits into a direct product of rings $A(G)_{q(H,p)}$, $(H) \in \phi(p)$, and these factors may also be written as $e(H)\ A(G)_{(p)}$ where $e(H)$ is a suitable indecomposable idempotent element of $A(G)_{(p)}$.

Proposition 7.8.1. Taking H_p-fixed points induces an isomorphism

$$A(H;FH,F'H) \cong A(H/H_p) \ .$$

Proof. Both groups have as an additive basis the H/K, $(H_p) \leq (K) \leq (H)$, and $H/K^{H_p} = H/K$.

Proposition 7.8.2. The following groups are canonically isomorphic

$$A(G)_{q(H,p)},\ A(G;FH)_{q(H,p)},\ A(G;FH,F'H)_{q(H,p)}$$

and $$A(G;FH,F'H)_{(p)}.$$

Proof. The quotient map $A(G;FH) \longrightarrow A(G;FH,F'H)$ becomes an isomorphism after localization at $q(H,p)$ because the kernel $A(G;F'H)$ is detected by fixed point mappings $\varphi_L : A(G;F'H) \longrightarrow Z$ with $q(L,p) \neq q(H,p)$ and therefore $Z^L_{q(H,p)} = 0$ where $Z^L = Z$ is an $A(G)$-module via φ_L. For a similar reason the inclusion $A(G;FH) \longrightarrow A(G)$ induces an isomorphism of its $q(H,p)$-localizations. The canonical map $A(G;F,F'H)_{(p)} \longrightarrow A(G;FH,F'H)_{q(H,p)}$ is an isomorphism by an argument and in the proof of 7.

The idempotent $e(H)$ is contained in $A(G;FH)_{(p)}$ and multiplication by $e(H)$ induces a split surjection $A(G)_{(p)} \longrightarrow A(G;FH,F'H)_{(p)}$ which corresponds to the canonical map $A(G)_{(p)} \longrightarrow A(G)_{q(H,p)}$ under the isomorphisms of 7.8.2. By the general theory we have an isomorphism

$$(7.8.3) \qquad A(G;FH,F'H)_{(p)} = A(H;FH,F'H)^{inv}_{(p)} .$$

Combining with 7.8.1 we obtain

Proposition 7.8.4. Taking H_p-fixed points for the various $(H) \in \phi(p)$ induces a ring isomorphism

$$A(G)_{(p)} \longrightarrow \prod_{(H) \in \phi(p)} A(H/H_p)^{inv}_{(p)}$$

and the corresponding map into the product without "inv" is a split monomorphism of rings.

7.9. Comments.

For localization of equivariant K-theory see Atiyah-Segal and Segal [142] ; for equivariant cohomology: Quillen [127] , Hsiang ; for bordism theory tom Dieck [53] , [58] , [59] Wilson [167] ; for cohomotopy and general theory: Kosniowski [105] , tom Dieck [56] , [57] , [60] . The presentation in this section is mainly drawn from the author's papers and unpublished manuscripts.