10. Geometric Modules over the Burnside Ring.

We investigate in this section stable equivariant homotopy sets of spheres. We consider them as modules over the Burnside ring using the fact the Burnside ring is isomorphic to stable equivariant homotopy of spheres in dimension zero. In order not to become involved in the homotopy group of spheres we mainly study those questions which only involve the concept of mapping degree. In particular we continue our study of homotopy equivalences between representations.

10.1. Local J-groups.

In order to prepare for the general study of vector bundles we study a somewhat weaker equivalence between representations than homotopy equivalence. In particular we recover results of Atiyah-Tall [14] , Lee-Wasserman [110], Snaith [151] .

We call real G-modules V and W locally J-equivalent, in symbols $V \sim_{loc} W$, if for each subgroup $H < G$ there exists a G-module U and G-maps

$$f : S(V \oplus U) \longrightarrow S(W \oplus U), \quad g : S(W \oplus U) \longrightarrow S(V \oplus U)$$

such that f^H and g^H have degree one. (Note that these degrees depend on the choice of orientations and are therefore only defined up to sign.) We put

$$(10.1.1) \qquad TO_G = \{ V-W \in RO(G) \mid V \sim_{loc} W \}$$

$$JO_G^{loc} = RO(G)/TO_G.$$

Note that we have a canonical quotient map

$$q(G) : JO_G^{loc} \longrightarrow RO(G)/RO_o(G) =: RO(G)_\Gamma ,$$

provided G is a finite group.

Theorem 10.1.2. *For every finite group* G *the map* q(G) *is an isomorphism*.

Proof. We have to show that for a G-module V and k prime to $|G|$ the relation $V \sim_{loc} \psi^k V$ holds. We can assume that k is an odd integer. We first show that there exist stable maps $f : V \longrightarrow \psi^k V$ such that for all $H < G$ the degree of f^H has the form k^t. (A stable map $f : V \longrightarrow W$ is a map $f : S(V \oplus U) \longrightarrow S(W \oplus U)$ for suitable U). If V is one-dimensional there is no problem. Next suppose that V is two-dimensional and irreducible; then $G/\ker V =: K$ is cyclic or dihedral and $V = \mathbb{C}$ with suitable action (see 9.7) and the map f can be taken as $z \longmapsto z^k$. In general, by a theorem of Brauer (Serre [147] , 12.6), we can write $V = \sum n_i \operatorname{ind}_{H_i}^G V_i$, $n_i \in Z$, V_i irreducible of dimension ≤ 2. Since G is prime to k induction commutes with ψ^k. Hence we have stable maps $\operatorname{ind}_{H_i}^G V_i \longrightarrow \operatorname{ind}_{H_i}^G \psi^k V_i$ of the required type. Moreover we can find an integer n such that $\psi^{k^n} V_i = V_i$ (choose n so that $k^n \equiv 1 \bmod |G|$). Hence we can find stable maps $\operatorname{ind}_{H_i}^G \psi^{k^{n-1}} V_i \longrightarrow \operatorname{ind}_{H_i}^G V_i$ so that negative n_i in the expression for V don't make trouble. Since we can find numbers k and l with $(k,l) = 1$ and $\psi^k V = \psi^l V$ suitable linear combinations of stable maps f,g with degrees $d(f^H) = k^t$, $d(g^H) = l^u$ give a map h with $d(h^H) = 1$; q. e. d.

10.2. Projective modules.

We recall some of the homotopy notions introduced in section 8. Let E and F be real G-modules, G being a compact Lie group. Put $\alpha = E-F \in RO(G)$ and let

$$\omega_\alpha = \omega_\alpha^G = \{ S^E, S^F \}$$

be the stable G-equivariant homotopy group of pointed stable G-maps $S^E \longrightarrow S^F$. Here S^E denotes the one-point compactification of E. The groups ω_α are the coefficient groups of an equivariant homology theory. When we need space for lower indices we write

$$\omega_\alpha = \omega^{-\alpha}$$

Smashed product of representatives induces a bilinear pairing

$$\omega_\alpha \times \omega_\beta \longrightarrow \omega_{\alpha+\beta} .$$

In particular ω_α is a module over ω_0, the stable equivariant homotopy ring of spheres in dimension zero. The pairing above induces a homomorphism

$$(10.2.1) \qquad m_{\alpha,\beta} : \omega_\alpha \otimes_{\omega_0} \omega_\beta \longrightarrow \omega_{\alpha+\beta} .$$

<u>Remark 10.2.2.</u> The modules ω_α are determined by α only up to <u>non-canonical</u> isomorphism because in general S^E has many homotopy classes of equivariant self-homotopy-equivalences. This causes difficulties if one has to use associativity or commutativity of the pairing $m_{\alpha,\beta}$. A way out of these difficulties is to choose canonical representatives α = E-F or extra structure (like suitable orientations).

<u>Theorem 10.2.2. Let</u> α = E-F <u>be in</u> $RO_0(G)$ <u>see</u> (9.1). <u>Then the following</u> holds:

(i) <u>The module</u> ω_α <u>is a projective</u> ω_0<u>-module of rank one.</u>

(ii) <u>For each</u> $\beta \in RO(G)$ <u>the pairing</u> (10.1.1)

$$\omega_\alpha \otimes_{\omega_0} \omega_\beta \longrightarrow \omega_{\alpha+\beta}$$

<u>is an isomorphism.</u>

(iii) <u>The</u> ω_o-<u>module</u> ω_α <u>is free if and only if</u> E <u>and</u> F <u>are stably</u> G-<u>homotopy equivalent</u> (<u>in the sense of</u> 9.1).

We split the proof into a sequences of Propositions. The whole section is concerned with the proof.

First recall the definition (and result): Let P be a module over the commutative ring R. Then P is a projective R-module of rank one if and only if P is finitely generated and for each maximal ideal q of R the localization P_q at q is a free R_q-module of rank one (see Bourbaki [33] , § 5 Théorème 2).

In the following we write

$$\omega = \omega_o, \quad \omega_\alpha = \omega^{-\alpha} .$$

We have shown in section 8 that ω is canonically isomorphic to the Burnside ring A(G). Using this isomorphism and the determination of the prime ideals of A(G) in 5.7 we can say:

Let $q \subset \omega$ be a maximal ideal. Then there exists a group $H < G$ (unique up to conjugation) such that NH/H is finite, the order of NH/H is prime to the characteristic $p \neq 0$ of ω/q and q is the kernel of mapping degree homomorphism d_H mod p where

(10.2.3)
$$d_H : \omega \longrightarrow Z$$
$$d_H [f] = \text{degree } f^H .$$

The corresponding ideal is then denoted q(H,p).

To define the mapping degree between different manifolds we need to choose orientations. Given E and F we choose orientations for S^E and S^F and define

$$(10.2.4) \qquad d_{\alpha,H} = d_H : \omega_\alpha \longrightarrow Z$$

by $d_{\alpha,H}[f]$ = degree f^H if $\dim E^H = \dim F^H$ and $= 0$ otherwise. Then we show

<u>Proposition 10.2.5.</u> <u>If</u> $\alpha = E-F \in RO_0(G)$ <u>then there exists for each</u> $H < G$ <u>with</u> NH/H <u>finite and</u> $|NH/H| \not\equiv 0 \bmod p$ an $x \in \omega_\alpha$ <u>such that</u>

$$d_H\, x \not\equiv 0 \bmod p\ .$$

(Note that this assertion is independent of the ambiguity in the definition of d_H).

<u>Proof.</u> An algebraic proof for finite G is given in Theorem 10.1.2. We give a topological proof for general G. We first show the existence of an H-map $f : S^E \longrightarrow S^F$ such that f^H has degree one. (Since we are only interested in stable maps we can assume that $\dim E^H = \dim F^H > 1$.) By the assumption $\alpha \in RO_0(G)$ we have $\dim E^H = \dim F^H$ and so we choose an H-map $f_1 : S^{E^H} \longrightarrow S^{F^H}$ of degree one. We extend f_1 to an H-map f using the obstruction theory of 8.3. The obstructions to extending over an orbit bundle lie in groups

$$H^i(X_n/G,\ X_{n-1}/G;\ \pi_{i-1}(Y^K))$$

where $X = S^E$, $Y = S^F$, $X_n \setminus X_{n-1} = X_{(K)}$ in an admissible filtration of X. Since $X_{(K)}/G = X_K/NK \subset X^K/NK$ and $\dim X^K = \dim Y^K$ we see that the obstruction groups vanish for dimensional reasons. Hence an f exists as claimed. We now apply the transfer homomorphism

$$t_H^G : \omega^H_{\alpha|H} \longrightarrow \omega^G_\alpha$$

which satisfies

$$d_{\alpha,K}(t_H^G\, y) = \chi(G/H^K)\; d_{\alpha|H,K}(y) .$$

The element $x = t_H^G[f]$ has the desired property.

Proposition 10.2.6. *For $q = q(H,p)$ and $\alpha \in RO_o(G)$ the module ω_q^α is a free ω_q-module on one generator. The element $x \in \omega_q^\alpha$ is a generator if and only if $d_{\alpha,H}(x) \not\equiv 0 \bmod p$.*

Proof. Take $x \in \omega^\alpha$, $y \in \omega^{-\alpha}$. Then multiplication with x resp. y, using the pairing 10.2.1, gives ω-linear maps

$$y_* : \omega^\alpha \longrightarrow \omega \quad , \; x_* : \omega \longrightarrow \omega^\alpha$$

respectively. The composition $y_*\, x_*$ is multiplication with $yx \in \omega$. By definition of $q(H,p)$ this element becomes a unit in ω_q if

$$d_H(yx) = \pm\, d_H(y)\; d_H(x) \not\equiv 0 \bmod p.$$

(Since d_H depends on the choice of orientations we have to put in a $\pm$.) A similar argument applies to $x_*\, y_*$. If xy is a unit in ω_q then x_{*q} is an isomorphism. By 10.2.5 we can find x,y such that xy becomes a unit in ω_q. This proves that ω_q^α is free with generator x. Since any other generator of ω_q^α differs from x by a unit of ω_q we also obtain the second assertion.

We now prove (ii) of the Theorem in case $\beta \in RO_o(G)$. Using a basic fact of commutative algebra (Bourbaki [33] , § 3.3.   ) we need only show that the localizations $(m_{\alpha,\beta})_q$ are isomorphisms, for each maximal ideal $q \subset \omega$. But then we are dealing with a map

$$\omega_q^{\alpha} \otimes \omega_q^{\beta} \longrightarrow \omega_q^{\alpha+\beta}$$

between free ω_q-modules of rank one (10.2.6), and the same Proposition tells us that the tensor product of the generators is mapped onto a generator.

We now finish the proof of (i) by showing

<u>Proposition 10.2.7.</u> <u>For</u> $\alpha \in RO_o(G)$ <u>the</u> ω-<u>module</u> ω_α <u>is finitely generated</u>.

<u>Proof.</u> By the remarks above we have an isomorphism $\omega_\alpha \otimes \omega_{-\alpha} \cong \omega$. Let the element $1 \in \omega$ correspond to $\sum_i m_i \otimes n_i$. Then ω_α is generated as ω-module by the m_i, namely for $x \in \omega_\alpha$

$$x = (\sum m_i \otimes n_i)x = \sum m_i(n_i x).$$

(This uses associativity of the pairings m).

<u>Remark 10.2.8.</u> If G is finite then $\omega_G^o(X;Y)$ is a finitely-generated ω_G^o-module if X and Y are finite G-CW-complexes. This follows by induction over the number of cells (using that ω_G^o is noetherian). What happens for G a compact Lie group?

In order to prove (ii) we note that an inverse to $m_{\alpha,\beta}$ is given by

$$\omega_{\alpha+\beta} \cong \omega_{\alpha} \otimes (\omega_{-\alpha} \otimes \omega_{\alpha+\beta}) \longrightarrow \omega_{\alpha} \otimes \omega_{\beta} .$$

Finally we show (iii). If E and F are stably G-homotopy equivalent then a stable equivalence induces an isomorphism $\omega_{\alpha} \cong \omega_{o}$. Conversely, assume that ω_{α} is free, with generator x say. Then $\omega_{-\alpha}$ is also free, because $\omega = \omega_{\alpha} \otimes \omega_{-\alpha} \cong \omega \otimes \omega_{-\alpha} \cong \omega_{-\alpha}$. Let y be a generator of $\omega_{-\alpha}$. The product $xy \in \omega$ is then a generator of this module, hence a unit of ω. This implies $d_H(xy) = \pm 1$ for all $H < G$ and therefore $d_H(x) = \pm 1$ for all $H < G$. By 8.2 x is represented by a G-homotopy equivalence.

10.3. The Picard group and invertible modules.

In order to use the results of 10.1 successfully we have to collect some facts about projective modules.

Let R be a commutative ring. The set of isomorphism classes of projective R-modules of rank one forms an abelian group under the composition law "tensor product over R". This group is called the Picard group of R

$$\mathrm{Pic}(R).$$

The inverse of an element is given by the dual module. Using the notations of section 9, part of 10.2.2 may be restated as follows

Proposition 10.3.1. The assignment $\alpha \longrightarrow \omega_{\alpha}$ *induces an injective* ring homomorphism

$$pO(G) : RO_o(G)/RO_h(G) \longrightarrow \mathrm{Pic}(\omega_o^G).$$

We are interested in the computation of $\mathrm{Pic}(\omega_o^G)$ and $pO(G)$. Since the results are interesting mainly for finite groups we assume from now on in this section that G is finite. This has the advantage that we can think of ω_o^G as a subring of a finite direct product of the integers.

The computation of Picard groups is facilitated by using the <u>Mayer-Vietoris sequence for</u> Pic.

<u>Proposition 10.3.2. Let</u>

$$\begin{tikzcd} R \arrow[r, "r_1"'] \arrow[d, "r_2"] & R_1 \arrow[d, "p_2"] \\ R_2 \arrow[r, "p_2"'] & S \end{tikzcd}$$

<u>be a pull-back diagram of commutative rings.</u>

<u>Suppose that</u> p_1 <u>is surjective. Then the following Mayer-Vietoris sequence is exact</u>

$$\mathrm{Pic}\, S \longleftarrow \mathrm{Pic}\, R_1 \oplus \mathrm{Pic}\, R_2 \longleftarrow \mathrm{Pic}\, R \xleftarrow{\ d\ } S^*$$

$$\longleftarrow R_1^* \oplus R_2^* \longleftarrow R^* .$$

Here S^* denotes the units of the ring S. We describe the maps in this sequence. If $f : R \longrightarrow S$ is a ring homomorphism we use f to view S as an R-module; if P is a projective R-module of rank one then $f_*P := P \otimes_R S$ is a projective S-module of rank one. The first two maps are given by $x \longmapsto (r_{1*}x, r_{2*}x)$ and $(y,z) \longmapsto p_{1*}y - p_{2*}y$ (consider Pic as additive group). The last two maps are given by similar formulas.

Now as to d. Given $e \in S^*$ let $l_e : S \longrightarrow S$ be the left translation $s \longmapsto es$. Let $M(e)$ be defined by the following pull-back diagram

(10.3.3)
$$\begin{tikzcd} M(e) \arrow[r] \arrow[d] & R_1 \arrow[d, "l_e p_1"] \\ R_2 \arrow[r, "p_2"'] & S \end{tikzcd} .$$

Then $M(e)$ is an R-module (an R-submodule of $R_1 \times R_2$). We need the following information about such modules. (We still assume the hypothesis of 10.3.2.)

Proposition 10.3.4. (i) $M(e_1) \oplus M(e_2) \cong M(e_1e_2) \oplus R$.

(ii) $M(e_1) \otimes M(e_2) \cong M(e_1e_2)$.

(iii) $M(e)$ is projective of rank one.

Proof. (i) The modules in question are given by the following pull-back diagrams

$$\begin{tikzcd} M(e_1) \oplus M(e_2) \arrow[r] \arrow[d] & R_1 \oplus R_1 \arrow[d, "h(p_1 x p_1)"] \\ R_2 \oplus R_2 \arrow[r, "p_2 \times p_2"'] & S \oplus S \end{tikzcd} \qquad \begin{tikzcd} M(e_1e_2) \oplus R \arrow[r] \arrow[d] & R_1 \oplus R_1 \arrow[d, "k(p_1 x p_1)"] \\ R_2 \oplus R_2 \arrow[r, "p_2 \times p_2"'] & S \oplus S \end{tikzcd}$$

where h is given by the matrix $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$ and k by the matrix

$\begin{pmatrix} e_1e_2 & 0 \\ 0 & 1 \end{pmatrix}$. Now h and k differ by the matrix

$\begin{pmatrix} e_2 & 0 \\ 0 & e_2^{-1} \end{pmatrix}$ which can be lifted to an invertible matrix over R_1 because p_1 is surjective; here one uses the formal identity

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Hence $h(p_1 \times p_2)$ is transformable into $k(p_1 \times p_1)$ by invertible matrices so that (by transitivity of pull-backs) the desired isomorphisms drops out.

(iii) We obtain from (i) that $M(e) \oplus M(e^{-1})$ is free. Hence $M(e)$ is projective. If we localize (10.3.3) at prime ideals q of R we see that $M(e)_q = 0$ hence $\mathrm{rank}_q\, M(e) \geqslant 1$. Since $\mathrm{rank}_q\, M(e) + \mathrm{rank}_q\, M(e) = \mathrm{rank}_q\, (M(e) \oplus M(e^{-1})) = 2$, by (i), we have $\mathrm{rank}_q\, M(e) = 1$.

(ii) Since $M(e)$ has rank one the second exterior power Λ^2 of $M(e)$ is zero. Now apply Λ^2 to (i) and (ii) drops out.

If view of 10.3.4 we can now define a homomorphism

$$d : S^* \longrightarrow \mathrm{Pic}\, R \quad \text{by} \quad d(e) = M(e) .$$

With these preparations 10.3.2 is easy to verify.

The Mayer-Vietoris sequence may be applied to the Burnside ring A as follows. Let c be a multiple of the group order $|G|$. Let

$$\varphi \;:\; A = A(G) \longrightarrow C = C(\Phi(G), Z)$$

be the standard map. Then the following diagram is a pull-back

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & C \\ \downarrow & & \downarrow \\ A/cC & \xrightarrow[\varphi \bmod c]{} & C/cC \end{array} .$$

Here the vertical maps are the canonical quotient maps. We regard φ as an inclusion. Since the cokernel of φ has exponent $|G|$ (section 1) we have $cC < A$ so that A/cC makes sense. We use the following facts.

<u>Proposition 10.3.6.</u> Pic $C = 0$. Pic $A/cC = 0$.

<u>Proof.</u> C is finite direct product of the integers, say $C = Z^n$. Since projective modules over Z are free we have Pic $Z = 0$. Using induction on n we obtain from 10.3.2 that Pic $Z^n = 0$.

In case of $R := A/cC$ we note that this ring is finite as an abelian group. Therefore R has a finite number of maximal ideals (is a <u>semi-local</u> ring). If $m_1, \ldots, m_n$ are the maximal ideals then $R \longrightarrow \prod A/m_i$ is surjective (Chinese remainder theorem) with kernel $m = m_1 \cap \ldots \cap m_n$ the radical. Since R is finite hence Artinian this radiacal equals the nil-radical nil R of R. The ring R/m is a product of fields hence Pic R/m is zero. We have proved Pic R = 0 if we use the following

<u>Proposition 10.3.7. Let</u> I <u>be an ideal in the commutative ring</u> R. <u>Then the canonical map</u>

$$\text{Pic } R \longrightarrow \text{Pic } R/I$$

<u>is injective if</u> I <u>is contained in the radical of</u> R <u>and bijective if</u> I <u>is nilpotent</u>.

Proof. The first statement follows from Bourbaki [33] , II § 3.2. Prop. 5. Now assume that I is nilpotent. We have to show that the map is surjective. A projective R/I-module of rank one is given as a direct summand of a finitely-generated free R/I-module hence is given by a certain idempotent matrix $A \in GL(n,R/I)$. We have to lift the matrix to an idempotent matrix $B \in GL(n,R)$. Once this is done the proof is finished because $\otimes_R R/I$ does not change the rank of a projective module. We define inductively a sequence of matrices as follows: Let $B_1 \in GL(n,R)$ be a lifting of A. Put $N_i = B_i^2 - B_i$ and $B_{i+1} = B_i + N_i - 2B_iN_i$. Then one checks that $N_{i+1} \in GL(n,I^{2^i})$ and that B_i is a lifting of A. For large i $N_i = 0$ and we are done.

Combining the previous results we obtain

Proposition 10.3.8. The following sequence is exact

$$0 \longleftarrow \text{Pic } A \longleftarrow (C/cC)^* \longleftarrow C^* \oplus A/cC^* \longleftarrow A^* .$$

In principal this sequence can be used to compute Pic A for the Burnside ring $A = A(G)$. But it is not easy to obtain the actual structure of the abelian group Pic A. We shall indicate later, how the congruences 1.3 for the Burnside ring can lead to a computation.

Remark 10.3.9. If G is a compact Lie group Proposition 10.3.8 is still valid with c being a common multiple of the $|NH/H|$, $(H) \in \phi(G)$. (See 5. 9. for the existence of such c.) One has the pull-back 10.3.5 and moreover Proposition 10.3.6. is still true.

We now continue with a pull back diagram as 10.3.5 where $C = Z^n$, is an inclusion of maximal rank. We consider C as an A-module via this inclusion. If $M,N \subset C$ are A-submodules we define their product

$$MN \subset C \tag{10.3.10}$$

to be the module generated by all elements mn, $m \in M$, $n \in N$. We call M _invertible_ if their exists N such that MN = A. (This is not quite the standard notion, e.g. as in Bourbaki [33], § 5.6, but exactly what we need. Therefore one should investigate a more general situation comprising both notions of invertible modules.) Let

$$\mathrm{Inv}\ (A)$$

be the set of invertible A-modules.

Proposition 10.3.11. (i) Inv A _is an abelian group under the composition law_ 10.3.10.

(ii) _Invertible modules are projective of rank one. Assigning to each invertible module its class in_ Pic A _we obtain a surjective homomorphism_

$$\mathrm{cl} : \mathrm{Inv}\ (A) \longrightarrow \mathrm{Pic}\ (A).$$

(iii) _There exists a canonical exact sequence_

$$0 \longrightarrow A^{*} \longrightarrow C^{*} \longrightarrow \mathrm{Inv}\ (A) \xrightarrow[\mathrm{cl}]{} \mathrm{Pic}(A) \longrightarrow 0.$$

(iv) _There exists a canonical exact sequence_

$$0 \longrightarrow (A/cC)^{*} \longrightarrow (C/cC)^{*} \longrightarrow \mathrm{Inv}\ (A) \longrightarrow 0.$$

Proof. (i) follows directly from the definition of Inv (A) because the existence of inverses was required.

(ii) Suppose MN = A. Then

$$CM = CMC \supset CMN = CA = C$$

hence $$CM = C .$$

Therefore $1 = \sum c_i m_i$ for suitable $c_i \in C$ and $m_i \in M$ and hence $c = \sum (cc_i)m_i$. But $cc_i \in A$ so that $c \in M$, hence $cA \subset M$. In particular $M \subset C$ is a subgroup of maximal rank with cokernel annihilated by c, and $M \otimes_Z Q \longrightarrow C \otimes_Z Q$ is an isomorphism.

If $1 = \sum m_i n_i$, $m_i \in M$, $n_i \in N$ then $f_i : M \longrightarrow A : m \longmapsto mn_i$ is A-linear and for each $x \in M$ we have $x = \sum f_i(x)m_i$. Therefore M is a finitely generated projective module. Let q be a maximal ideal of A. Then M_q is a free A_q-module. Since $M_q \otimes \mathbb{Q} \cong C_q \otimes \mathbb{Q}$ as $A_q \otimes Q$-modules M_q must have rank one.

Finally given a projective module of rank one M. By 10.3.8 this module is isomorphic to a module of type $M(e)$, $e \in (C/cC)^*$. We give another description of this module. Let $e' \in C$ be a lifting of e. Then $M(e)$ can be identified with

(10.3.12) $$M'(e') := \{x \in C \mid e'x \in A\} .$$

Choose $f' \in C$ such that $e'f' = 1 + c^2z$ for an $z \in C$. Then $f' \in M'(e')$, $e' \in M'(f')$ and $e'f' = 1 + c^2z \in M'(e')\,M'(f') \subset M'(e'f')$. But $c \in M'(e')$ and $cz \in M'(f')$ hence $c^2z \in M'(e')\,M'(f')$ hence $A \subset M'(e')\,M'(f')$. On the other hand $M'(e'f') = M'(1+c^2z) = M'(1) = A$. Therefore $M'(e')$ is invertible and cl is surjective. From 10.3.4 (ii) we see that cl is a homomorphism.

(iii) Suppose that $M \in \mathrm{Inv}(A)$ is free, with generator x say. If $MN = A$ we must have an identity of the form $1 = \sum (a_i x)n_i$, so that $x \in C^*$ and $M = M'(x)$. If $M'(x) = M'(y)$ for $x,y \in C^*$ then $x = ay$ for $a \in A$; hence $a \in A^*$.

(iv) Let $r : C \longrightarrow C/cC$ be the quotient map. Let $C' = r^{-1}(C/cC^{*})$. If $r(e) = r(f)$ then $M'(e) = M'(f)$: Let $e = f+ch$. Then $x \in M'(e) \Rightarrow ex \in A \Rightarrow x(f+ch) \in A$. Since $cC \subset A$ we conclude that $xch \in A$ and therefore $xf \in A$, so that $M(e) \subset M(f)$. We can therefore define a map $(C/cC)^{*} \longrightarrow Inv(A)$ by $r(e) \longmapsto M'(e)$. To show that this is a homomorphism we note that $M'(e)\, M'(f) \subset M'(ef)$ which follows from the definition. This is an inclusion of invertible modules. Thus we have to show that any such inclusion $M \subset N$ must be an equality. Let q be a maximal ideal of A. By the Cohen-Seidenberg theorem (Atiyah-Mac Donald [11] , 5.) there exists a ring homomorphism $\varphi : C \longrightarrow Z$ such that $q = \{a \in A \mid \varphi(a) \equiv 0 \bmod p\}$ for some prime p. Therefore $x \in M$ is a generator of the localized module M_q if and only if $\varphi(x) \not\equiv 0 \bmod p$. Therefore $M_q \subset N_q$ maps a generator onto a generator, hence is an isomorphism. By commutative algebra, $M \subset N$ is an isomorphism.

The exactness of the sequence (iv) is implied by (iii) and 10.3.8.

We now prove a recognition principle for invertible modules.

Proposition 10.3.13. Let M be invertible. Suppose $e \in M$ and $r(ef) = 1$. Then $M = M'(f)$.

Proof. If $x \in M'(f)$ then $xf \in A$ and therefore $xef \in M$. Since $cC \subset M$ we obtain $x \in M$ hence $M'(f) \subset M$. By the previous proof this inclusion must be an equality.

We conclude with a geometric application. Let $\alpha \in R_O(G)$. The module ω_α is contained via the mapping degree of fixed point mappings in $C(\phi(G),Z) = C$, see 8. 5 . We use this inclusion as an identification.

Proposition 10.3.14. (i) Let $\alpha \in R_O(G)$. Then $\omega_\alpha \subset C$ is invertible.

(ii) <u>The assignment</u> $\alpha \longmapsto \omega_\alpha$ <u>induces a homomorphism</u>

$$R_o(G) \longrightarrow \mathrm{Inv}(\omega_o^G).$$

(iii) <u>For</u> $\alpha \in R_o(G)$ <u>the module</u> ω_α <u>is equal to</u> ω_o <u>if and only if</u> $\alpha \in R_h(G)$.

<u>Proof.</u> (i) We know already that ω_α is projective of rank one (10.2.2), but not every such submodule of C is invertible. The pairing 10.2.1

$$\omega_\alpha \otimes \omega_{-\alpha} \longrightarrow \omega_o$$

shows, by passing to fixed point degrees, that

$$\omega_o \supset \omega_\alpha \omega_{-\alpha} \ni 1$$

so that $\omega_o = \omega_\alpha \omega_{-\alpha}$.

(ii) The pairing 10.2.1 also shows $\omega_\alpha \omega_\beta \subset \omega_{\alpha+\beta}$. This being an inclusion of invertible modules is an equality by the proof of 10.3.11. (iv).

(iii) If $\omega_\alpha = \omega_o$ then $1 \in \omega_\alpha$. A map representing 1 is an oriented stable homotopy equivalence. Conversely $1 \in \omega_\alpha$ implies $\omega_\alpha = \omega_o$, by 10.3.13.

We restate 10.3.14 as follows

<u>Proposition 10.3.15. The assignment</u> $\alpha \longmapsto \omega_\alpha$ <u>induces an injective homomorphism</u>

$$p(G) : R_o(G)/R_h(G) \longrightarrow \mathrm{Inv}\,(\omega_o^G).$$

10.4 Comments.

This section is based on tom Dieck-Petrie [69] , where further information may be found. Generalizations to real G-modules are in Tornehave [160] . A more conceptual proof of the main result of section 9 using section 10 and the theory of p-adic λ-rings may be found in tom Dieck [68] . These one also finds a computation of Pic A(G) for abelian G and an indication now Pic A(G) may be computed in general. For homotopy equivalent G-modules for compact Lie groups G see Traczyk [161] . For G-maps $S(V) \longrightarrow S(W)$ of specific degree see Lee-Wasserman [110] and Meyerhoff-Petrie [114] . An interesting and difficult problem is the study of homotopy equivalences between products $S(V) \times S(W)$. For the homeomorphism problem for the S(V) see Schultz [139] .