9. Homotopy Equivalent Group Representations.

We are concerned in this section with the homotopy theory of group representations. If G is a compact Lie group and E and F are orthogonal real representations so that the unit spheres S(E) and S(F) are preserved by the G-action, we ask: When does there exist a G-map $f : S(E) \longrightarrow S(F)$ which has a G-homotopy inverse?

It turns out that homotopy equivalences between different representations can essentially only occur for finite groups. Therefore we shall only consider finite groups and restrict our attention to stable homotopy equivalences. Later we shall deal with the unstable situation and compact Lie groups.

9.1. Notations and results.

Let G be a finite group. If V is a (real or complex) G-module we denote by S(V) its unit sphere with respect to some G-invariant inner product. Two real G-modules V and W are called homotopy equivalent if the G-spaces S(V) and S(W) are G-homotopy equivalent. If V and W (resp. V_1 and W_1) are homotopy equivalent, then $V \oplus V_1$ and $W \oplus W_1$ are homotopy equivalent because $S(V \oplus V_1)$ is G-homeomorphic to the join $S(V) * S(V_1)$ and we can use the join of the individual homotopy equivalences. Two real G-modules V and W are called stably homotopy equivalent if for some real G-module U the modules $V \oplus U$ and $W \oplus U$ are homotopy equivalent. Let R(G) resp. RO(G) denote the Grothendieck ring of complex resp. real G-modules (identified with the corresponding character ring). Elements $x \in RO(G)$ are formal differences $x = V-W$ of real G-modules V and W. The $x = V-W$ such that V and W are stably homotopy equivalent form, by the remark above about joins, an additive subgroup of RO(G), denoted $RO_h(G)$. If we deal with complex G-modules we call V and W oriented homotopy equivalent if there exists a G-homotopy equivalence

$f : S(V) \longrightarrow S(W)$ such that for each subgroup H of G the induced map $f^H : S(V)^H \longrightarrow S(W)^H$ on the H-fixed point sets has degree one with respect to the coherent orientations that $S(V)^H$ and $S(W)^H$ inherit from the complex structure on V^H and W^H. We let $R_h(G)$ be the additive subgroup of R(G) consisting of x = V-W such that V and W are oriented stably homotopy equivalent.

If $S(V \oplus U)$ and $S(W \oplus U)$ are G-homotopy equivalent then the H-fixed points are homotopy equivalent. In particular the spheres $S(V)^H$ and $S(W)^H$ then have the same dimension (or are both empty). Let $R_o(G)$ be the additive subgroup of the V-W such that for all subgroups $H < G$ we have $\dim V^H = \dim W^H$. Let $RO_o(G)$ be the analogous subgroup of RO(G). Since $R_h \subset R_o$ and $RO_h \subset RO_o$ we introduce the groups

(9.1.1) $$j(G) = R_o(G)/R_h(G), \quad jO(G) = RO_o(G)/RO_h(G).$$

If G has order $g = |G|$ then G-modules are realisable over the field Q(u) where u is a primitive g-th root of unity. The Galois group Γ of Q(u) over Q acts on R(G) and RO(G) via its action on character value (see 3.5). Actually Γ acts on the set

$$\mathrm{Irr}\ (G,C) \quad \text{resp.} \quad \mathrm{Irr}\ (G,\mathbb{R})$$

of complex resp. real irreducible G-modules. Let $Z[\Gamma]$ be the integral group ring of Γ and $I(\Gamma)$ its augmentation ideal. Then we have

<u>Proposition 9.1.2. The following equalities hold</u>

$$R_o(G) = I(\Gamma)R(G), \quad RO_o(G) = I(\Gamma)RO(G).$$

The need for the following objects will become clear in a moment:

$$(9.1.3) \qquad R_1(G) = I(\Gamma)R_o(G),\ RO_1(G) = I(\Gamma)RO_o(G)$$
$$i(G) = R_o(G)/R_1(G),\ iO(G) = RO_o(G)/RO_1(G) .$$

We shall obtain the following results.

Theorem 9.1.4. For all finite groups G *we have*

$$R_1(G) \subset R_h(G) \quad \text{and} \quad RO_1(G) \subset RO_h(G).$$

Using this theorem we can consider the canonical quotient maps

$$(9.1.5) \qquad t(G) : i(G) \quad j(G),\ tO(G) : iO(G) \quad jO(G).$$

Theorem 9.1.5. Let G *be a* p-*group. Then* t(G) *and* tO(G) *are isomorphisms.*

The plan of the demonstration of 9.1.4 and 9.1.6 is as follows: We begin with a recollection of some representation theory in 9.2, proving 9.1.2 and giving a detailed analysis of i(G) and iO(G). In 9.3 we shall prove 9.1.4 and in 9.6 we shall prove 9.1.6 using the functorial properties of 9.1.5. In subsequent section we discuss various extensions and refinements: Nilpotent and hyperelementary groups; maps between unstable modules; connections with the Burnside ring and rational characters.

9.2. Dimension of fixed point sets.

The number of irreducible complex representations of G equals the number of conjugacy classes of elements of G (see Serre [147], Théorème 7), in symbols

$$|\mathrm{Irr}\ (G,C)| = |\mathrm{Conj}\ (G)| .$$

Let $\Gamma = \Gamma(m)$ be the Galois group of Q(u) over Q where u is a primitive m-th root of unity and m is a multiple of $|G|$. The group Γ may be identified with the group of units in the ring Z/m. The group Γ acts on Irr(G,C). Let $X = X(G) = Irr(G,C)/\Gamma$ be the orbit set of this action (it is independent of m). Then the elements

$$x_A = \Sigma_{y \in A}\, y \,, \quad A \in X(G)$$

form a Z-basis of the invariants

(9.2.1) $$R(G)^{\Gamma} .$$

The rational representation ring R(G;Q) is contained in $R(G)^{\Gamma}$ as a subgroup of maximal rank but in general different from it. There exists an integer n_A (the Schur-index, see 9. 3.) such that $n_A x_A$ is represented by an irreducible rational representation (Serre [147], 12.) Hence

(9.2.2) $$|X(G)| = \mathrm{Rank}_Z\, R(G;Q)$$

and this rank is equal to the number of conjugacy classes of cyclic subgroups (Serre [147], Théorème 29). Let $\zeta(G)$ be the set of conjugacy classes of cyclic subgroups of G and let $C(\zeta(G),Z)$ be the ring of functions $\zeta(G) \longrightarrow Z$. We obtain an additive map

(9.2.3) $$d : R(G) \longrightarrow C(\zeta(G),Z) \qquad d(x)(C) = \dim_C x^C .$$

Since $\dim V^H = |H|^{-1} \Sigma_{h \in H} V(h)$ and the left hand side is Galois invariant we see that $I(\Gamma)R(G) \subset R_o(G) \subset$ kernel d. Hence we obtain a surjection

(9.2.4) $$R(G)_{\Gamma} := R(G)/I(\Gamma)R(G) \longrightarrow R(G)/\mathrm{Ker}\ d$$

which is compatible with the restriction to subgroups.

<u>Proposition 9.2.5. The map</u> 9.2.4 <u>is injective, i. e.</u>

$$I(\Gamma)R(G) = R_o(G) = \{V-W \mid \dim V^C = \dim W^C,\ C < G \text{ cyclic}\} .$$

<u>Proof.</u> We show that

$$R(G)_\Gamma \longrightarrow \prod_C R(C)_\Gamma$$

is injective, where C runs through the cyclic subgroups of C and the map is restriction. The group $R(G)_\Gamma$ is free abelian, a basis consisting of representatives for the Γ-orbits $\mathrm{Irr}(G,\mathbb{C})/\Gamma$. The assignement $x \longmapsto \sum_{\gamma\in\Gamma} \gamma x$ induces a homomorphism $t : R(G)_\Gamma \longrightarrow R(G)$ which, composed with $R(G) \longrightarrow R(G)_\Gamma$, is multiplication by $|\Gamma|$. Hence t is injective. Since $R(G) \longrightarrow \prod R(C)$ is injective the map above must be injective. We now have a commutative diagram

$$\begin{array}{ccc} R(G) & \longrightarrow & \prod R(C)_\Gamma \\ \downarrow & & \downarrow \\ R(G)/\mathrm{Ker}\ d & \longrightarrow & \prod R(C)/\mathrm{Ker}\ d \end{array}$$

and it remains to be shown that for cyclic C the map $R(C) \longrightarrow R(C)/\mathrm{Ker}\ d$ is injective which is easily done by the reader.

Exactly the same argument shows

<u>Proposition 9.2.6. For every finite group</u> G

$$I(\Gamma)RO(G) = RO_o(G) = \{V-W \mid \dim V^C = \dim W^C,\ C < G \text{ cyclic}\} .$$

We therefore obtain from 9.2.3 and its real analog injective maps

(9.2.7)
$$d : R(G)_{\Gamma} \longrightarrow C(\zeta(G),Z)$$
$$dO : RO(G)_{\Gamma} \longrightarrow C(\zeta(G),Z)$$

with image group of maximal rank, i. e. the cokernel is a finite group. We want to compute the order of the cokernel. It would be interesting to know the actual structure of the cokernel.

We begin with a series of reductions. Let $V_1,\ldots,V_r$ be a system of representatives of $\mathrm{Irr}(G,C)/\Gamma$ and $H_1,\ldots,H_r$ a system of representatives for $\zeta(G)$. Then

(9.2.8)
$$|\mathrm{Cok}\ d| = \det(a_{ij})$$
$$a_{ij} = \dim \mathrm{Fix}\ (H_j,V_i).$$

Using $|H| \dim V^H = \Sigma_{h\in H} V(h)$ we obtain

(9.2.9)
$$|\mathrm{Cok}\ d| \ \pi_j\ |H_j| = |\det(\Sigma_{h\in H_j} V_i(h))|.$$

Let H^* denote the set of generators of the cyclic group H.

Lemma 9.2.10. We have

$$\det(\Sigma_{h\in H_j} V_i(h)) = \det(\Sigma_{h\in H_j^*} V_i(h)) .$$

Proof. Choose an indexing such that $(H_i) \leq (H_k)$ implies $k \leq i$. Put $b^*_{ij} = \Sigma_{h\in H_j^*} V_i(h)$ and $b_{ij} = \Sigma_{h\in H_j} V_i(h)$. Then

$$b_{ij} = b^*_{ij} + \Sigma_{l<j}\ e_l\ b_{il}$$

where e_1 = 1 or 0, independent of i. Subtracting suitable "earlier" columns from "later" one's we can transform the matrix (b_{ij}) into (b^*_{ij}).

We now observe that we can identify $\Gamma = Z/m^*$ in such a way that

$$\gamma V(g) = V(g^{\gamma})$$

so that Γ acts on each set H^*_j. We choose for each j an element $g_j \in H^*_j$ and let Γ_j be the isotropy group of the Γ-action at g_j. Then

$$(9.2.11) \qquad b^*_{ij} = |\Gamma_j|^{-1} \sum_{\gamma \in \Gamma} \gamma V_i(g_j).$$

Hence, if we put $IV = \sum_{\gamma \in \Gamma} \gamma V$, then we obtain from 9.2.10 and 9.2.11

$$(9.2.12) \qquad \det(b_{ij}) \prod_j |\Gamma_j| = \det(IV_i(g_j)).$$

In order to compute this determinant we make the following remark: Let W be a complex vector space with hermitian form $\langle -,- \rangle$ and orthogonal basis $e_1,\dots,e_r$. Given $a_i = \sum c_{ik} e_k$, $1 \leq i \leq r$, then

$$(9.2.13) \qquad \det \langle a_i, a_j \rangle = (\det(c_{ik}))^2 \prod_j \langle e_j, e_j \rangle .$$

We shall compute $\det^2(IV(g_j))$ in this way. Consider IV_i as function on G. Put

$$G = C_1 \cup \dots \cup C_r$$

where $g \in C_j$ if and only if g generates a group conjugate to H_j. Then IV_j belongs to the space of functions which are constant along the sets

C_j. Denote the characteristic function of C_j with the same letter. Then

$$(9.2.14) \qquad IV_i = \Sigma_j \, IV_i(g_j) \, C_j \, .$$

We use the standard hermitian form on the space of functions $G \longrightarrow C$. Then $\langle C_j, C_j \rangle = |C_j|$. Using 9.2.13 we get

$$(9.2.15) \qquad (\Pi_j \, C_j) \det^2 (IV_i(g_j)) = \det \langle IV_i, IV_j \rangle \, .$$

The orthogonality relations for characters yield

$$(9.2.16) \qquad \langle IV_i, IV_j \rangle = G \, |\Gamma| \, |\Gamma^i| \, \delta_{ij}$$

where Γ^i is the isotropy group of the Γ-action on Irr(G,C) at V_i. Collecting our results we obtain

$$|\mathrm{Cok}\ d| = \Pi_j |H_j|^{-1} \cdot |\det(b_{ij})| \qquad (9.2.9)$$

$$= \Pi_j (|H_j| \, |\Gamma_j|)^{-1} \cdot |\det IV_i(g_j)| \qquad (9.2.12)$$

$$= \Pi_j (|H_j| \, |\Gamma_j| \, |C_j|^{1/2})^{-1} \, |\det \langle IV_i, IV_j \rangle| \qquad (9.2.15)$$

$$= \frac{|\Gamma|^{r/2} |G|^{r/2} \, \Pi |\Gamma^j|^{1/2}}{\Pi \, (|H_j| \, |\Gamma_j| \, |C_j|^{1/2})} \qquad (9.2.16)$$

If we note that $|\Gamma_j| \, |H_j^*| = |\Gamma|$ and $|C_j| = |H_j^*| \, |G/NH_j|$ we finally obtain

<u>Proposition 9.2.17.</u>

$$|\mathrm{Cok}\ d| = \frac{\Pi \, |NH_j|^{1/2}}{\Pi \, |H_j|} \cdot \frac{\Pi \, |\Gamma^j|^{1/2}}{\Pi \, |\Gamma_j|^{1/2}}$$

It is not obvious a priori that the right hand side of 9.2.17 is an integer. In certain cases the formula simplifies. The Γ-factors disappear for abelian groups G.

Proposition 9.2.18. Let G be a p-group, $p \neq 2$. Then Irr(G,C) and Conj(G) are isomorphic Γ-sets.

Proof. Let V_1 and V_2 be the permutation representations associated to the Γ-sets Irr(G,C) and Conj(G), respectively. We show that V_1 and V_2 are isomorphic Γ-representations. Since in our case Γ is cyclic and for such groups $A(\Gamma) \longrightarrow R(\Gamma)$ is injective we conclude that the Γ-sets in question are isomorphic. The isomorphism of V_1 and V_2 is given by identifying linear combination of elements of Irr(G,C) as usual with functions Conj(G) $\longrightarrow$ C. The formula 3.5.1 for the action of the Adams operations on characters shows that this is an isomorphism of Γ-modules.

If Γ'_j denotes the isotropy group of the conjugacy class of g_j and ZH_j the centralizer of H_j in G then

$$(9.2.19) \qquad |\Gamma'_j| \, |ZH_j| = |NH_j| \, |\Gamma_j| \, .$$

Using 9.2.17 - 19 we obtain

Proposition 9.2.20. Let G be a p-group, $p \neq 2$ a prime. Then the order of the cokernel of d is

$$\prod_j |NH_j/H_j| \, |ZH_j|^{-1/2} \, .$$

Let $c : R(G;Q) \longrightarrow C(\mathfrak{z}(G),Z)$ be the ring homomorphism which associates with each $Q[G]$-module V the function c(V) such that c(V)(C) is the value of the character V at a generator of C. This is an inclusion of maximal rank. One would like to compute the cokernel; this would give congruences expressing conditions for functions to be a rational characters. Arguments as in the proof of 9.2.17 allow to compute the order of cokernel c. Let n_i be the Schur index of V_i.

<u>Propositione 9.2.21.</u> $|\mathrm{Cok}\ c| = \prod_j n_j\, |NH_j|^{1/2}$.

<u>Proof.</u> $|\mathrm{Cok}\ c| = |\det W_i(g_j)|$ where $W_j = n_j\ |\Gamma^j|^{-1}\ IV_j$ is the irreducible rational representation belonging to V_j. Now use the calculations above.

<u>Problem 9.2.22.</u> <u>Compute</u> <u>the</u> <u>groups</u> Cok c <u>and</u> Cok d. (The results of section 10 should be helpful.)

<u>9.3. The Schur index.</u>

We collect the classical results about the Schur index with emphasis on p-groups. We always work with subfields of the complex numbers. General references for the following are: Lang [107], Ch XVII; Curtis-Reiner [48], § 70; Roquette [135].

Let $k \subset C$ be a field. The group algebra $k[G]$ is semi-simple and decomposes into a product of simple algebras A_i

$$k[G] = A_1 \oplus \dots \oplus A_r .$$

The corresponding decomposition $1 = e_1 + \dots + e_r$ yields the <u>indecomposible</u> <u>central</u> <u>idempotents</u> e_i of $k[G]$. By the theorem of Wedderburn each A_i is isomorphic to a full matrix algebra

$$A_i = M_{n_i}(D_i)$$

over a division algebra D_i. If V_i is a minimal left ideal of A_i, then V_i is an irreducible k [G] -module and every irreducible k [G] -module is isomorphic to one of this form. The endomorphismring of V_i is a division algebra, and in fact

$$D_i = \mathrm{Hom}_{k[G]} (V_i, V_i) \ .$$

The degree of D_i over its center K_i is a square m_i^2 where $m_i = [E_i, K_i]$ and E_i is a maximal field contained in K_i. The integer m_i is called the <u>Schur</u> <u>index</u> of V_i or A_i.

If V is an irreducible k(G)-module we let

$$A_V = A = \text{image } (k(G) \longrightarrow \mathrm{Hom}_k (V,V))$$

be the k-algebra generated by maps $1_g : v \longmapsto gv$. Then V is a faithfull irreducible A-module and since A is semisimple (being a quotient of k [G]) A must be simple. Hence $A = M_n(D)$ for some division algebra D whose center contains k.

If A is a simple algebra with center k then an extension field E of k is called a <u>splitting</u> <u>field</u> <u>for</u> A if $A \otimes_k E$ is a full matrix algebra over E. If A is a matrix algebra over the division algebra D then E is a splitting field if and only if E is a splitting field for A. If [D:k] is finite then a maximal subfield E of D is a splitting field for D and $[D : k] = [E : k]^2$. If L is any other splitting field for D which is a finite algebraic extension of k then [E : k] divides [L : k].

Applying these results to the algebra $A = A_V$ above, assuming that k is the center of A (= center of D), then for any splitting field F of D one has

$$A \otimes_k F \cong M_{mn}(F)$$

where $m^2 = [D : k]$, $n^2 = [A : D]$. If U is an irreducible F(G)-module given by a minimal left ideal $A \otimes_k F$ then

$$V \otimes_k F \cong m\,U$$

which shows that mU is realisable over k. If tU is realisable over k then $m|t$.

If U is an irreducible $\mathbb{C}[G]$-module we let $A_{k,U}$ be the k-algebra spanned by the $1_g \in \mathrm{Hom}_{\mathbb{C}}(U,U)$ which is a simple k-algebra. The center of this algebra is $k(\chi_U)$, this meaning k with character values $\chi_U(g)$ adjoined. The representation U is realisable over $F \supset k(\chi_U)$ if and only if F is a splitting field for $A_{k,U}$. The Schur index of $A_{k,U}$ is the minimal value m such that mU is realisable over $k(\chi_U)$ and there exists an extension F of degree m of $k(\chi_U)$ such that U is realisable over F. We therefore call $m = m_k(U)$ the _Schur index of_ U _with respect to_ k.

We call E a _splitting field for_ G if every irreducible $\mathbb{C}[G]$-module is realisable over E. If k is given one can always find a finite algebraic extension E of k which is a splitting field for G. By a famous theorem of Brauer $E = Q(u)$ is a splitting field for G if u is a primitive m-th root of unity and m is the last common multiple of the orders of elements in G.

Let V be an irreducible $k[G]$-module. Let E be a splitting field for G which is a finite Galois extension of k. Then $V \otimes_k E$ splits

$$V \otimes_k E = m(U_1 \oplus \dots \oplus U_t)$$

where the U_i are irreducible $E[G]$-modules. Moreover $U_i \otimes_E \mathbb{C}$ is an irreducible $\mathbb{C}[G]$-module and $m = m_k(U_i \otimes_E C)$ for $i = 1,\dots,t$. The $U_1,\dots,U_t$ form an orbit under the action of the Galois group Gal(E : k) on the irreducible $E[G]$-modules. The number t above equals $k(\chi_1):k$ where χ_1 is the character of U_1.

For later reference we now collect what happens for p-groups. We follow Roquette [135].

Proposition 9.3.1. *Let* G *be a* p-*group. Then for each irreducible* $\mathbb{C}[G]$-*module* V:

i) *If* $p \neq 2$ *then* $m_Q(V) = 1$.

ii) *If* $p = 2$ *then* $m_Q(V) = m_{\mathbb{R}}(V)$ *is* 1 *or* 2.

Proof. Roquette [135] shows i) and $m_Q(V) = 1$ or 2. We make the additional remark that $m_Q = m_{\mathbb{R}}$. (This was communicated by J. Tornehave.) Roquette shows that in the case $m_Q(V) = 2$ the division algebra associated to $A_{Q,V}$ (in the notation above) is the ordinary quaternionic extension of its center $Q(\chi_V)$. Since $A_{Q,V} \otimes_{Q(\chi_V)} \mathbb{R} \cong A_{\mathbb{R},V}$ and $\mathbb{R}$ does not split the quaternionic extension of $Q(\chi_V)$ we must have that $A_{\mathbb{R},V}$ is a matrix algebra over the quaternions, hence $m_{\mathbb{R}}(V) = 2$. Clearly $m_Q(V) = 1$ implies $m_{\mathbb{R}}(V) = 1$.

Corollary 9.3.2. *Let* G *be a* p-*group. Then*:

i) *If* $p \neq 2$ *then* $R(G,Q) \cong R(G)^{\Gamma}$.

ii) *For arbitrary* p $R(G,Q) = RO(G)^{\Gamma}$.

Proposition 9.3.3. (Tornehave) *Let* V *be an irreducible complex representation of a* 2-*group* G *with* $\dim V^H$ *even for every subgroup* H *of* G. *Then* V *is quaternionic*.

Proof. (Tornehave) Let χ be the character of V and let $\mathrm{ind}_H^G 1_H$ be the character induced from the trivial character of H. Then by Frobenius reciprocity (Serre [147], 7.2) and the orthogonality relations

$$\langle \chi , \mathrm{Ind}_H^G 1_H \rangle = \dim V^H .$$

So the assumption on V means that χ has even multiplicity in every virtual permutation character. By Segal's theorem (section 4) we find that $\langle \chi , \xi \rangle$ is even whenever ξ is the character of a Q [G] -module. There is a unique irreducible Q [G] -module whose character ξ satisfies $\langle \chi , \xi \rangle \neq 0$. The even integer $m = \langle \chi , \xi \rangle$ is the Schur-index $m_Q(\chi)$. But $m_Q(\chi) = m_{\mathbb{R}}(\chi)$, and if this number is even V must be quaternionic.

9.4. The groups i(G) and iO(G)

The proof of the main theorem 9.1.6 will use induction over the order of the group. In this section we prepare this induction by presenting the relevant algebraic facts about i(G) and iO(G), in particular for p-groups.

For each orbit $A \in X = \mathrm{Irr}(G,\mathbb{C})/\Gamma$ we let F(A) be the free abelian group on its element. Then (additively) $R(G) = \oplus_{A\in X} F(A)$ and if we put $F_o(A) = R_o(G) \cap F(A)$ then $R_o(G) = \oplus_{A\in X} F_o(A)$. Moreover

$$F_o(A) = \left\{ \sum_{a\in A} n_a a \;\middle|\; \sum n_a = 0 \right\}.$$

Since Γ is abelian the isotropy group of the Γ-action on A at $a \in A$ is independent of $a \in A$. Therefore we call this isotropy group Γ_A. We put $F_1(A) = I(\Gamma)F_o(A)$ and obtain $R_1(A) = \oplus_{A\in X} F_1(A)$ and

$$i(G) = \bigoplus_{A \in X} F_o(A)/F_1(A) .$$

The map

$$\Gamma / \Gamma_A \longrightarrow F_o(A)/F_1(A) : \gamma \longmapsto (1-\gamma)V$$

for $V \in A$ is independent of V and is seen to be an isomorphism. Thus we obtain a canonical isomorphism

(9.4.1) $$i(G) \cong \bigoplus_{A \in X} \Gamma / \Gamma_A$$

which we sometimes regard as an identification.

We need some functional properties of this map. The group $\Gamma = \Gamma(m)$ is not uniquely determined by G because m could be any multiple of $|G|$. If we are dealing with several groups we want m to be a multiple of all their orders. For a more functorial treatment one should use instead of Γ a profinite group, e. g. the Galois group of the field generated by all roots of unity over Q. This point of view is not so important for us. Nevertheless Γ / Γ_A is, by elementary Galois-theory, in a canonical way independent of m.

The restriction of the group action to a subgroup H induces a homomorphism

$$res_H : i(G) \longrightarrow i(H).$$

We need a description of res_H in terms of the isomorphism 9.4.1. If $V \in A \in X(G)$ then $res_H V$ splits into irreducible H-modules, say

$$\mathrm{res}_H\, V = \bigoplus_{i=1}^{t} \left(\bigoplus_{j=1}^{n(t)} W_{ij} \right)$$

where the index i collects all those summands which belong to the same Γ-orbit, A(i) say, of Irr(H,$\mathbb{C}$). Then res_H is the direct sum of the maps

(9.4.2)
$$\Gamma/\Gamma_A \longrightarrow \bigoplus_{i=1}^{t} \Gamma/\Gamma_{A(i)}$$

$$\gamma \longmapsto (\gamma^{n(1)}, \ldots, \gamma^{n(t)})$$

This is easy to verify.

The computation of i(G) above can be done in a completely analogous manner for iO(G). We obtain an isomorphism as in 9.4.1.

We now come to another description of i(G) and iO(G), valid for p-groups. We need an elementary Lemma. Let a cyclic group Γ act on a free abelian group A as a group of automorphism. Let $\gamma_o \in \Gamma$ be a generator of this group. Put $A_\Gamma = A/(1-\gamma_o)A$, $(1-\gamma_o)^i A_{i-1}$ for $i \geqslant 1$, $i(A) = A_o/A_1$.

Lemma 9.4.3. The following sequence is exact

$$0 \longrightarrow A^\Gamma \longrightarrow A_\Gamma \xrightarrow[1-\gamma_o]{} i(A) \longrightarrow 0 .$$

Proof. Suppose $a \in A^\Gamma$ maps to zero in A_Γ. Then $a = (1-\gamma_o)b$ and therefore $|\Gamma| a = \sum_{\gamma \in \Gamma} \gamma a = \sum \gamma(1-\gamma_o)b = 0$. Since A is free we must have a = o, hence the map $A^\Gamma \longrightarrow A_\Gamma$ is injective. By definition $A_\Gamma \longrightarrow i(A)$ is surjective (and well-defined). If a is in the kernel of this map then $(1-\gamma_o)a = (1-\gamma_o)^2 b$ and therefore the element

$c = a-(1-\gamma_0)b$, which represents the same element as a in A_Γ, satisfies $c = \gamma_0 c$ and therefore lies in A^Γ because γ_0 is a generator.

Now we note that our group Γ can be taken to be cyclic if G is a p-group ($p \neq 2$) and $\Gamma / \{\pm 1\}$ is also cyclic for p = 2. Therefore the Lemma yields

Proposition 9.4.4. *Let* G *be a* p-*group. The following sequences are exact:*

(1) $$0 \longrightarrow RO(G)^\Gamma \longrightarrow RO(G)_\Gamma \xrightarrow[s_k]{} iO(G) \longrightarrow 0$$

and similar sequences with RO *replaced by* RSO *or the augmentation ideals* IO *and* ISO.

(2) (For $p \neq 2$)

$$0 \longrightarrow R(G)^\Gamma \longrightarrow R(G)_\Gamma \xrightarrow[s_k]{} i(G) \longrightarrow 0$$

and similarly for the augmentation ideal I(G) *instead of* R(G).

For the rest of this section G will be a p-group.

Let V be an irreducible G-module with kernel H. We call V *primitive* if G/H is a cyclic, dihedral, or generalized quaternion group, and imprimitive otherwise. Let X'(G) be the set of Γ-orbits of imprimitive G-modules. Let i'(G) be the subgroup of i(G) that corresponds to $\bigoplus_{A \in X'(G)} \Gamma / \Gamma_A$ under the isomorphism 9.4.1. We define analogously $iO'(G) \subset iO(G)$. The importance of the primitive modules comes from the following variant of Blichfeldt's theorem which we state for later use as a Lemma.

Let V be an irreducible complex G-module which is isomorphic to its dual V^*. Then there exists a conjugate linear map $J : V \longrightarrow V$ with either $J^2 = \mathrm{id}$ (V of real type) or $J^2 = -\mathrm{id}$ (V of quaternionic type).

Lemma 9.4.5. An imprimitive G-module V of real (resp. quaternionic) type is induced from a real (resp. quaternionic) module of a proper subgroup.

Proof. We give a proof in the quaternionic case. (The real case is analogous.) Assume that V as a quaternionic G-module is not induced from a proper subgroup. We may assume that V is faithful and want to show that G is cyclic or generalized quaternion, in this case. Let K be a maximal normal abelian subgroup of G. If the restriction $\mathrm{res}_K V$ would contain two non-isomorphic irreducible quaternionic modules then V would not be irreducible. (See Curtis-Reiner [48], § 49 - 50, and note that the considerations apply to quaternionic modules.) Therefore $\mathrm{res}_K V \cong V_o + \dots + V_o$ with some irreducible quaternionic K-module V_o. Since V_o is faithful and K is abelian we must have that K is cyclic and $\dim_{\mathbb{H}} V_o = 1$ ($\mathbb{H}$ = quaternions). Since K was a maximal abelian normal subgroup, G/K acts via conjugation faithfully on K. The module V_o is a complex K-module of the form $W_o \oplus W_o^*$. If $g \in G \setminus K$ and $k \in K$ is a generator then $gkg^{-1} \neq k$. Therefore conjugation by g interchanges W_o and W_o^* and acts as $gkg^{-1} = k^{-1}$ because V_o is a faithful K-module. This implies that the order of G/K is at most 2 and therefore that G is either cyclic (G = K) or dihedral or generalized quaternion. But a dihedral group has no quaternionic irreducible modules.

Let $\mathrm{res} : i(G) \longrightarrow \prod_H i(H)$ be the product of the restriction maps res_H where H runs through the maximal proper subgroups of G. We also let res be the restriction of this map to $i'(G)$. We have a similar map in the real case.

<u>Proposition 9.4.6. The map</u>

$$res : iO'(G) \longrightarrow \prod_H iO(H)$$

<u>is injective. The map</u>

$$res : i'(G) \longrightarrow \prod_H i(H)$$

<u>is injective if</u> G <u>has odd order</u>.

The rest of this section is concerned with the proof of this Proposition. The essential fact is isolated in Lemma 9.4.7 which implies the Proposition easily if we use the isomorphism 9.4.1 and the commutative diagram

$$\begin{array}{ccc} i'(G) & \xrightarrow{\text{res}} & \prod_H i(H) \\ \downarrow \cong & & \downarrow \cong \\ \prod_{A \in X'(G)} \Gamma/\Gamma_A & \xrightarrow{\text{res}} & \prod_H (\prod_{D \in X(H)} \Gamma/\Gamma_D) \end{array}$$

where the description of the bottom map is given in 9.4.2. Similarly in the real case.

Now suppose $x = (\gamma_A \in \Gamma/\Gamma_A \mid A \in X'(G))$ is given.

<u>Lemma 9.4.8. Assume</u> $p \neq 2$ <u>in the complex case. For each</u> $A \in X'(G)$ <u>there exists a maximal proper subgroup</u> H <u>of</u> G <u>and a</u> $C \in X(H)$ <u>such that the following holds</u>:

i) <u>For</u> $A \neq B \in X'(G)$ <u>the</u> C-<u>component of</u> $res_H \gamma_B \in \prod_{D \in X(H)} \Gamma/\Gamma_D$ <u>is zero</u>.

ii) $: \Gamma/\Gamma_A \xrightarrow[\mathrm{res}_H]{} \prod_{D \in X(H)} \Gamma/\Gamma_D \xrightarrow[\mathrm{pr}_C]{} \Gamma/\Gamma_C$

<u>is injective</u>.

<u>Proof.</u> We begin with the complex case and allow also $p = 2$ in the following recollection of representation theory.

Let $V \in A \in X'(G)$. Since V is imprimitive we have $\dim_{\mathbb{C}} V > 1$. By the theorem of Blichfeldt (Serre [147], 8.5) we can find a proper subgroup H of G such that V is induced from an irreducible H-module W, notation: $V = \mathrm{ind}_H^G W$. By transitivity of induction we can moreover assume that H is a maximal proper subgroup of G. Then H is normal in G with index p. We choose H and $W \in C \in X(H)$ with these properties to prove the assertion of the Lemma.

We have a splitting $\mathrm{res}_H V \cong W_1 \oplus \ldots \oplus W_p$ with $W_1 = W$, say, and the W_i are pairwise non-isomorphic (Serre [147], 7.4). If U is irreducible and W is a direct summand of $\mathrm{res}_H U$, then by Frobenius reciprocity

$$0 \neq \langle \mathrm{res}_H U, W \rangle_H = \langle U, \mathrm{ind}_H^G W \rangle = \langle U, V \rangle$$

and hence $U \cong V$. This proves i). We note that $V \cong \mathrm{ind}_H^G W_i$. For the proof of ii) we consider several cases.

<u>First case.</u> The W_i belong to different Γ-orbits. Since induction is compatible with the Γ-action we obtain $\Gamma_C \subset \Gamma_A$. But if $\gamma \in \Gamma_V$ then

$$W_1 \oplus \ldots \oplus W_p \cong \gamma W_1 \oplus \ldots \oplus \gamma W_p$$

and therefore $\gamma W_i = W_i$ for all i because the W_i belong to different

Γ-orbits. Hence also $\Gamma_A \subset \Gamma_C$ and the map ϱ is the identity in this case.

Second case. There exists $\gamma_o \in \Gamma$ with $\gamma_o W_i \cong W_j$ for a pair $i \neq j$. Then $V = \gamma_o V$ and therefore $\gamma_o \in \Gamma_V$ permutes $W_1, \ldots, W_p$. This has to be a cyclic permutation. Hence $\gamma_o^p \in \Gamma_C$ and Γ_A/Γ_C has exponent p. From 9.4.2 we see that ϱ is given by $\varrho(\gamma) = \gamma^p$. If p is odd, Γ is cyclic of order $(p-1)p^k$ for a suitable k and ϱ must be injective ($p \neq 2$).

If $p = 2$ then $\Gamma = Z/2 \times Z/2^k$ for a suitable k. If $Z/2 \subset \Gamma_A$ this means $V = V^*$. Then either $W_1 = W_1^*$, $W_2 = W_2^*$ or $W_1 = W_2^*$, $W_2 = W_1^*$. In the first case ϱ is still injective, reasoning as for $p \neq 2$. By 9.4.5 we can avoid the case $W_1 = W_2^*$. If Z/2 is not contained in Γ_A then this factor of Γ is contained in the kernel of ϱ.

We now turn to real G-modules. Then Γ/Γ_A is always cyclic. If $res_H V$ splits into p non-isomorphic irreducible real H-modules the same proof as above works. We look at the irreducible real G-modules according to their endomorphism ring which is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The cases End(V) = $\mathbb{R}$, $\mathbb{H}$ can only occur for 2-groups (Serre [147], p. 122).

End(V) = $\mathbb{C}$. Then V is obtained by restriction of scalars from a complex G-module U with $U \neq U^*$, notation: $rU = V$. Then

$$res_H V = res_H rU = r\, res_H U = rU_1 \oplus \ldots \oplus rU_p .$$

A relation $U_i = U_j$ would imply $U = U^*$. Hence the $U_1, \ldots, U_p, U_1^*, \ldots, U_p^*$ are all distinct and therefore $rU_i = V_i$ are distinct real G-modules. If V_i is a direct summand in $res_H V'$ for an irreducible real G-module V' then Frobenius reciprocity again would imply that $V' = V$.

End(V) = $\mathbb{R}$. Then the complexification cV of V is irreducible. Since $\dim_{\mathbb{C}} V > 1$ we have $\mathrm{res}_H\, cV = W_1 \oplus W_2$ for a suitable subgroup H of index 2 in G. We must have $(W_1 \oplus W_2)^* = W_1 \oplus W_2$ and therefore $W_1 = W_1^*$, $W_2 = W_2^*$ or $W_1 = W_2^*$, $W_2 = W_1^*$. By 9.4.5 we can avoid the second case, hence we still have $W_i = cV_i$ with irreducible V_i and V_1, V_2 are not isomorphic.

End(V) = $\mathbb{H}$. Then V is obtained by restriction of scalars from an irreducible quaternionic G-module U, notation: rU = V. Again by 9.4.5 we can assume that $\mathrm{res}_H\, U$ splits into two non-isomorphic H-modules for suitable H and therefore $\mathrm{res}_H\, V$ splits into two non-isomorphic irreducible H-modules.

9.5. Construction of homotopy-equivalences.

We prove Theorem 9.1.4, namely the inclusions

$$R_1(G) \subset R_h(G), \quad RO_1(G) \subset RO_h(G).$$

We begin with an example due to Ted Petrie.

Let G be the cyclic group of order n with generator g. Let V^a be the $\mathbb{C}[G]$-module $\mathbb{C}$ with g acting as multiplication with $\exp(2\pi ia/n)$. Let a and b be integers, relatively prime and prime to n. Choose integers p,q such that $-ap + bq = 1$. The map

(9.5.1)
$$f : V^a \oplus V^b \longrightarrow V^1 \oplus V^{ab}$$
$$(x,y) \longmapsto (x^p \bar{y}^q,\ x^b + y^a)$$

is a G-map. We claim that f has degree one. Consider the value (1,0).

It is easy to see that $f(x,y) = (1,0)$ implies $(x,y) = ((-1)^q, (-1)^p)$. One calculates the jacobian point to be $a^2p^2 + b^2q^2$. If this would be zero then we would obtain, using $-ap + bq = 1$, that $-2abpq = 1$ which is impossible because a,b,p,q are integers. Since f is a proper map it induces a map of degree one between the one-point compactifications. Also a G-map between unit spheres

$$h : S(V^a \oplus V^b) \longrightarrow S(V^1 \oplus V^{ab})$$

$$h(x,y) = f(x,y)/\| f(x,y)\|$$

is induced. We can see that h has degree one: The radial extension of h to a map $h_1 : V^a \oplus V^b \longrightarrow V^1 \oplus V^{ab}$ has the same degree as h, and h_1 is properly homotopic to f. Since h is a G-map between free G-spaces which is an ordinary homotopy equivalence, it is a G-homotopy equivalence by Proposition 8.2.1.

Now given $E-F \in R_1(G)$ for a cyclic group G. Then E-F is an integral linear combination of elements $(1-\psi^a)(1-\psi^b)U$ where a and b are prime to $|G|$. If $(a,b) = 1$ then the example of Petrie above shows that $(1-\psi^a)(1-\psi^b)U \in R_h(G)$ because we actually have constructed an oriented homotopy equivalence. If a and b are not relatively prime than we replace b by a suitable b+kn such that $(a,b+kn) = 1$. Hence we have shown that $R_1(G) \subset R_h(G)$ for cyclic G.

We use induced representations to prove the general result. If $H < G$ and $\mathrm{ind}_H^G : R(H) \longrightarrow R(G)$ is the homomorphism given by induced representations then

(9.5.2) $$\mathrm{ind}_H^G(R_h(H)) \subset R_h(G).$$

(9.5.3) $$\mathrm{ind}_H^G(R_i(H)) \subset R_i(G), \quad i = 0,1.$$

The relation 9.5.3 follows from the fact that ind_H^G commutes with the Γ-action; and to prove 9.5.2 we note that

$$S(ind_H^G W) \cong *_{gH \in G/H} S(gH \times_H W),$$

so that homotopy equivalences for H-modules induce homotopy equivalences for the induced G-modules by taking suitable maps on the join. By the result above for cyclic G and 9.5.2 - 3 we see that $R_1(G) \subset R_h(G)$ whenever irreducible G-modules are induced from one-dimensional G-modules. This holds for p-groups and more generally for supersolvable groups (Serre [147], 8.5. Thèorème 16), and in particular for extensions of cyclic groups by p-groups. Now we can apply a general induction theorem of Dress [80] to conclude that $R_1(G) \subset R_h(G)$ for general G (see also section 6): The functors R_1 and R_h are compatible with restriction and induction (9.5.2 - 3). They are therefore sub-Mackey-functors of the representation ring functor. Therefore elements in $R_1(G)$ are induced from hyperelementary subgroups H of G (i. e. $0 \to S \to H \to P \to 0$, S cyclic, P a p-group). But for such groups H we know already that $R_1(H) \subset R_h(H)$. This proves Theorem 9.1.4 in the complex case.

In the real case we again need only consider groups G which are extensions of cyclic groups by p-groups. Using induction we reduce to the case of a real faithful irreducible G-module M which is not induced from a proper subgroup. The arguments of Dress [81] , p. 318, then show that either G is cyclic and $\dim_{\mathbb{R}} M \leq 2$ or G is dihedral and $\dim_{\mathbb{R}} M = 2$. If G is cyclic and $\dim_{\mathbb{R}} M = 1$ then (M being faithful) $G = Z/2$ and the Γ-action is trivial. If G is cyclic and $\dim_{\mathbb{R}} M = 2$ then M is obtained from a complex G-module by restriction of scalars. The restriction is compatible with the Γ-action, hence $(1-\gamma)(1-\sigma) M \in RO_h(G)$ follows in this case from the analogous statement for complex modules. If G is dihedral with generators g,t and relations $g^n = gtgt = t^2 = 1$ then the possible M have the form: M = $\mathbb{C}$, g acts through multiplication with $\exp(2\pi ij/n)$, $(j,n) = 1$, and t acts as complex conjugation. In this case 9.5.1 still works. This finishes the proof in the real case.

Remark. A different proof for Theorem 9.1.4 will be given in section 10. This proof uses the Galois invariance of certain stable homotopy modules over the Burnside ring.

9.6. Homotopy equivalences for p-groups.

We prove Theorem 9.1.6. This Theorem tells which representations of p-groups are (oriented) stably homotopy equivalent. The proof will be done by induction over the order of the group. Later we shall present a more conceptual proof which also gives better results. We assume in this section that 9.1.6 holds for cyclic, dihedral, and quaternionic groups; this is essentially classical (see de Rahm [132] ,) and will be reproved in 9.7 after we have developed some general facts from equivariant K-theory.

Let G be a p-group. Let S(G) be the set of normal subgroups of G. If a G-module V is given we write

$$V = \bigoplus_{H \in S(G)} V(H)$$

where V(H) collects the irreducible submodules of V which are lifted from faithful irreducible G/H-modules (i. e. have kernel H).

Lemma 9.6.1. *If* $x = V-W \in R_h(G)$ (*resp.* $RO_h(G)$) *then for all* $H \in S(G)$ *we have* $x(H) := V(H) - W(H) \in R_h(G)$ (*resp.* $RO_h(G)$). (*Here* G *can be an arbitrary group.*)

<u>Proof.</u> Let $f : S(V \oplus U) \longrightarrow S(W \oplus U)$ be a G-homotopy equivalence. If $H \in S(G)$ is a maximal proper subgroup of G (among the isotropy groups on V) then $S(V \oplus U)^H = S(V^G \oplus V(H) \oplus U^H)$ and therefore f^H gives a stable homotopy equivalence between V(H) and W(H), which is oriented if f was oriented. But because $R_h(G)$ is a subgroup of R(H) we can subtract x(H) from x and use the same argument for x - x(H). Downward induction over the $H \in S(G)$ gives the result.

We let j(K,f) be the j-group built from faithful irreducible K-modules, i. e. $j(K,f) = R_o(K,f)/R_h(K,f)$ where $R_o(K,f)$ is the set of x = V-W with V and W direct sums of faithful irreducible K-modules and $R_h(K,f)$ the subgroup of those $x = V-W \in R_o(K,f)$ such that V and W are oriented stably homotopy equivalent. We have similar groups i(K,f), iO(k,f), and jO(K,f). Lemma 9.6.1 tells us that we have a splitting

$$(9.6.2) \qquad s : j(G) \cong \prod_{H \in S(G)} j(G/H,f)$$

mapping x to $(x(H) | H \in S(G))$. The isomorphism 9.4.1 yields a similar splitting for i(G). The map t(G) is compatible with this splitting, it is therefore a direct sum of maps

$$t(G/H,f) : i(G/H,f) \longrightarrow j(G/H,f) .$$

It is enough to study the maps t(K,f) and similarly defined maps tO(K,f). They are surjective by definition. Our assumption in the beginning of this section was that these maps are injective if K is cyclic, or if K is a dihedral or generalized quaternion 2-group. By Proposition 9.4 and induction over the group order, t(G/H,f) and tO(G/H,f) is injective if we deal with imprimitive modules ($p \neq 2$ in the complex case). By 9.4 the possible kernel of t(G) for 2-groups G may be described as follows: It is generated by elements $V-V^*$, where V is an irreducible G-module with $V \neq V^{*}$ and $\dim V^{H} \equiv 0 \bmod 2$ for all $H < G$. But by 9.3.3 this case cannot occur. This finishes the proof of 9.1.6.

9.7. Equivariant K-theory and fixed point degrees.

Let V and W be complex G-modules. Let $f : V^{c} \longrightarrow W^{c}$ be a pointed G-map between their one-point-compactifications. In this section G is a compact Lie group, if not otherwise specified. We apply equivariant complex K-theory to f and obtain an induced homomorphism

$$f^{*} : \widetilde{K}_{G}(W^{c}) \longrightarrow \widetilde{K}_{G}(V^{c}) .$$

By the equivariant Bott-isomorphism (Atiyah [10]) $\widetilde{K}_{G}(V^{c})$ is a free R(G)-module with generator $\lambda(V)$, the Bott class. Therefore f defines an element $z_f = z \in R(G)$ by $f^{*} \lambda(W) = z\, \lambda(V)$. We think of z being a character, i. e. a function on G. We want to compute this character.

Let $C < G$ be a topologically cyclic subgroup with generator g (i. e. powers of g are dense in C). Consider the following diagram (with $K_G(V)$ for $\widetilde{K}_{G}(V^{c})$)

$$\begin{array}{ccc}
K_G(W) & \xrightarrow{\quad f^{*} \quad} & K_G(V) \\
\downarrow r & & \downarrow r \\
K_C(W^{C}) & \xrightarrow{\quad (f^{C})^{*} \quad} & K_C(V^{C})
\end{array}$$

where the vertical maps are given by restriction to C and its fixed point sets. Since C acts trivially on V^{C} and W^{C} we have

$$(f^{C})^{*} \lambda(W^{C}) = d(f^{C}) \lambda(V^{C}),$$

$d(f^C)$ = degree of f^C. We put $d(f^C) = 0$ if $\dim W^C \neq \dim V^C$. Moreover from elementary properties of Bott-classes we have

$$r\ \lambda(W) = \lambda_{-1}(W_C)\ \lambda(W^C)$$

where W_C is a complement of W^C in W (as C-module) and λ_{-1} is the alternating sum $\sum (-1)^i \lambda^i$ of the exterior powers. If we put this together we obtain

$$(9.7.1) \qquad \lambda_{-1}(W_C)\ d(f^C) = \mathrm{res}_C z \cdot \lambda_{-1}(V_C).$$

If C is a torus we can solve for $\mathrm{res}_C z$ because R(C) has no zero-divisors. In general we evaluate characters at the generator $g \in C$, observing that $\lambda_{-1}(V_C)(g) \neq 0$. Therefore we obtain the following expression for the character z

<u>Proposition 9.7.2.</u> <u>The character</u> z_f <u>has values</u>

$$z_f(g) = d(f^C)\ \lambda_{-1}(W_C - V_C)(g)$$

<u>where</u> C <u>is the closed subgroup generated by</u> $g \in G$.

<u>Remark 9.7.3.</u> In particular the right hand side of the equation in 9.7.2 is a character of G. This is in general not obvious and gives conditions on the degrees $d(f^C)$. We exploit this fact in section 10.

<u>Corollary 9.7.4.</u> <u>If</u> $V-W \in R_h(G)$ <u>then</u>

$$g \longmapsto \lambda_{-1}(W_g - V_g)(g)$$

<u>is a character of</u> G. (<u>Here</u> $W_g := W_C$)

We shall see, especially in section 10, that 9.7.4 is a strong condition for V-W to lie in $R_h(G)$, but it is awkward to work with and therefore we derive a simpler criterion using the θ_k-operations of section 3. Namely if $k \in Z$ and $W = \psi^k V$ then we have

Proposition 9.7.5. The function

$$u(g) = k^{\dim V^g} \quad \lambda_{-1}(W_g - V_g)(g)$$

is a character of G, namely the character of $\theta_k(V)$.

Proposition 9.7.6. If V and $\psi^k V$ are oriented stably homotopy equivalent then

$$e : g \longmapsto k^{\dim V^g}$$

ia a character of G.

Proof. 9.7.4 and 9.7.5.

We use the last Proposition to do some explicit calculations. Namely we prove the results missing in 9.6.

Proposition 9.7.7. The maps t(K,f) and tO(K,f) are injective if K is an arbitrary cyclic group, or if K is a dihedral or generalized quaternion 2-group.

Proof. Cyclic groups. Let K be the cyclic group of order n with generator g. Let V be the standard irreducible K-module with g acting as multiplication with $u_n = \exp(2\pi i/n)$. We have $i(K,f) \cong Z/n^*$, V- $\psi^k V$ corresponding to k mod n. Injectivity of t(K,f) means in this case:

$V - \psi^k V \in R_h(K)$ if and only if $k \equiv 1 \bmod n$. Proposition 9.7.6 says in this case: $e(1) = k$, $e(x) = 1$ for $x \neq 1$ is a character of K. For any character e of a group G we have $|G|^{-1} \sum_{x \in G} e(x) \in Z$ because this is the multiplicity of the trivial character in e. Hence $\sum_{x\in G} e(x) \equiv 0 \bmod |G|$ In our case this yields $k+(n-1) \equiv 0 \bmod n$, i. e. $k \equiv 1 \bmod n$ as was to be shown.

In the case of real representations we allow also degrees -1. Hence we have to see wether $e(1) = k$, $e(x) = -1$ for $x \neq 1$ defines a character of G. This gives $k \equiv -1 \bmod n$, in accordance with $iO(K,f) = (Z/n)^*/\{\pm 1\}$.

Generalized quaternion groups. Let K be the group of order 2^{n+1} given by generators A,B and relations $BAB^{-1} = A^{-1}$, $A^{2^{n-1}} = B^2$, $n \geq 2$. The faithful irreducible representations of K are given as follows. We put $m = 2^n$.

$$V_k(A) = \begin{pmatrix} u_m^k & 0 \\ 0 & u_m^{-k} \end{pmatrix}, \quad V_k(B) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $1 \leq k \leq 2^{n-1} - 1$ and $k \equiv 1 \bmod 2$. One has $\psi^k V_1 = V_k$. Moreover

$i(K,f) \cong (Z/m)^*/\{\pm 1\}$, $V_1 - V_k \longmapsto k \bmod m$. Proposition 9.7.6 says that $e(1) = k^2$, $e(x) = 1$ for $x \neq 1$, shall be a character of K if $V_1 - V_k \in R_h$. This implies $k^2 + (2^{n+1} - 1) \equiv 0 \bmod 2^{n+1}$ and hence $k \equiv \pm 1 \bmod m$, q. e. d.

In the real case the only new condition to be considered is $k^2 \equiv -1 \bmod 2^{n+1}$ which is impossible. Restriction of scalars defines an isomorphism $i(K,f) = iO(K,f)$ and $tO(K,f)$ is injective.

<u>Dihedral groups.</u> Let K be the group of order 2^{n+1} with generators A,B and relations $A^{2^n} = ABAB = B^2 = 1$. The faithful irreducible representations are given as follows. We put $m = 2^n$.

$$V_k(A) = \begin{pmatrix} \cos 2\pi k/m & -\sin 2\pi k/m \\ \sin 2\pi k/m & \cos 2\pi k/m \end{pmatrix}$$

$$V_k(B) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $1 \leq k \leq 2^{n-1} -1$ and $k \equiv 1 \bmod 2$. We have $\psi^k V_1 = V_k$ and $i(K,f) \cong (Z/m)^*/\{\pm 1\}$. Proposition 9.7.6 says that $e(1) = k^2$, $e(A^i) = 1$ for $1 \leq i < m$, must be a character if $V_1 - V_k \in R_h$. One obtains $k^2+(m-1) + km \equiv 0 \bmod 2m$. This gives mod m $k \equiv \pm 1, \pm 1 + 2^{n-1}$ and only $k \equiv \pm 1$ lifts to a solution mod 2m. Whence injectivity of t(K,f).

Since the faithful irreducible real K-modules have no complex structure we use an ad hoc argument. The restriction to the cyclic subgroup C generated by A induces an isomorphism iO(H) = iO(C). But tO(C) is injective.

9.8. Exercises

1. Show that the functors $G \longmapsto j(G)$, $G \longmapsto jO(G)$ are modules over the Green functor "rational representation ring". Deduce that they satisfy hyperelementary induction.

2. Let V,W be complex G-modules which are oriented stably homotopy equivalent. Show that they are oriented homotopy equivalent. (Does an analogous assertion hold for real modules?)

3. Show by an example that $R_1(G) = R_h(G)$ is in general not true for non-p-group.