11. Homotopy-equivalent stable G-vector bundles. *)

The aim of this section is to extend some of the previous results and techniques from representations to vector bundles. The group G will always denote a finite p-group and we are concerned with the question: When are the sphere bundles of two G-vector bundles stably G-fibre-homotopy equivalent?

11.1. Introduction and results about local J-groups.

One of the basic questions in the homotopy theory of vector bundles is the following: Given two vector bundles over a space X, when are the associated sphere bundles fibre-homotopy-equivalent?

The question has been answered, for stable bundles, by Adams in his series of papers on the groups J(X) [2] , together with the affirmative solution of his famous conjecture (Quillen [128] , Sullivan [157] , Becker-Gottlieb [19]).

We shall extend some of these results to G-vector bundles. We consider G-vector bundles over finite G-CW-complexes. If $p : E \longrightarrow X$ is such a bundle we can choose a G-invariant Riemannian metric on E and consider the unit-sphere bundle $S(E) \longrightarrow X$. If V is a real G-module we also let V denote the product bundle $V \times X \longrightarrow X$. If $p_i : E_i \longrightarrow X$ are G-vector bundles a <u>stable map</u> $f : S(E_1) \longrightarrow S(E_2)$ shall be a fibrewise G-map $S(E_1 \oplus V) \longrightarrow S(E_2 \oplus V)$ for some G-module V. Two G-vector bundles $p_i : E_i \longrightarrow X$ over X are called <u>stably-homotopy-equivalent</u>, notation $E_1 \sim E_2$, if for some G-module V there exists a G-fibre-homotopy-equivalence $f : S(E_1 \oplus V) \longrightarrow S(E_2 \oplus V)$. If E and F are G-vector bundles over X then $S(E \oplus F)$ is G-homeomorphic over X to the <u>fibrewise join</u> $S(E) * S(F)$. Using this it is easy to see that $E_1 \sim E_2$, $F_1 \sim F_2$ implies $E_1 \oplus F_1 \sim E_2 \oplus F_2$. Let $KO_G(X)$ be the Grothendieck ring

*) This section contains joint work with H. Hauschild.

of real G-vector bundles over X. Then the previous remark shows that

$$(11.1.1) \qquad TO_G(X) = \{E_1 - E_2 \in KO_G(X) \mid E_1 \sim E_2\}$$

is well-defined and an additive subgroup of $KO_G(X)$. We pose the problem: Describe $TO_G(X)$ as a subgroup of $KO_G(X)$. The solution uses the computation of the J-groups

$$(11.1.2) \qquad JO_G(X) = KO_G(X)/TO_G(X).$$

We now introduce some intermediate J-groups where homotopy-equivalence is replaced by weaker conditions. Note that a G-fibre-homotopy-equivalence $f : S(E_1 \oplus V) \longrightarrow S(E_2 \oplus V)$ induces an ordinary fibre-homotopy-equivalence f^H for all H-fixed point bundles ($H < G$ a subgroup of G). We therefore consider the following local condition: Two G-vector bundles E and F are called stably locally homotopy-equivalent, notation $E \sim_{loc} F$, if for every $H < G$ there exists a G-module V and fibrewise G-maps $f : S(E \oplus V) \longrightarrow S(F \oplus V)$ and $g : S(F \oplus V) \longrightarrow S(E \oplus V)$ such that f^H and g^H are ordinary fibre-homotopy-equivalences. As before it is seen that

$$(11.1.3) \qquad TO_G^{loc}(X) = \{E_1 - E_2 \in KO_G(X) \mid E_1 \sim_{loc} E_2\}$$

is well-defined and an additive subgroup of $KO_G(X)$. We study this subgroup via a computation of

$$(11.1.4) \qquad JO_G^{loc}(X) = KO_G(X)/TO_G^{loc}(X).$$

The introduction of these local J-groups may seem artificial at first sight. We offer some justification. Obviously we have a surjective homomorphism $JO_G(X) \longrightarrow JO_G^{loc}(X)$. If X is a point one obtains from

Atiyah-Tall [14] and tom Dieck [67] that this map is not an isomorphism: For p-groups it measures the difference between G-homotopy-equivalence and G-maps of degree one. It turns out that a computation of 11.1.4 will yield the main part of 11.1.2. Moreover $JO_G^{loc}(X)$ is actually computable using the action of the Adams operations on $KO_G(X)$ in the same way as the non equivariant J-groups are computed. So also from this point of view 11.1.3 is just the correct object to consider.

We now state our results on the computation of the local J-groups 11.1.4. It is expedient to consider the localizations

(11.1.5) $$JO_G^{loc}(X)_q = KO_G(X)_q / TO_G^{loc}(X)_q$$

where the index q indicates that we have localized at the rational prime q.

Given q let r(1),...,r(n) be a set of integers (depending on q and p) generating the q-adic units (modulo $\pm$ 1 if q = 2) and generating the units $Z/|G|Z^*$ of the integers modulo $|G|$. If q = p then we take n = 1 and r = r(1) = 3 if p = 2, and r a generator of Z/p^2Z^* if $p \neq 2$. Our main result is the

<u>Theorem 11.1.6.</u> <u>Let</u> G <u>be</u> <u>a</u> <u>finite</u> p-<u>group</u>. <u>Then</u> $TO_G^{loc}(X)_q$ <u>is generated as abelian group by elements of the form</u> $x - \psi^{r(i)}x$, $x \in KO_G(X)_q$ $i = 1,\dots,n$, <u>where</u> ψ^r <u>denotes the</u> r-<u>th Adams operation</u>.

The proof naturally splits into two parts. First we consider the case p = q. Here we prove an equivariant analogue of the Adams conjecture by elementary methods. We use the device of Becker-Gottlieb [19] but apply it to the universal example: orthogonal representations. We thus generalize the method which Adams [2] used for two-dimensional bundles. Moreover the main theorem of Atiyah-Tall [14] on p-adic λ-rings is used as well as the completion theorem of Atiyah-Segal [12]. The second part of the proof is essentially concerned with the situation where the order of the group is invertible. Here we can use the localization and splitting theorems of section 8 to decompose K-theory into simpler pieces for which the problem can easily be solved. We should point out that our exposition contains a computation of the non-equivariant J-groups which seems somewhat simpler than other published versions: We neither need Quillens computations nor infinite loop spaces.

11.2 Mapping degrees. Orientations.

This section contains some technical preparation. In particular we show that it suffices to consider orientable bundles.

An n-dimensional real G-vector bundle $E \longrightarrow X$ is called orientable if the n-th exterior power $\Lambda^n E$ is isomorphic to $X \times \mathbb{R} \longrightarrow X$ with trivial G-action on $\mathbb{R}$. Bundles E_1 and E_2 of dimension n are said to have the same orientation behaviour if $\Lambda^n E_1$ and $\Lambda^n E_2$ are isomorphic G-bundles. We put

$$(11.2.1) \qquad KSO_G(X) = \{ E_1 - E_2 \in KO_G(X) \mid E_i \text{ orientable} \} .$$

By a theorem of Dold [71] a fibrewise map $S(E) \longrightarrow S(F)$ is a fibre homotopy equivalence if and only if it is a homotopy equivalence on each fibre, i. e. has degree ± 1 on each fibre. It is therefore reasonable to ask for the existence of fibrewise G-maps with prescribed degree on the fibres.

Let $S \subset \mathbb{Z}$ be a set of prime numbers. If E and F are G-vector bundles over X we write

(11.2.2) $$E_1 \leq_S E_2$$

if there exists a stable map $f : S(E) \longrightarrow S(F)$ with fibre degree prime to all elements of S. We write

(11.2.3) $$E \sim_S F \quad \text{if} \quad E \leq_S F \quad \text{and} \quad F \leq_S E.$$

We put

(11.2.4) $$TO_{G,S}(X) = \{E - F \in KO_G(X) \mid E \sim_S F\}$$

(11.2.5) $$JO_{G,S}(X) = KO_G(X)/TO_{G,S}(X) \ .$$

If S is the set of all primes then $E \sim_S F$ means that there exist stable maps $S(E) \longrightarrow S(F)$ and $S(F) \longrightarrow S(E)$ of degree ± 1 on the fibres.

<u>Lemma 11.2.6.</u> *Suppose there exists a fibrewise G-map* $f: S(E) \longrightarrow S(F)$ *of odd degree. Then*

$$E - F \in KSO_G(X).$$

<u>Proof.</u> Since Stiefel-Whitney classes are modulo 2 fibre-homotopy invariant we have $w_1(E) = w_1(F)$. If $w_1(E) \neq 0$ and $\Lambda^n E$ is the determinant bundle of E we have a fibrewise G-map $S(E \oplus \Lambda^n E) \longrightarrow S(F \oplus \Lambda^n E)$ of odd degree. We can therefore assume without loss of generality that E and F are orientable as bundles without group action. To show the determinant bundles are equal in this case we need only show that the G-action on each fibre is the same. But $g \in G$ acts as identity on the determinant bundle if it preserves the orientation and as minus identity otherwise and this distinction is preserved by a map of odd degree.

Corollary 11.2.7. $TO_G(X) \subset TO_G^{loc}(X) \subset KSO_G(X)$.

Let $B(G,O(1)) = B$ be the classifying space for one-dimensional G-bundles (tom Dieck). Then assigning to each bundle E its determinant bundle induces a split surjective homomorphism

$$(11.2.8) \qquad \det : KO_G(X) \longrightarrow [X,B]_G$$

with kernel $KSO_G(X)$; here $[-,-]_G$ denotes the set of G-homotopy classes. Using Corollary 11.2.7 we therefore obtain natural splittings

$$(11.2.9) \qquad JO_G(X) \cong JSO_G(X) \oplus [X,B]_G ,$$

with JSO = KSO/TO; and similarly for the local J-groups.

11.3. Maps between representations and vector bundles.

In this section we construct certain equivariant maps between orthogonal representations. The construction is a simple application of the methods in Becker-Gottlieb [19] and Meyerhoff-Petrie [114], and is essentially well known. These maps between representations will then give us maps between vector bundles.

Proposition 11.3.1. Let $\mathbb{R}^{2n}$ be the standard $O(2n)$-representation. Let k be a positive integer. Then there exist stable $O(2n)$-maps $S(\mathbb{R}^{2n}) \longrightarrow S(\psi^k \mathbb{R}^{2n})$ with degree a divisor of k^t for some $t \in \mathbb{N}$ if k is odd. (Otherwise for $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$.)

Remark. $\psi^k \mathbb{R}^{2n}$ may be a virtual $O(2n)$-module $V - W$, of course. The Proposition has to be read that there exists stable $O(2n)$-maps

$S(\mathbb{R}^{2n} \oplus W) \longrightarrow S(V)$. We use similar notations for vector bundles.

Proof. Let $T < O(2n)$ be a maximal torus with normalizer NT. Then $NT = S_n \times_s O(2)^n$, where S_n is the symmetric group and $\times_s$ means semi-direct product with respect to the permutation action of S_n on $O(2)^n$. We first show the existence of an NT-map of the required degree. Let

$$H = \{(s;x_1,\dots,x_n) \in S_n \times_s O(2)^n \mid s(1) = 1\}.$$

One obtains a homomorphism

$$h : H \longrightarrow O(2) : (s;x_1,\dots,x_n) \longmapsto x_1$$

and an associated 2-dimensional H-module V. The group H has finite index in NT, namely $[NT : H] = n$. Therefore one can consider induced representations ind_H^{NT}. One has

(11.3.2) $$\mathrm{ind}_H^{NT} V \cong W$$

where W is the standard NT-module (restriction of the standard $O(2n)$-module). See Becker-Gottlieb [19] for a proof of 11.3.2. If k is odd there is an $O(2)$-map $g : S(V) \longrightarrow S(\psi^k V)$; if $V = \mathbb{C}$ this is simply the map $z \longmapsto z^k$ (see Adams [2]). If k is even then $\psi^k(V) = \mu_k - \lambda_2 + 1$, where λ_2 is the determinant representation associated to the standard $O(2)$-action on $\mathbb{R}^2$ and where μ_k is $\mathbb{C}$ with $z \in S^1 = SO(2)$ acting as multiplication by z^k and $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ acting as conjugation. There exists an $O(2)$-map $g' : S(V) \longrightarrow S(\mu_k)$, the map $z \longmapsto z^k$ as before. Since λ_2 and $\mathbb{R}$ have different orientation behaviour there does not exist a stable Z/2-map $S(\lambda_2) \longrightarrow S(\mathbb{R}^1)$. But $\lambda_2 \oplus \lambda_2$ and $\mathbb{R} \oplus \mathbb{R}$ have the same orientation behaviour and therefore we can find a stable Z/2-map (and hence $O(2)$-map)

$S(\lambda_2 \oplus \lambda_2) \longrightarrow S(\mathbb{R} \oplus \mathbb{R})$ of degree 2. Put together we see that there exists a stable H-map $g : S(V \oplus V) \longrightarrow S(\Psi^k(V \oplus V))$ whose degree divides some power of k.

Induction ind_H^{NT} yields a stable G-map

(11.3.3) $\quad \mathrm{ind}_H^{NT}(g) : S(\mathrm{ind}_H^{NT} V) = S(W) \longrightarrow S(\mathrm{ind}_H^{NT} \Psi^k V).$

In order to finish the proof we need a stable NT-map

(11.3.4) $\quad h : S(\mathrm{ind}_H^{NT} \Psi^k V) \longrightarrow S(\Psi^k(\mathrm{ind}_H^{NT} V))$

of suitable degree. For a prime p let $(NT/T)_p$ be the Sylow-p-group of NT/T and N_pT its counter-image in NT. If p is prime to k then $\mathrm{ind}_H^{NT}(\Psi^k V)$ and $\Psi^k(\mathrm{ind}_H^{NT} V)$ are isomorphic as N_pT-modules; this follows from two facts:

(11.3.5) If k is prime to the index $[G : H]$ then in general $\Psi^k \mathrm{in}_H^G = \mathrm{ind}_H^G \Psi^k$.

(11.3.6) $\mathrm{res}_{N_pT}^{NT} \mathrm{ind}_H^{NT}$ is by the double coset formula of representation theory a direct sum with summands of the form $\mathrm{ind}_K^{N_pT} \mathrm{res}_K^H$; and since $T < K$ the index $[N_pT : K]$ is prime to k.

Using this isomorphism of N_pT-modules we can find a stable NT-map h_p in 11.3.4 of degree $|NT/N_pT|$. Since the greatest common divisor of all the $|NT/N_pT|$ with p prime to k involves only prime divisors of k we can form a suitable linear combination of the h_p (in the homotopy group of stable maps) to produce an NT-map h whose degree divides a power of k.

As a consequence of Proposition 11.3.1 we obtain stable maps between vector bundles as follows. Let $E \longrightarrow B$ be a real G-vector bundle of dimension n (with Riemannian metric). The associated principal O(2n)-bundle $P \longrightarrow B$ is in fact a (G,O(2n))-bundle (see tom Dieck [50]). We have the following isomorphisms of G-vector bundles

$$E \cong P \times_{O(2n)} \mathbb{R}^{2n}, \quad \psi^k E = P \times_{O(2n)} \psi^k \mathbb{R}^{2n}.$$

Hence we obtain from Proposition 11.3.1.

Proposition 11.3.7. *Let G be a compact Lie group and let $E \longrightarrow B$ be an orthogonal G-vector bundle. Then there exist stable G-maps $S(E) \longrightarrow S(\psi^k E)$ if k is odd $(S(E \oplus E) \longrightarrow S(\psi^k(E \oplus E))$ if k is even) of fibre-degree dividing a power of k.*

One actually would like to have an information about the degrees on fixed point sets. By the methods of Quillen [128] one can prove the following equivariant version of the Adams conjecture.

Theorem 11.3.8. *There exist stable G-maps $f : S(E) \longrightarrow S(\psi^k E)$ such that f^H has for all $H < G$ a degree which divides a power of k. (k prime to $|G|$).*

By the results of section 9 and 10 this is easy to see for bundles with finite structure group.

11.4. Local J-groups at p.

Let G be a finite p group and let $r \in \mathbb{N}$ be an odd generator of the p-adic units (mod ± 1 if $p = 2$). Let X be a finite connected G-CW-complex. The main result of this section is

Theorem 11.4.1. The following sequence is exact

$$KO_G(X)_p \xrightarrow{1-\Psi^r} KO_G(X)_p \xrightarrow{J} JO_G^{loc}(X)_p.$$

(*The map* J *is the quotient map*.)

The proof consists in a sequence of Propositions. Recall definition (2.5) for the next result. Let S be the set of all primes.

Proposition 11.4.2. The canonical quotient map

$$B : JO_G^{loc}(X)_p \longrightarrow JO_{G,\{p\}}(X)_p$$

is an isomorphism.

Proof. Suppose $B(E - F) = 0$. Then we can find stable G-maps $f : SE \longrightarrow SF$ and $g : SF \longrightarrow SE$ of degree prime to p. By a theorem of Adams [2] , we can find a stable map $h:S(kE) \longrightarrow S(kF)$ of degree one, where $(k,p) = 1$. Hence (using induction) there exists a stable G-map $h' : S(kE) \longrightarrow S(kF)$ of degree $p^n = |G|$. Since $(\deg(f), \deg(h')) = 1$ a suitable linear combination of f and h' will yield a stable G-map $v : S(kE) \longrightarrow S(kF)$ of degree 1. The same reasoning can be applied to g, and to fixed point mappings. Hence E-F is zero in $JO_G^{loc}(X)_p$.

We now have to consider fibrewise localizations of sphere bundles in the sense of Sullivan [157] . In order to talk about something definite we use the following construction for such localizations. Let $E \longrightarrow B$ be an orthogonal G-vector bundle and $P \longrightarrow B$ be the associated principal $(G,O(n))$-bundle. Let $O(n)$ act on $\mathbb{R}^n \oplus \mathbb{R}^k$, $k \geq 3$, through the standard action on $\mathbb{R}^n$. Let $S(\mathbb{R}^n \oplus \mathbb{R}^k)_p$ be the p-local sphere obtained from a telescope-construction applied to a diagram

$$S(\mathbb{R}^{n+k}) \xrightarrow[f_1]{} S(\mathbb{R}^{n+k}) \xrightarrow[f_2]{} \cdots$$

where the maps f_i are the identity on $S(\mathbb{R}^n)$ in $S(\mathbb{R}^n \oplus \mathbb{R}^k) = S(\mathbb{R}^n) * S(\mathbb{R}^k)$. Then $S(\mathbb{R}^{n+k})_p$ still carries an $O(n)$-action and

$$P \times_{O(n)} S(\mathbb{R}^{n+k})_p$$

is our stable representative for the p-local sphere bundle associated to $E \longrightarrow B$. By abuse of notation we denote this bundle $S(E)_p$. We use the fact that $S(E)_p \longrightarrow B$ is a G-fibration (G-homotopy lifting property for all spaces) if $E \longrightarrow B$ is a numerable bundle.

Proposition 11.4.3. *Suppose* r *is odd and prime to* p. *Let* G *be a p-group and* X *a finite G-CW-complex. Then*

$$(1-\psi^r)\, KO_G(X)_p \subset TO_{G,\{p\}}(X)_p .$$

Proof. By Proposition 11.3.7 there exists a stable G-map $f : S(E) \longrightarrow S(\psi^r E)$ of degree prime to p. Since G is a p-group we have $\deg f^H \not\equiv 0 \bmod p$ for all $H < G$. The induced map

$$f_p^H : S(E)_p^H \longrightarrow S(\psi^r E)_p^H$$

is therefore a fibrewise map and a homotopy-equivalence on each fibre. By a theorem of Dold [71] f_p^H is a fibre-homotopy-equivalence. By 8.2.4 f_p is a G-homotopy-equivalence and by the equivariant analogue of Dold [71] therefore a G-fibre-homotopy-equivalence, with inverse $g_p : S(\psi^r E)_p \longrightarrow S(E)_p$ say. Since X is compact the composition

$$S(\psi^r E) \xrightarrow[i]{} S(\psi^r E)_p \xrightarrow[g_p]{} S(E)_p,$$

where i, a the canonical map into the telescope, has an image which is contained in a finite piece of the telescope. Therefore we obtain a stable G-map $g : S(\psi^r E) \longrightarrow S(E)$ of degree prime to p. This shows $E \sim_{\{p\}} \psi^r E$.

We remark that the proof above actually shows the following

Proposition 11.4.4. Suppose $f : S(E) \longrightarrow S(F)$ is a stable G-map such that the fibre degrees of f^H divide a power of k. Then there exists a stable G-map $g : S(F) \longrightarrow S(E)$ with the same property.

Proof of Theorem 11.4.1. By Proposition 11.4.2 and 11.4.3 we know that $J \circ (1-\psi^r)$ is zero. Hence we have to show that the induced map

$$Q : KO_G(X)_p/(1-\psi^r) \longrightarrow JO_G^{loc}(X)_p$$

is injective. We use the results of Atiyah-Tall [14] on p-adic λ-rings which we have presented in section 3. We let A_p be the p-adic completion of the abelian group A.

Let $\tilde{K}SO_G(X)$ be the subgroup of elements of dimension zero. By the results of 11.2, in particular Lemma 11.2.6, we need only show that the map

$$\tilde{Q} : \tilde{K}SO_G(X)_p/(1-\psi^r) \longrightarrow \tilde{J}SO_G^{loc}(X)_p$$

is injective.

By Atiyah-Tall [14] , III. Proposition 3.1, the p-adic and I(G)-adic topologies on KO_G(Point) coincide. This implies that the p-adic and I(G)-adic topologies on $KSO_G(X)$ coincide, if X is a finite G-CW-complex (use Atiyah-MacDonald [11] , 10.13). By the version for orientable vector bundles of the Atiyah-Segal completion theorem [12] one has an isomorphism

$$\alpha : \widetilde{KSO}_G(X)^\wedge_p \longrightarrow \widetilde{KSO}(X_G),$$

where $X_G = EG \times_G X$, EG the universal free G-space.

We now consider the following diagram whose ingredients we explain in a moment.

$$\begin{array}{ccccc}
\widetilde{KSO}_G(X)_p/1-\psi^r) & & \xrightarrow{\quad\tilde{Q}\quad} & & \widetilde{JSO}_G^{loc}(X)_p \\
\downarrow i_\Gamma & & (*) & & \\
\widetilde{KSO}_G(X)^\wedge_{p,\Gamma} & \xrightarrow{\sigma_{r,\Gamma}} & (1+\widetilde{KSO}_G(X)^\wedge_p)_\Gamma & & \swarrow \theta_r \\
\cong \downarrow \alpha_\Gamma & (**) & \alpha_\Gamma \searrow \cong & & \\
\widetilde{KSO}(X_G)_\Gamma & \xrightarrow{\tau_{r,\Gamma}} & & (1+\widetilde{KSO})X_G))_\Gamma &
\end{array}$$

The index Γ indicates that we factor out the image of $1-\psi^r$. The ring $\widetilde{KSO}_G(X)^\wedge_p$ is an orientable p-adic γ-ring; we therefore have the map ς^{or}_r, as defined in 3.10.7. The map $\sigma_{r,\Gamma}$ is induced by ς^{or}_r on the quotients. Similarly α_Γ is induced by α and $\tau_{r,\Gamma}$ is defined so as to make (**) commutative. The inclusion $i : \widetilde{KSO}_G(X)_p \longrightarrow \widetilde{KSO}_G(X)^\wedge_p$ induces an injective map i_Γ because p-adic completion is exact on finitely-generated Z_p-modules. Since $\sigma_{r,\Gamma}$ is an isomorphism by

3.14.10 we need only demonstrate the existence of a homomorphism Θ_r which makes the diagram commutative.

Suppose $f : S(E) \longrightarrow S(F)$ is a stable G-map of degree zero. Then $EG \times_G S(E)$ and $EG \times_G S(F)$ are fibre homotopy equivalent hence have the same Stiefel-Whitney classes. We therefore may and will assume that they both have a Spin(8n)-structure and hence a K-theory Thom-class. Applying $id \times_G f$ to these Thom-classes and using 3.15 one obtains

$$\tilde{\Theta}_r(EG \times_G E)\ \psi^r(z) = z\ \tilde{\Theta}_r(EG \times_G F)$$

with a suitable $z \in 1 + \widetilde{KSO}(X_G)$ and this yields the desired factorisation.

11.5. Local J-groups away from p.

We now assume that q is a prime different from p and compute the J-groups localized at q.

To begin with let C be a cyclic group and Y a trivial C-space. We can compute $JO_C^{loc}(Y)_q$ as follows.

Since Y is a trivial C-space vector bundles over Y split according to the irreducible C-modules (see Segal [142], Remark on p. 133). Since C is a cyclic p-group the splitting of vector bundles according to the kernels of the irreducible C-modules is preserved by JO^{loc}-equivalence and by Adams operations. Hence it suffices to discuss that direct summand of $JO_C^{loc}(Y)_q$ which comes from C-vector bundles whose fibre representations only contain faithful C-modules. We claim that forgetting the group action induces an isomorphism of this direct summand with $JO(Y)_q$ (if $q \neq 2$) and with $J(Y)_q$ (if $q = 2$ and C non-trivial). Moreover $JO_C^{loc}(Y)_q$ can be computed as in 11.1.6 in this case. We prove all this.

Let $(r,pq) = 1$. Then there is a stable C map $S(E \otimes V) \longrightarrow S(\psi^r E \otimes \psi^r V)$ of degree t dividing r^n, where V is a faithful C-module and E a bundle with trivial C-action. As in the proof of 11.4.2 we see that there exists a C-map $S(t^i(E \otimes V)) \longrightarrow S(t^i(\psi^r E \otimes \psi^r V))$ for suitable i. Since $(t,q) = 1$ we have that $(1-\psi^r)(E \otimes V)$ is zero in $JO_C^{loc}(X)_q$ (use also 11.4.4).

Now suppose that $E_1 - E_2$ maps to zero in $JO(Y)_q$. For each r generating the q-adic units there exists an F such that $E_1 - E_2 = (1-\psi^r)F$, by the non-equivariant computation of $JO(Y)_q$ which is a special case of the results in 11.4. Hence also $F \otimes V - \psi^r F \otimes V^r$ in $JO_C(Y)_q$. (We can actually work with complex vector bundles, because $J(Y)_q \cong JO(Y)_q$ if $q \neq 2$ and if $q = 2$ then C is not a 2-group and the faithful representations of C are of complex type.) If we choose r such that $V^r = V$ then we see that $E_1 \otimes V - E_2 \otimes V = (1-\psi^r)(F \otimes V)$ maps to zero in $JO_C^{loc}(Y)_q$ is of the form as claimed in 11.1.6. In general if $E_1 - E_2 =$ $= (1-\psi^s)F_1$ then $E_1 \otimes V - E_2 \otimes V^s = F_1 \otimes V - \psi^s F_1 \otimes V^s$ $= (F_1 \otimes (V - V^s)) + ((F_1 - \psi^s F_1) \otimes V^s)$ shows that $F_1 \otimes (V - V^s)$ is also contained in the subgroup generated by the $(1 - \psi^{r(i)})$ of 11.1.6. This settles the case of cyclic p-groups C and trivial C-spaces Y.

We now prove 11.1.6 in general for $q \neq p$. By 7.7 we have a natural transformation

$$KO_G(X) \longrightarrow \bigoplus_{(C)} KO_C(X^C)$$

where (C) runs over the conjugacy classes of cyclic subgroups of G. This transformation has a natural splitting which is compatible with the action of the Adams operations. Let $JO'_G(X)$ denote the quotient of $KO_G(X)$ by the subgroups generated by $(1-\psi^{r(i)})x$ as in 11.1.6. Then we have the diagram

$$\begin{array}{ccc}
KO_G(X)_q & \longrightarrow & \bigoplus_{(C)} KO_C(X^C)_q \\
\downarrow & & \downarrow \\
JO'_G(X)_q & \xrightarrow[(2)]{} & \bigoplus_{(C)} JO'_C(X^C)_q \\
\downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (3)} \\
JO^{loc}_G(X)_q & \xrightarrow[(4)]{} & \bigoplus_{(C)} JO^{loc}_C(X^C)_q
\end{array}$$

The maps (1) and (3) are surjective by construction. The map (2) is split injective by the splitting theorem just quoted. The map (3) is bijective by the proof above. Hence (1) is also injective hence an isomorphism. This finishes the proof of Theorem 11.1.6.

11.6. Projective modules.

We are going to discuss the difference between JO^{loc}_G and JO_G.

Let E and F be G-vector bundles over X. Let $[S(E),S(F)]$ be the set of G-fibre homotopy classes $S(E) \longrightarrow S(F)$. Fibrewise suspension defines a map $[S(E),\ S(F)] \longrightarrow [S(E \oplus V),\ S(F \oplus V)]$. We take the direct limit over such suspension maps and call the limit $\omega^{o}_{G}(E,F)$, which is the set of G-homotopy classes of stable maps $S(E) \longrightarrow S(F)$. We list some of the standard properties of this construction.

(11.6.1) $\omega^{o}_{G}(E,F)$ is an abelian group and in fact a module over $\omega^{o}_{G}(X)$.

(11.6.2) A G-map $f : Y \longrightarrow X$ induces a homomorphism

$$f^* : \omega_G^0(E,F) \longrightarrow \omega_G^0(f^*E, f^*F).$$

(11.6.3) Composition of mappings defines a pairing

$$\omega_G^0(E,F) \times \omega_G^0(F,H) \longrightarrow \omega_G^0(E,H)$$

which is $\omega_G^0(X)$-bilinear.

(11.6.4) Whitney sum defines a pairing

$$\omega_G^0(E_1,F_1) \times \omega_G^0(E_2,F_2) \longrightarrow \omega_G^0(E_1 \oplus E_2, F_1 \oplus F_2)$$

which is ω_G^0-bilinear.

(11.6.5) There are canonical isomorphisms of $\omega_G^0(X)$-modules

$$\omega_G^0(E,E) \cong \omega_G^0(X)$$

$$\omega_G^0(E,E) \cong \omega_G^0(E \oplus F, E \oplus F)$$

Proposition 11.6.7. Suppose $E - F \in TO_G^{loc}(X)$. _Then_ $\omega_G^0(E,F)$ _is a projective_ $\omega_G^0(X)$-_module of rank one and_ $\omega_G^0(F,E)$ _is its inverse in the Picard group of_ $\omega_G^0(X)$. _The module is free if and only if_ $E - F \in TO_G(X)$.

Proof. We have determined the prime ideals q of $\omega_G^0(X)$ in We localize at q and show that $\omega_G^0(E,F)_q$ is a free $\omega_G^0(X)_q$-module of rank one and that $\omega_G^0(E,F) \otimes \omega_G^0(F,E) \longrightarrow \omega_G^0(E \oplus F, F \oplus E) \cong \omega_G^0(X)$ induces an isomorphism after localization at q. But by the definition of $TO_G^{loc}(X)$ we have for a given H a stable G-map $f : S(E) \longrightarrow S(F)$ such that f^H has fibre degree one. Now proceed as in 10.2.6.

From 11.6.7 we obtain an injective homomorphism $pO_X(G) : TO_G^{loc}(X)/TO_G(X) \longrightarrow \mathrm{Pic}\ \omega_G^0(X)$. Note that the source of $pO_X(G)$ is precisely the kernel of $JO_G^{loc} \longrightarrow JO_G$. The Picard group $\omega_G^0(X)$

does not change if we divide out the nilradical of $\omega^{o}_{G}(X)$. We have seen that $\omega^{o}_{G}(X)/\mathrm{Nil}$ only depends on the orbit category of X. In particular if all the fixed point sets of X are non-empty and connected then we obtain a natural splitting $JO_G(X) \cong JO_G^{loc}(X) \oplus jO(G)$.