## 1. The Burnside ring of finite G-sets.

In this section let G denote a finite group. In order to motivate some of the subsequent investigations we give an introduction to the Burnside ring of a finite group. Later we generalize this to compact Lie groups by geometric methods which in case of a finite group are not always suitable for the applications of the Burnside ring in representation theory. The material in this section is mainly due to Andreas Dress.

### 1.1. Finite G-sets.

A finite G-set S is a finite set together with a left action of G on this set. A finite G-set is the disjoint union of its orbits. The orbits are transitive G-sets and are G-isomorphic to homogeneous G-sets $G/H = \{gH \mid g \in G\}$. The G-sets G/H and G/K are isomorphic if and only if H is conjugate to K in G. The set of G-isomorphism classes of finite G-sets becomes a commutative semi-ring $A^+(G)$ with identity with addition induced by disjoint union and multiplication induced by cartesian product with diagonal action. The non-triviality of the multiplication arises from the decomposition of $G/H \times G/K$ into orbits. These orbits correspond to the double cosets HgK, $g \in G$, which can be identified with the orbit space of G/K under the left H-action. This correspondence can be described as follows: If X is an H-space the H-orbits of X correspond to the G-orbits of $G \times_H X$. If moreover X is a G-space then we have the G-isomorphism $G/H \times X \longrightarrow G \times_H X : (g,x) \longmapsto (g,g^{-1}x)$. We apply this to X = G/K. Explicitely, the double coset HgK corresponds to the orbit through (1,g).

### 1.2. The Burnside ring A(G).

The Grothendieck ring constructed from the semi-ring $A^+(G)$ is denoted A(G) and will be called the Burnside ring of G. If S is a finite G-set let $[S]$ or $S$ be its image in $A(G)$. Additively, $A(G)$ is the free abelian group on isomorphism classes of transitive G-sets. Equivalently, an additive Z-basis is given by the $[G/H]$ where $(H)$ runs through the set $C(G)$ of conjugacy classes of subgroups of G. The multiplication comes from the decomposition of $G/H \times G/K$ into orbits. The ring $A(G)$ is commutative with unit $[G/G]$ .

Example 1.2.1.

Let G be abelian. Then, since generally the isotropy group of $G/H \times G/K$ at $(g_1H, g_2K)$ is $g_1Hg_1^{-1} \cap g_2Kg_2^{-1}$, all isotropy groups are $H \cap K$ in the abelian case. Therefore $[G/H] \cdot [G/K] = a [G/H \cap K]$ where $a \in Z$ is obtained by counting the number of elements on both sides. In particular $[G/H]^2 = |G/H| \; [G/H]$ , where $|S|$ is the cardinality of S. We see that for abelian G the $[G/H]$ are almost idempotent.

If $H < G$ and S,T are finite G-sets then we have for the cardinality of the H-fixed point sets $|S^H + T^H| = |S^H| + |T^H|$ and $|(S \times T)^H| = |S^H| \; |T^H|$. Hence $S \longmapsto |S^H|$ extends to a ring homomorphism

$$\varphi_H : A(G) \longrightarrow Z .$$

Conjugate subgroups give the same homomorphism so that we have one $\varphi_H$ for each $(H) \in C(G)$. We let

$$\varphi = (\varphi_H) : A(G) \longrightarrow \prod_{(H) \in C(G)} Z$$

be the product of the $\varphi_H$.

Proposition 1.2.2.

$\varphi$ is an injective ring homomorphism.

Proof.

By definition $\varphi$ is a ring homomorphism. Suppose $x \neq o$ is in the kernel of $\varphi$. We write x in terms of the basis $x = \sum a_H[G/H]$. We have a partial ordering on the $[G/H]$, namely $[G/H] \leq [G/K]$ if and only if H is subconjugate to K. Let $[G/H]$ be maximal among the basis elements with $a_H \neq o$. Then $G/K^H \neq \emptyset$ implies $[G/H] \leq [G/K]$. Hence $o = \varphi_H x =$ $= a_H\ [G/H^H] = a_H\ |NH/H| \neq 0$, a contradiction.

Since $\varphi$ is an injection of a subgroup of maximal rank the cokernel is a finite group. We want to compute its order. We consider the diagram of injective ring homomorphisms

$$\begin{array}{ccc} A(G) & \xrightarrow{\ \varphi\ } & \prod Z \\ \downarrow & & \downarrow \\ A(G) \otimes Q & \xrightarrow{\ \varphi_Q\ } & \prod Q \end{array}$$

where the lower $\varphi_Q$ is the rational extension of the upper $\varphi$.

Recall that $WH = NH/H$ acts freely on G/H as the group of G-automorphisms: The action is given by $WH \times G/H \longrightarrow G/H\ :\ (wH,gH) \longmapsto gw^{-1}H$. Hence it acts freely on any fixed point set $G/H^K$. In particular $|G/H^K|$ is divisible by $|WH|$. Therefore $\varphi_Q(\ [G/H] \otimes |WH|^{-1})$ is contained $\prod Z$.

Proposition 1.2.3.

The elements $\varphi_Q(\ [G/H] \otimes |WH|^{-1}) =: x_H$ form a Z-basis of $\prod Z$. The order of cokernel $\varphi$ is $\prod_{(H)\in C(G)} |WH|$.

Proof.

The first assertion implies the second one. We view elements in $\prod Z$ as row vectors. Then the $x_H$ form (suitably ordered) a triangular matrix with one's on the diagonal. Hence they must be a basis.

Remark 1.2.4.

The homomorphism $\varphi$ may be discovered from the ring structure of A(G) as follows. An element $x \in A(G)$ is a non-zerodivisor if and only if $\varphi x$ has no zero component. Therefore $A(G) \otimes Q$ is the total quotient ring of A(G) (i.e. all non-zero-divisors made invertible). If $x \in A(G) \otimes Q$ is integral over A(G) then the components of $\varphi_Q x$ are integral over Z hence integers. Conversely $\prod Z$ is integral over $\varphi A(G)$, e. g. because $\prod Z$ is generated by idempotent elements which are integral over any subring. Hence $\varphi$ may be identified with the inclusion of A(G) into the integral closure in its total quotient ring. (For the notion of integral ring extension see Lang [107], Chapter IX; Bourbaki [33] , Ch. 5.)

1.3. Congruences between fixed point numbers.

We have seen in 1.2. that $\varphi A(G)$ is a subgroup of maximal rank in $\prod Z$. How can we describe its image? If $G = Z/pZ$ is the cyclic group of prime order p then $|S| \equiv |S^G| \bmod p$ because the orbits of $S \setminus S^G$ have cardinality p. Hence this congruence gives a condition for elements to be in the image of $\varphi$ . The reader can easily check that this is the only condition, for $G = Z/pZ$. We generalize such congruences.

Let S be a finite G-set and let V(S) be the complex vector space spanned by the elements of S. The G-action on the basis S of V(S) induces a linear action on V(S). The resulting G-module V(S) is called the permutation representation associated to S. The character of V(S) is a function on G; it will be denoted with the same symbol. The orthogonality relations for characters say in particular that for any complex G-module V the number $|G|^{-1} \sum_{g \in G} V(g)$ is the dimension of $V^G$. Hence

$$(1.3.1) \qquad \sum_{g \in G} V(S)(g) \equiv 0 \mod |G| .$$

Now note that

$$V(S)(g) = \mathrm{Trace}(1_g : V(S) \longrightarrow V(S) : v \longmapsto gv) = |S^g|$$

(look at the matrix of $1_g$ with respect to the basis S). Therefore 1.3.1 can be rewritten

$$(1.3.2) \qquad \sum_{g \in G} \varphi_{\langle g \rangle}(x) \equiv 0 \mod |G|$$

for any $x \in A(G)$, where $\langle g \rangle$ denotes the cyclic group generated by g. If H is a cyclic subgroup of G the number of elements g with $\langle g \rangle$ conjugate to H is

$$|H^*| \, |G/NH|$$

where $H^*$ is the set of generators of H and $|G/NH|$ is the number of groups conjugate to H. Therefore (1.3.2) can be rewritten

$$(1.3.3) \qquad \sum_{(H) \text{ cyclic}} |H^*| \, |G/NH| \, \varphi_H(x) \equiv 0 \mod |G|$$

where now the summation is taken over conjugacy classes of cyclic subgroups of G.

We now apply the same argument to $V(S^H)$ considered as NH/H-module and obtain

$$\sum_{(K)} |NK/NH \cap NK|\,|K/H^{*}| \quad \varphi_K(x) \equiv 0 \mod |NH/H|$$

where we sum over NH-conjugacy classes K such that H is normal in K and K/H is cyclic. This may also be written in the form

(1.3.4) $$\sum_{(K)} n(H,K) \quad \varphi_K(x) \equiv 0 \mod |NH/H|$$

where the n(H,K) are certain integers with n(H,H) = 1 and the sum is over the G-conjugacy classes (K) such that H is normal in K and K/H is cyclic.

For the next Proposition we view elements of $\prod Z$ as functions $C(G) \longrightarrow Z$.

Proposition 1.3.5.

The congruences 1.3.4 are a complete set of congruences for image $\varphi$, i. e. $x \in \prod Z$ is contained in the image of $\varphi$ if and only if

$$\sum_{(K)} n(H,K) \; x(K) \equiv 0 \mod |NH/H|$$

for all $(H) \in C(G)$.

Proof.

We have already seen that the elements in the image of $\varphi$ satisfy these congruences. The congruences 1.3.6 are independent because they are given by a triangular matrix with one's on the diagonal. Hence they describe a subgroup A of index $\prod |NH/H|$ . By Proposition 1.2.3 therefore $A = \text{im}\ \varphi$ .

Remark 1.3.7.

A slightly different set of congruences is obtained if one considers $V(S^H)$ as $N_pH/H$-module where $N_pH/H$ is a Sylow p-group of NH/H. This yields a set of p-primary congruences which may be used instead of 1.3.4. These congruences are useful when localizations of A(G) are considered; e. g. for $A(G)_{(p)}$, the Burnside ring localized at p, only p-primary congruences are valid.

## 1.4. Idempotent elements.

Idempotent elements in $\prod Z$ are the functions with values 0 and 1. We use 1.3 to see when such functions come from A(G). We consider A(G) as subring of $\prod Z$ via $\varphi$ .

A subgroup H of G is called perfect if it is equal to its commutator subgroup. Each $H < G$ has a smallest normal subgroup $H_s$ such that $H/H_s$ is solvable. One has $(H_s)_s = H_s$. A subgroup H is perfect if and only if $H = H_s$. Let P(G) be the subset of C(G) represented by perfect subgroups.

Proposition 1.4.1.

*An idempotent $e \in \prod Z$ is contained in A(G) if and only if for all $(H) \in C(G)$ the equality $e(H) = e(H_s)$ holds.*

Proof.

Suppose $e \in A(G)$. Then e satisfies 1.3.6. Given $K < G$. Choose $K_s = K^n \triangleleft K^{n-1} \triangleleft \dots \triangleleft K^0 = K$ such that $K^i/K^{i+1}$ is cyclic of prime order p(i). Then by 1.3.6 applied to the group $K^{i+1}$ we have $e(K^i) \equiv e(K^{i+1})$ mod p(i). Since the values of e are 0 or 1 we must have $e(K^i) = e(K^{i+1})$ and therefore $e(K_s) = e(K)$. Conversely assume that $e(K_s) = e(K)$ for all K. Then we must have $e(H) = e(K)$ for all $H \triangleleft K$ with K/H cyclic so that e satisfies the congruences 1.3.6.

Corollary 1.4.2.

*The set of indecomposable idempotents of* A(G) *corresponds bijectively to* P(G). *In particular* G *is solvable if and only if* 0 *and* 1 *are the only idempotents in* A(G).

Remark 1.4.3.

Let $P \subset Z$ be a set of prime numbers. Let $A(G)_P$ be the localization of A(G) at P, i. e. the primes *not* in P are made invertible. Then one can show as in the proof of Proposition 1.4.1 that the idempotents of $A(G)_P$ are the functions e with $e(H) = e(H_P)$ where $H_P$ is the smallest normal subgroup of H such that $H/H_P$ is solvable of order involving only primes in P.

1.5. Units.

If A is a commutative ring we let $A^*$ be the multiplicative group of its units.

Let $e \in A$ be an idempotent. Then $1-2e = u$ is a unit. Conversely it can happen that for a unit u the element $(1-u)/2 = e$ is contained in A. Then e is an idempotent, because $(1-u)^2 = 2(1-u)$ for any unit u. In case of the Burnside ring $(1-u)/2$ is contained in $\Pi Z$ but not in general in A(G) as we shall see in a moment. But if G has odd order then coker $\varphi$ is odd and hence $1-u \in A(G)$ and $(1-u)/2 \in \Pi Z$ implies $(1-u)/2 \in A(G)$. Since a non-solvable group has non-trivial idempotents, by 1.4.2, we obtain

Proposition 1.5.1.

*If* G *is non-solvable then* $A(G)^* \neq \{\pm 1\}$ . *If* G *is solvable of odd order then* $A(G)^* = \{\pm 1\}$ .

Let H be a subgroup of index 2 in G. Then $H \lhd G$, $[G/H]^2 = 2\,[G/H]$

and therefore $u(H) := 1 - [G/H] \in A(G)^*$. Note that $(1-u(H))/2$ is not in $A(G)$. The subgroups of index 2 are precisely the kernels of non-trivial homomorphisms $G \longrightarrow Z/2Z$. Hence we obtain an injective map $j : Hom(G,Z/2Z) \longrightarrow A(G)^*$ given by $j(f) = 1-G/ker(f)$. The image of j is in general not a subgroup.

Problem 1.5.2.

Determine the structure of $A(G)^*$ in terms of the structure of G. (Of course one knows by the famous theorem of Feit - Thompson that groups of odd order are solvable. Therefore the 2-primary structure of G is relevant. In particular $A(G)^*$ for 2-groups would be interesting. (See also the next remark.)

Remark 1.5.3.

We shall prove later by geometric methods that for a real representation $V$ the function $(H) \longmapsto (-1)^{\dim V^H}$ is contained in A(G). This function is then a unit in A(G). It would be interesting to see units which are not of this form (2-groups?).

1.6. Prime ideals.

Since $\Pi Z$ is integral over A(G) by the "going-up theorem" of Cohen-Seidenberg (see Atiyah-Mac Donald [11] , p. 62) every prime ideal of A(G) comes from $\Pi Z$ hence has the form

$$q(H,p) := \{ x \in A(G) \mid \varphi_H(x) \equiv 0 \mod p \}$$

for a subgroup H of G and a prime ideal (p) of Z. The elementary proof of Dress [79] for this fact shall be given later (section 5) in the slightly more general context of compact Lie groups. The prime ideals $q(H,o)$ are minimal; the ideals $q(H,p)$, $p \neq 0$, are maximal with residue field $Z/pZ$. If $q(H,p) = q(K,q)$ then $p = q$ and

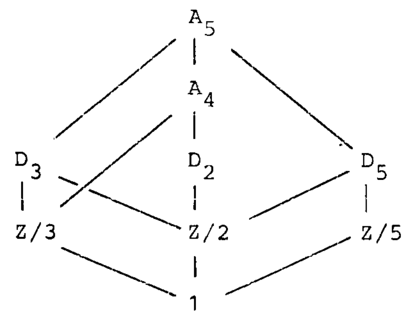
(i) $(H) = (K)$ if $p = 0$

(ii) $(H_p) = (K_p)$ if $p \neq 0$.

Here $H_p$ is the smallest normal subgroup of H such that $H/H_p$ is a p-group. If q is a prime ideal of A(G) then there exists a unique minimal (H) such that $[G/H] \notin q$. Moreover for this H one has $q = q(H,p)$ where p is the characteristic of the ring $A(G)/q$. Finally this (H) is the maximal (H) for which $q = q(H,p)$. All this is proved in Dress [79] and will later be proved for compact Lie groups.

## 1.7. An example: The alternating group $A_5$.

The diagram of conjugacy classes of subgroups of $A_5$ is




Here $D_n$ is the dihedral group of order 2n. The groups $A_5$, $A_4$, $D_5$, $D_3$ are their own normalizers while $N(Z/n) = D_n$ and $N(D_2) = A_4$. $A(A_5)$ is the set of functions $z : C(G) \longrightarrow Z$ satisfying

(i) $z(H)$ arbitrary for $H = A_5, A_4, D_5, D_3$.

(ii) $z(Z/n) \equiv z(D_n) \bmod 2$ for $n = 3,5$.

(iii) $z(D_2) \equiv z(A_4) \bmod 3$.

(iv) $z(1) + 20z(Z/3) + 15z(Z/2) + 24z(Z/5) \equiv 0 \bmod 60$.

The ring $A(A_5)$ contains the following units:

| 1 | Z/2 | Z/3 | Z/5 | $D_2$ | $D_3$ | $D_5$ | $A_4$ | $A_5$ |
|---|---|---|---|---|---|---|---|---|
| a | a | a | a | b | c | d | b | e |

Here a,b,c,d,e $\in \{1,-1\}$ and the second line gives the value of the function $u : C(G) \longrightarrow Z$ at the element indicated in the first line. The congruences (i) - (iv) show that there are no conditions for a unit u at $A_5$, $A_4$, $D_3$. From (iii) we obtain $u(D_2) = u(A_4)$. Considering (iv) mod 3, mod 4, and mod 5 we obtain

$$u(1) = u(Z/2) = u(Z/3) = u(Z/5).$$

The subgroups 1 and $A_5$ are perfect. Therefore $A(A_5)$ contains the idempotents 0,1,e,1-e where $\varphi_{A_5}(e) = 1$, $\varphi_H(e) = 0$ for $H \neq A_5$.

1.8. Comments.

The Burnside ring was introduced by Dress [79] where also the prime ideal spectrum was determined. The Burnside ring plays an important role in the axiomatic representation theory (Green [88] , Dress [80] ) in particular in the general theory of induction theorems (Dress [80] ). The Burnside ring, as a functor, is universal among the Mackey functors of Dress, see the cited references.

We shall demonstrate in these lectures the topological significance of the Burnside ring. At this point we only mention that a finite simplicial complex with simplicial G-action is a combinatorial object built from finite G-sets. So one expects some basic invariants of simplicial G-complexes to lie in the Burnside ring, e.g. the "Euler-Characteristic": the alternating sum $\sum (-1)^i S_i$ of the G-sets $S_i$ of i-simplices.

The Burnside ring codifies in a convenient frame-work some basic properties of the lattice of subgroups of a given group. Given G, the

G-transformation groups are governed by the internal relations of the Burnside ring. This influence of the Burnside ring is more transparent when we have shown that the ring is isomorphic to equivariant stable homotopy of sphere in dimension zero (Segal [145] ) so that in particular stable equivariant homotopy groups are modules over the Burnside ring.

The description of the Burnside ring using congruences among cardinalities of fixed point sets is based on an oral communication by Dress. These congruences are generalized in tom Dieck-Petrie [69] where also various geometrical applications are given.

1.9. Exercises.

1. Let G and H be finite groups whose orders are relatively prime. Show that

$$A(G \times H) \cong A(G) \otimes A(H)$$

2. For $i \not\equiv 0 \mod p$ let

$$M(i) = \{(a,b) \mid ai \equiv b \mod p\} \subset Z \times Z.$$

Show that M(i) is a projective module over A(Z/pZ). Classify projective modules over A(Z/pZ).

3. Show that G is perfect if and only if A(G) contains the idempotent e such that

$$\varphi_H e = 0 \quad \text{for } H \neq G, \quad \varphi_G e = 1.$$

4. Let G be a p-group of order $p^n$ (p a prime). Let $m \subset A(G)$ be the ideal

$$m = \{x \mid \varphi_{\{1\}} x \equiv 0 \mod p\}.$$

Show that $m^{n+1} \subset p\, A(G)$. (In particular: The p-adic and the m-adic topologies coincide.)

5. Let G be a 2-group and let $|A(G)^{*}| = 2^{n}$. Show that n is not greater than the number of conjugacy classes (H) such that $|NH/H| = 2$.