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The Morava K-theory and Brown-Peterson Cohomology of Spaces related to BP

By

Takuji KASHIWABARA and W. Stephen WILSON*

Abstract

We calculate the Morava K-theory of the spaces in the Omega spectra for $BP\langle q\rangle$. They fit into an exotic array of short and long exact sequences of Hopf algebras. We apply this to calculate the p-adically completed Brown-Peterson cohomology, as well as all of the intermediary cohomology theories, E, of these spaces. We give two descriptions of the answer, both of which turn out to be surprisingly nice. One part of our first description is just the image in the E cohomology of the corresponding space in the Omega spectrum for BP, which is as big as it could possibly be and which we show how to calculate. The other part is just the E cohomology of several copies of Eilenberg-MacLane spaces, something which is already known. Our second description is inductive and gives us a new way of looking at the Brown-Peterson cohomology of Eilenberg-Mac Lane spaces. The Brown-Comenetz dual of $BP\langle q\rangle$ shows up in our calculations and so we take up the study of this spectrum as well. It was already known that the Morava K-theory of the spaces in the Omega spectrum for the Brown-Comenetz dual of $BP\langle q\rangle$ made it look like a product of Eilenberg-Mac Lane spaces and we find, somewhat to our surprise, that the same is true for the BP cohomology. In order to state our answers we set up the foundations for the category of completed Hopf algebras.

1. Introduction

The purpose of this paper is to understand, (in particular, to calculate) various generalized cohomology theories of the spaces in the Omega spectra for $BP\langle q\rangle$, where $BP\langle q\rangle$ is the spectrum with coefficient ring

$$BP\langle q\rangle_* \simeq \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_q],$$

and the degree of v_i is $2(p^i - 1)$, see [Wil75] and [JW73]. Recall $BP^* \simeq \mathbb{Z}_{(p)}[v_1, v_2, \dots]$, and let $I_m = (p, v_1, \dots, v_{m-1})$. Most of our paper is spent working with Morava K-theory but our main application is easy to state and

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gives various cohomologies of the spaces $\underline{BP\langle q\rangle}_r$. We usually work with the p-adically completed version of BP cohomology, $BP^{\wedge *}_p(-)$, so that we can avoid the problems associated with phantom maps. We consider this theory, the theories $P(m)^*(-)$ with $P(m)^* \simeq BP^*/I_m$ [JW75], and the theories $E(m,n)^*(-)$, with $E(m,n)^* \simeq v_n^{-1}BP\langle n\rangle^*/I_m$ where $0 \leq m \leq n$, $0 < n$ [RWY98]. When $m = 0$ we always mean the p-adically complete version of the theory (unless explicitly stated otherwise) and we can think of $P(0)$ as $BP^{\wedge}_p$. Note that when $m = n$, $E(n,n) = K(n)$, the n-th Morava K-theory. We let E denote any of these theories. For all of the spaces we consider, we show that there is an E-cohomology Künneth isomorphism (using a completed tensor product). This is not normally the case. Since all of our spaces are also homotopy commutative H-spaces we have all of the maps associated with Hopf algebras (replacing tensor product with completed tensor product). The topology (on the cohomology groups) prevents this from being an abelian category but we can still talk about kernels, cokernels and short exact sequences of completed Hopf algebras.

We let $\{\underline{G}_*\}$ denote the spaces in the Omega spectrum for a spectrum G. Let $g(q) = 2(p^{q+1}-1)/(p-1) = 2(1+p+p^2+\cdots+p^q)$ throughout the paper where p is the prime associated with E and the spectrum $BP\langle q\rangle$. Let $I = (i_1, i_2, \dots, i_q)$ with $i_k \geq 0$ and let $d(I) = \sum 2i_k(p^k-1)$. We give two quite different descriptions of $E^*(\underline{BP\langle q\rangle}_*)$ and we need to define a special map for one of them. We have:

$$\underline{BP\langle q\rangle}_r \xrightarrow{\Delta_q} \prod_{0<i\leq q} \underline{BP\langle q\rangle}_{r-|v_i|} \xrightarrow{\Delta_{q-1}} \prod_{0<i_1<i_2\leq q} \underline{BP\langle q\rangle}_{r-|v_{i_1}|-|v_{i_2}|}.$$

The first map is just $\prod(-1)^{1+i}v_i$ and the second map just jiggles the signs a bit, as in a Koszul complex, so that $v_iv_j = v_jv_i$ makes the above composition trivial. Using the product and H-space structures on the target space we need only define the map on the individual terms. From $\underline{BP\langle q\rangle}_{r-|v_i|}$ to $\underline{BP\langle q\rangle}_{r-|v_{i_1}|-|v_{i_2}|}$ we use the trivial map unless i is i_1 or i_2. If $i = i_1$ we use $(-1)^{i_2}v_{i_2}$ and if $i = i_2$ we use $(-1)^{1+i_1}v_{i_1}$.

The main application of our work is the following. Recall that our kernel and cokernel are as completed Hopf algebras.

Theorem 1.1. *Let $r = g(q) + k$ and $k_I = k - d(I)$. Let E be any of $BP^{\wedge *}_p(-)$, $P(m)^*(-)$, or $E(m,n)^*(-)$ where $0 \leq m \leq n$ and $0 < n$. Then $E^*(\underline{BP\langle q\rangle}_r)$ is $P(m)^*$ flat for the category of $P(m)^*P(m)$ modules which are finitely presented over $P(m)^*$ and algebraically determined by its isomorphism to both:*

Description 1:

$$\widehat{\bigotimes_{d(I)<k}} E^*(K(\mathbb{Z}_{(p)}, q+2+k_I))$$

$$\widehat{\bigotimes} \operatorname{Ker} \left\{ E^*(\underline{BP}_r) \xrightarrow{\widehat{\otimes} v_i^*} \widehat{\bigotimes_{i>q}} E^*(\underline{BP}_{r+|v_i|}) \right\}$$

and, when $k > 0$, Description 2:

$$E^*(K(\mathbb{Z}_{(p)}, q+2+k)) \widehat{\bigotimes}$$
$$\operatorname{Coker} \left\{ \widehat{\bigotimes_{0<i\le q}} E^*(\underline{BP\langle q\rangle}_{r-|v_i|}) \xleftarrow{\Delta_{q-1}^*} \widehat{\bigotimes_{0<i_1<i_2\le q}} E^*(\underline{BP\langle q\rangle}_{r-|v_{i_1}|-|v_{i_2}|}) \right\}.$$

There are a number of observations to be made about this result. First, note that E can be $K(n)$. The Morava homology K-theory is just dual to this result: use regular tensor products instead of completed tensor products, change the direction of the arrows and switch Coker and Ker.

The flatness referred to in the theorem is *Landweber flatness.* Landweber analyzes this flatness in [Lan76] for $m = 0$ and Yosimura [Yos76] and Yagita [Yag76] do it for $m > 0$. They show it to be equivalent to v_i being monomorphic on $M/(v_m, \dots, v_{i-1})M$ for all $m \le i$ (with $v_0 = p$). Thus our E cohomology behaves quite nicely.

There is no condition on k in the first description. When $k \le 0$ the first term with the Eilenberg-Mac Lane spaces goes away and we are left with only the second term which we knew anyway because $\underline{BP\langle q\rangle}_r$ has no torsion and splits off of $\underline{BP}_r$ in this range. For arbitrary k, this second term is just the image $E^*(\underline{BP\langle q\rangle}_r) \to E^*(\underline{BP}_r)$ and we have

$$E^*(\underline{BP\langle q\rangle}_r) \longrightarrow E^*(\underline{BP}_r) \xrightarrow{\widehat{\otimes} v_i^*} \widehat{\bigotimes_{i>q}} E^*(\underline{BP}_{r+|v_i|}) \tag{1.2}$$

is "exact", as completed Hopf algebras, at the middle term. Because $\underline{BP}_r$ is a space with no torsion, its E cohomology is as nice as can be and our second part is contained in it. As for the actual evaluation of the kernel, in principle, the work of [RW77] gives all necessary information. The formulas in [BJW95] make that principle a reality. This second part of our first description might reasonably be expected, or at least hoped for, because it makes some sense. In fact, this shows that it is as big as possible because the composition

$$\underline{BP}_{r+|v_i|} \xrightarrow{v_i} \underline{BP}_r \longrightarrow \underline{BP\langle q\rangle}_r \tag{1.3}$$

is trivial for $i > q$.

The first part of our first description is, however, a surprise. It is nice because we know the E cohomology of these Eilenberg-Mac Lane spaces since they are completely described in [RWY98, Theorem 1.14], where, together with [Wil99a] and [Kasb], the ability to go from Morava K-theory to Brown-Peterson cohomology was developed. This first part of this answer is intimately tied up with the Brown-Comenetz dual of $BP\langle q\rangle$. Unstably it is a finite Postnikov system which maps to $\underline{BP\langle q\rangle}_r$ and carries all of the homotopy of the first part of our result. The Morava K-theory of such a space always splits up in this manner, [HRW98], but one doesn't get the same for the Brown-Peterson cohomology and the others in general. In this special case, it does split and although none of the homotopy from this space shows up in $\underline{BP\langle q\rangle}_r$, much of its BP cohomology does. The homotopy of the first part shows up in a second space which $\underline{BP\langle q\rangle}_r$ maps to and which realizes the second part of our first description. The point here is that there is some interesting topology underlying the first description.

The second description is an inductive description. For $k \leq 0$ we have known the answer (second part of first description) for decades. The cokernel part comes complete with a description of the maps v_i^* and v_i^* is trivial on the Eilenberg-Mac Lane part. Related, we have a description of the cohomology of Eilenberg-Mac Lane spaces.

Corollary 1.4. *For $k \geq 0$, and E as in Theorem* 1.1,
(i)

$$E^*(K(\mathbb{Z}_{(p)}, q+2+k)) \simeq E^*(\underline{BP\langle q\rangle}_{g(q)+k})/(v_1^*, v_2^*, \dots, v_q^*).$$

(ii)

$$E^*(K(\mathbb{Z}/(p^c), q+1+k)) \simeq E^*(\underline{BP\langle q\rangle}_{g(q)+k})/(p^{c*}, v_1^*, v_2^*, \dots, v_q^*).$$

This generalizes the $k=0$ version of this from [RWY98, Theorem 1.14] where the use of $v_1, v_2, \dots, v_{q-1}$ was found to be unnecessary. For $k > 0$ they become necessary.

Remark 1.5. If we are looking only at the theories $E = E(m,n)$, then, since $K(n)^*(K(\mathbb{Z}/(p^c), q+1+k))$ is finitely generated and free over $K(n)^*$, the proofs in [RWY98] give us that $E^*(K(\mathbb{Z}/(p^c), q+1+k))$ is also finitely generated and free over E^*. Although free algebraically, the topology on each summand can be quite different.

To simplify our notation a bit we remind the reader that $g(q) = 2(p^{q+1}-1)/(p-1) = 2(1+p+p^2+\cdots+p^q)$ and we let $g_\delta(q) = g(q)-(q+1) = \sum_{i=0}^q(|v_i|-1)$ and $g_v(q) = g(q)-2(q+1) = \sum_{i=0}^q 2(p^i-1) = \sum_{i=0}^q |v_i|$. These numbers are used throughout.

The spaces in the Omega spectrum for the Brown-Comenetz dual of $BP\langle q\rangle$, $IBP\langle q\rangle$, arise naturally in our study and so we turn our attention towards them now. The connection to our work is the existence of a stable cofibration:

$$\Sigma^{-g_\delta(q)} IBP\langle q\rangle \longrightarrow BP\langle q\rangle \longrightarrow L_q BP\langle q\rangle. \tag{1.6}$$

L_n is Bousfield localization, [Bou79], with respect to the theory $E(n)$ (our $E(0,n)$, from [JW73, Remark 5.13, p. 347]). This is explicit in Mahowald and Rezk's work, [MR]. In our preliminary Section 3, we review Ravenel's functors, N_n and L_n on the stable category, [Rav84], Mahowald and Rezk's work, Mahowald and Sadofsky's work, [MS95], and generalize this to fit our needs. In traditional notation,

$$\pi_* IBP\langle q\rangle \simeq \Sigma^{g_v(q)} BP\langle q\rangle_*/(p^\infty, v_1^\infty, v_2^\infty, \dots, v_q^\infty). \tag{1.7}$$

Rephrased, the homotopy group $\pi_{-j} IBP\langle q\rangle$ is a finite number of copies of $\mathbb{Q}/\mathbb{Z}_{(p)}$. The number of copies of $\mathbb{Q}/\mathbb{Z}_{(p)}$ is the same as the $\mathbb{Z}_{(p)}$ rank of $\pi_j BP\langle q\rangle$, which is zero, by "sparseness", unless $2(p-1)$ divides j. Strictly for degree reasons we get a split short exact sequence:

$$0 \longrightarrow \pi_* BP\langle q\rangle \longrightarrow \pi_* L_q BP\langle q\rangle \longrightarrow \pi_* \Sigma^{-g_\delta(q)+1} IBP\langle q\rangle \longrightarrow 0. \tag{1.8}$$

For any space X (all of our spaces are infinite loop spaces), we let $X^{(s)}$ denote the s-connected cover of X and let $X^{[s]}$ denote the corresponding space with the same homotopy groups as X up to degree s so that we have a fibration $X^{(s)} \to X \to X^{[s]}$. We also let $X^{(t,s]}$ be ${X^{(t)}}^{[s]} = {X^{[s]}}^{(t)}$. Observe that $\pi_k(X^{(t,s]}) = \pi_k(X)$ if $k \in (t, s]$ and 0 otherwise. Note that for $r > s$, $\underline{BP\langle q\rangle}_r$ is the same as $\underline{BP\langle q\rangle}_r^{(s)}$.

Because of this connection, equation (1.6), we have an interest in the Brown-Comenetz dual of $BP\langle q\rangle$, in particular, with the spaces $\underline{IBP\langle q\rangle}_r$ and $\underline{IBP\langle q\rangle}_r^{(s)}$. Note that since the top homotopy group of $\underline{IBP\langle q\rangle}_r$ is in degree r, it is a finite Postnikov system. As such, its Morava K-theory is the same as if it were a product of Eilenberg-Mac Lane spaces with the same homotopy, [HRW98], and these are understood by [RW80]. What was unanticipated is that the same is true for all of the other theories we use. The homology, $H_*(-; \mathbb{Z}_{(p)})$, of these spaces is not of finite type over $\mathbb{Z}_{(p)}$. Although that does not really present a serious problem for us it is perhaps easier to use a finite type approximation, i.e. the fibre of multiplication by p^c, which gives us a stable triangle:

$$I^{\mathbb{Z}/(p^c)} BP\langle q\rangle \longrightarrow IBP\langle q\rangle \xrightarrow{p^c} IBP\langle q\rangle. \tag{1.9}$$

This new spectrum is just the Brown-Comenetz dual of $BP\langle q\rangle$ modulo p^c. This gives us a short exact sequence on homotopy groups. Not only is $\underline{I^{\mathbb{Z}/(p^c)} BP\langle q\rangle}_r$ a finite Postnikov system but all of the homotopy groups are finite, i.e. we replace each $\mathbb{Q}/\mathbb{Z}_{(p)}$ by a $\mathbb{Z}/(p^c)$.

Much is known about our cohomology theories applied to finite Postnikov systems.

Theorem 1.10. *Let E be as in Theorem* 1.1 *and let X be a homotopy commutative H space which has a finite Postnikov system. Then*

$$E^* \longleftarrow E^*(X^{(s)}) \longleftarrow E^*(X) \longleftarrow E^*(X^{[s]}) \longleftarrow E^*$$

is a short exact sequence of completed Hopf algebras. In particular, the images of the $E^*(X^{[s]})$ filter $E^*(X)$ with quotients given by the short exact sequences of completed Hopf algebras:

$$E^* \longleftarrow E^*(K(\pi_s(X), s)) \longleftarrow E^*(X^{[s]}) \longleftarrow E^*(X^{[s-1]}) \longleftarrow E^*.$$

Since our category is not abelian we need to explain what we mean by short exact: the surjection is the cokernel and the injection is the kernel (see 6.7). For $E = K(n)$ this result is in [HRW98]. For our more general E the injections and surjections are shown in [RWY98] as well as the (algebraic) cokernel part. This follows from a general theorem which tells how Hopf algebra kernels in Morava homology K-theory give rise to cokernels in E cohomology, a theorem used over and over again in applications in [RWY98]. Results about the cokernel were expanded in [Wil99a] and [Kasb] but applications in the three papers never called for a general theorem about kernels. Here we show how Hopf algebra cokernels in Morava homology K-theory give rise to kernels in E cohomology. This is essential for the first description in Theorem 1.1 and, together with the completed Hopf algebra language developed here, allows us to state the above theorem. In [HRW98] it was further proved that the above short exact sequences all split for Morava K-theory, i.e. the middle term is the completed tensor product of the two end terms. It seems highly unlikely that this is true for the more general E although to be honest we do not have a counter example. Our intuition against such a splitting is also very much against the existence of Theorem 1.10 and so is suspect. We are, however, able to prove such a splitting in all the cases of interest to us in this paper.

Theorem 1.11. *For $r \geq 0$, $q \geq 0$, and $c > 0$, let E be as in Theorem 1.1, then the filtration of Theorem 1.10 for $\underline{I^{\mathbb{Z}/(p^c)}BP\langle q\rangle}_r$ (and also $\underline{I^{\mathbb{Z}/(p^c)}BP\langle q\rangle}_r^{(s)}$) splits, i.e.*

$$E^*(\underline{I^{\mathbb{Z}/(p^c)}BP\langle q\rangle}_r) \simeq \widehat{\bigotimes_{0\le i\le r}} E^*(K(\pi_i(\underline{I^{\mathbb{Z}/(p^c)}BP\langle q\rangle}_r), i))$$
$$\simeq \widehat{\bigotimes_{0\le d(I)\le r}} E^*(K(\mathbb{Z}/(p^c), r-d(I)))$$

as completed Hopf algebras. (For the s-connected case the first tensor product is over $s < i \leq r$ and the second over $0 \leq d(I) < r-s$.) The p-adic completion of E is not necessary for this result since all of the homotopy groups are finite.

Remark 1.12. This result can be expanded to $\underline{IBP\langle q\rangle}_r$ and $\underline{IBP}_r$. First note that the last two spaces are the same when $2(p^{q+1}-1) > r$. Now we observe that our space is the direct limit of the spaces $\underline{I^{\mathbb{Z}/(p^c)}BP\langle q\rangle}_r$. Since these spaces are torsion spaces the Brown-Peterson cohomology is the same as the p-adically complete Brown-Peterson cohomology and so their inverse limit is the same. Likewise, their $\lim^1$s must be the same but this is zero for the p-adically completed version so it is zero for the non-completed version as well.

$$BP^*(\underline{IBP\langle q\rangle}_r) \simeq BP^{\hat{}*}_p(\underline{IBP\langle q\rangle}_r) \simeq \lim{}^0 BP^*(\underline{I^{\mathbb{Z}/(p^c)}BP\langle q\rangle}_r)$$

and a similar splitting follows. There are no odd degree elements.

This theorem helps in Description 1 of Theorem 1.1 but it is really a side interest for us in this paper. In fact, it is quite easy to prove from [RWY98] and to make it easy on readers only interested in the rather appealing Brown-Comenetz dual we have separated the proof of this (and the proof of the next result) out in its own Section 7 and made it reasonably self contained.

Related to the last theorem, and perhaps also of more general interest, is a splitting for another class of spectra.

Using the Baas-Sullivan theory of manifolds with singularities, [Baa73] (and now [EKMM96]), we can construct spectra which have a finite number of homotopy groups. Let $I = (i_0, i_1, \ldots, i_q)$ or $(i_1, \ldots, i_q)$ with $i_k > 0$ for all $k \leq q$. The Baas-Sullivan theory gives us BP module spectra, $BP\langle q\rangle_I$, with homotopy

$$BP\langle q\rangle_{I*} \simeq BP\langle q\rangle_*/(p^{i_0}, v_1^{i_1}, \ldots, v_q^{i_q})$$

(with no p^{i_0} if i_0 is not defined).

Theorem 1.13. *Let E be as in Theorem 1.1. Then the filtration of Theorem 1.10 splits for the spaces in the Omega spectrum for $BP\langle q\rangle_I$. The p-adic completion is not necessary if i_0 is defined.*

The BP cohomology is trivial stably because it is for Eilenberg-Mac Lane spaces.

We say an algebraic object is *completely algebraically determined* if we can give a purely algebraic construction of it. For example, in [RW77] the Hopf ring $E_*\underline{BP}_*$ was constructed algebraically for all complex oriented homology theories $E_*(-)$. This includes the E of present interest. We note that when r is even $E_*\underline{BP}_r$ is a polynomial algebra on even degree generators and concentrated in even degrees. When r is odd, $E_*\underline{BP}_r$ is an exterior algebra on odd degree generators. Furthermore, because $\underline{BP\langle q\rangle}_r$ splits off of $\underline{BP}_r$ when $r \leq g(q)$ ([Wil75], [BJW95] and [BWa]), $E_*\underline{BP\langle q\rangle}_r$ is completely algebraically determined in this case. All one needs to do is set the $[v_i] = [0]$ for $i > q$ in $E_*\underline{BP}_*$. The proper way to say this now is to use the Goerss-Hunton-Turner generalized tensor product, [Goe99] and [HT98], to write

$$E_*[BP\langle q\rangle^*]\overline{\otimes}_{E_*[BP^*]}E_*\underline{BP}_r \simeq E_*\underline{BP\langle q\rangle}_r$$

when $r \leq g(q)$. By duality, since these are all free, we get an algebraic determination of $E^*(\underline{BP\langle q\rangle}_*)$. The Morava K-theory of Eilenberg-Mac Lane spaces is completely algebraically determined in [RW80, Corollaries 11.3 and 12.2] and then again in [RWY98, Proposition 1.16] by (1.26). The paper, [HRW98, Theorem 2.1], gives a complete algebraic determination of the Morava K-theory of all of the finite Postnikov systems which are homotopy commutative H-spaces. The E cohomology of Eilenberg-Mac Lane spaces was algebraically determined in [RWY98, Theorem 1.14]. Our terminology is a bit of a misnomer when we are working with cohomology groups because they come with a topology on them. When we say "algebraically determined" in this paper, we mean we have all of the structure, including the topology. In [RWY98], cohomologies were algebraically determined, but the topology was not proven to be determined. Our work in this paper is a significant improvement and allows us to upgrade the concept of "algebraically determined" to include the topology. This is done mainly by our introduction of completed Hopf algebras and our theorems about them. This work should have been made more prominent in this paper. However, the depths of the problems created by the topology, and our solutions to them, came late in the game. Consequently this work is buried in Section 6.

The results of [RWY98], [Wil99a], [Kasb] and this paper have made calculating our $E^*(-)$ cohomology groups (including $BP^*(-)$) possible for lots of examples as corollaries of calculating the Morava K-theory (homology) of related spaces. Morava K-theory is relatively easy to work with because it has a Künneth isomorphism and because the category of Hopf algebras we work in is abelian. Most of our work in this paper is done with Morava K-theory (homology) and our other results are applications of these calculations.

Whenever possible in this paper we will use $K_*(-)$ to denote the Morava K-theory, $K(n)_*(-)$, in order to suppress the n from our notation, i.e. $K_* = K(n)_* \simeq \mathbb{F}_p[v_n, v_n^{-1}]$. Much of the notation used in this paper is quite unpleasant. As we inherit some of it from the literature we are not entirely to blame.

Our main computation is really to show the following result for Morava K-theory. Note that the first space is trivial for $r \leq g(q)$.

Theorem 1.14. *For $q+2 < r$ the fibration*

$$\underline{IBP\langle q\rangle}^{(q+1)}_{r-g_\delta(q)} \longrightarrow \underline{BP\langle q\rangle}_r \longrightarrow \underline{L_q BP\langle q\rangle}^{(q+2)}_r$$

gives rise to a split short exact sequence of Hopf algebras in Morava K-theory

$$K_* \longrightarrow K_*\underline{IBP\langle q\rangle}^{(q+1)}_{r-g_\delta(q)} \longrightarrow K_*\underline{BP\langle q\rangle}_r \longrightarrow K_*\underline{L_q BP\langle q\rangle}^{(q+2)}_r \longrightarrow K_*.$$

The first term is always even degree and because it is a finite Postnikov system splits up further. The last term is completely algebraically determined by the Goerss-Hunton-Turner generalized tensor product:

$$\begin{aligned} K_*\underline{L_q BP\langle q\rangle}^{(q+2)}_r &\simeq image\ \{K_*\underline{BP}_r \to K_*\underline{BP\langle q\rangle}_r\} \\ &\simeq K_*[BP\langle q\rangle^*]\overline{\otimes}_{K_*[BP^*]} K_*\underline{BP}_r \end{aligned}$$

and so is in even degrees if r is even. This splitting gives a completely algebraic determination for $K_*\underline{BP\langle q\rangle}_r$. When $n \leq q+1$ the first term is trivial and we get the nice

$$K(n)_*[BP\langle q\rangle^*]\overline{\otimes}_{K(n)_*[BP^*]} K(n)_*\underline{BP}_r \simeq K(n)_*\underline{BP\langle q\rangle}_r.$$

The proof of the above is a bit involved and requires a number of variations on the Koszul complex. First, we start with the usual resolution of $\mathbb{Z}_{(p)}$ over

$$BP\langle q\rangle_* \simeq \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_q].$$

We define

$$\mathcal{KZ}_j^{BP\langle q\rangle_*} \simeq \bigoplus_{0<i_1<i_2<\ldots<i_j\le q} \Sigma^{\sum_s 2(p^{i_s}-1)} BP\langle q\rangle_*.$$

We then define the maps

$$\mathcal{KZ}_j^{BP\langle q\rangle_*} \longrightarrow \mathcal{KZ}_{j-1}^{BP\langle q\rangle_*}$$

summand by summand using $(-1)^{1+t}v_{i_t}$ to map from the summand $\Sigma^{\sum_s 2(p^{i_s}-1)} BP\langle q\rangle_*$ in $\mathcal{KZ}_j^{BP\langle q\rangle_*}$ to the similar summand in $\mathcal{KZ}_{j-1}^{BP\langle q\rangle_*}$ with no i_t. This realizes the graded version of the Koszul complex. Its homology is a $\mathbb{Z}_{(p)}$ concentrated in degree zero of the zeroth homology. $\mathcal{KZ}_*^{BP\langle q\rangle_*}$ is a finite resolution of $\mathbb{Z}_{(p)}$ by $BP\langle q\rangle_*$ with all maps split over $\mathbb{Z}_{(p)}$. This construction can now be mimicked with the spectrum $BP\langle q\rangle$ to obtain $\mathcal{KZ}_*^{BP\langle q\rangle}$. The homotopy groups of $\mathcal{KZ}_*^{BP\langle q\rangle}$ give the corresponding algebraic resolution, $\mathcal{KZ}_*^{BP\langle q\rangle_*}$. If we take the Omega spectrum for all of the spectra in $\mathcal{KZ}_*^{BP\langle q\rangle}$ we get unstable versions $\mathcal{KZ}_*^{\underline{BP\langle q\rangle}_*}$. The minus signs in the maps must be interpreted as the H-space inverse. The indexing which we will use frequently is $\mathcal{KZ}_j^{\underline{BP\langle q\rangle}_{2(q+1)+k}}$ which is just

$$\prod_{0<i_1<i_2<\ldots<i_j\le q} \underline{BP\langle q\rangle}_{\sum_s 2(p^{i_s}-1)+2(q+1)+k}.$$

Note that when we are working with the final space in this complex we have

$$\mathcal{KZ}_q^{\underline{BP\langle q\rangle}_{2(q+1)+k}} \simeq \underline{BP\langle q\rangle}_{g(q)+k} \tag{1.15}$$

which is convenient for us.

Since we are working with BP module spectra we can make similar definitions for Koszul complexes $\mathcal{KZ}_*^{L_qBP\langle q\rangle}$ and $\mathcal{KZ}_*^{IBP\langle q\rangle}$. We can also make this unstable with $\mathcal{KZ}^{\underline{L_qBP\langle q\rangle}_*}$, $\mathcal{KZ}^{\underline{L_qBP\langle q\rangle}^{(s)}_*}$, $\mathcal{KZ}^{\underline{IBP\langle q\rangle}_*}$, and $\mathcal{KZ}^{\underline{IBP\langle q\rangle}^{(s)}_*}$, etc. We can then apply $K_*(-)$ to all of these complexes, noting that -1 becomes the Hopf algebra conjugation, and we have $\mathcal{KZ}^{K_*\underline{L_qBP\langle q\rangle}_*}$, etc. In this case and in all others, when we have a sequence of K_*-Hopf algebras, all of the undefined ones are assumed to be the trivial Hopf algebra, K_*. In addition, we have Koszul complexes for $\mathbb{Z}/(p^c)$, $\mathcal{K}^{\mathbb{Z}/(p^c)}\mathcal{Z}_*^{(-)}$, for all of the above. We just index over $0\le i_0<i_1<\ldots<i_j\le q$ and let the map v_0 be p^c. Note that the length of the complex is now $q+1$ and we give up our splittings.

Because $v_iv_j=v_jv_i$, the composition of any two maps,

$$\mathcal{KZ}_{j+1}^{(-)} \longrightarrow \mathcal{KZ}_j^{(-)} \longrightarrow \mathcal{KZ}_{j-1}^{(-)}$$

is always trivial which gives sense to the term "complex". Since the category of Hopf algebras we work in is abelian, [Bou96b], [Bou96a], [HRW98] and [SW98], we can talk about exactness and homology of complexes of Morava K-theory Hopf algebras.

Recall the stable cofibration

$$\Sigma^{2(p^n-1)}BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \to BP\langle n-1\rangle \tag{1.16}$$

gives rise to the (BP module) boundary map

$$BP\langle n-1\rangle \xrightarrow{\delta} \Sigma^{2p^n-1}BP\langle n\rangle.$$

Iterating these we have

$$\begin{aligned}\delta\ :\ K(\mathbb{Q}/\mathbb{Z}_{(p)}, 2(0)+q+1) \to K(\mathbb{Z}_{(p)}, 2(1)+q) \simeq \Sigma^{2(1)+q}BP\langle 0\rangle \to \\ \Sigma^{2(1+p)+q-1}BP\langle 1\rangle \to \Sigma^{2(1+p+p^2)+q-2}BP\langle 2\rangle \to \cdots \to \Sigma^{g(q)}BP\langle q\rangle.\end{aligned} \tag{1.17}$$

We abuse notation a bit and allow δ to be any iterated boundary map. Which it is will be uniquely determined by the source and target.

We will call $\mathcal{EKZ}_*^{BP\langle q\rangle}$ the extended Koszul complex when we tack on a $q+1$ term to $\mathcal{KZ}_*^{BP\langle q\rangle}$, i.e.

$$\begin{aligned}\mathcal{EKZ}_{q+1}^{BP\langle q\rangle} = K(\mathbb{Z}_{(p)}, -q) \xrightarrow{\delta} \\ \Sigma^{g(q)-2(q+1)}BP\langle q\rangle = \mathcal{KZ}_q^{BP\langle q\rangle} = \mathcal{EKZ}_q^{BP\langle q\rangle}.\end{aligned} \tag{1.18}$$

Because δ is a BP module map and the v_i act trivially on Eilenberg-Mac Lane spaces, this is still a complex. (Note we are referring here to v_i acting on the spaces, not on BP cohomology.) We get corresponding unstable versions, $\mathcal{EKZ}_*^{\underline{BP\langle q\rangle}_*}$ and $\mathcal{EKZ}_*^{K_*\underline{BP\langle q\rangle}_*}$. We have a similar extended Koszul complex, $\mathcal{EK}^{\mathbb{Z}/(p^c)}\mathcal{Z}_*^{BP\langle q\rangle}$, with corresponding unstable versions. Here,

$$\mathcal{EK}^{\mathbb{Z}/(p^c)}\mathcal{Z}_{q+2}^{BP\langle q\rangle} = K(\mathbb{Z}/(p^c), -(q+1)).$$

We also have $\mathcal{EKZ}_*^{IBP\langle q\rangle}$ with $\mathcal{EKZ}_{q+1}^{IBP\langle q\rangle} = K(\mathbb{Q}/\mathbb{Z}_{(p)}, g_v(q))$ and of course there is a $\mathbb{Z}/(p^c)$ version as well. All of these definitions are leading up to our next theorem, the proof of which is thoroughly linked to the proof of our short exact sequence, Theorem 1.14, which is just the $\mathcal{EKZ}_q$ part of the following with $r = g_v(q)+t$.

Theorem 1.19. *For $t > 2(q+1)$ there is a short exact sequence of long exact sequences with all maps split as algebra maps*:

$$\begin{aligned}K_* \longrightarrow \mathcal{EKZ}_*^{K_*\underline{IBP\langle q\rangle}^{(q+1)}_{t-g_\delta(q)}} \longrightarrow \mathcal{EKZ}_*^{K_*\underline{BP\langle q\rangle}_t} \\ \longrightarrow \mathcal{KZ}_*^{K_*\underline{L_qBP\langle q\rangle}^{(q+2)}_t} \longrightarrow K_*\end{aligned}$$

For the term on the right, this Koszul complex inductively determines the last term $K_*\underline{L_qBP\langle q\rangle}^{(q+2)}_{t+g_v(q)}$ which is also determined by the generalized tensor product in Theorem 1.14.

There is yet another way to view our results. In particular this gives another perspective on the once mysterious way the Morava K-theory of Eilenberg-Mac Lane spaces shows up in our answers.

We have used the Goerss-Hunton-Turner generalized tensor product to describe our answer. Hunton and Turner go further with this and define the derived functors which they call CTor where CTor_0 is just this new tensor product. In their paper, [HT98], they do precisely the algebra we need. This is developed even further in [Kasa]. Our CTor is still a Hopf algebra in our category and so it splits into an exterior algebra part and an even degree part, [HRW98]. A more detailed statement will follow in the final section, but for now we will just observe:

Theorem 1.20. *Let E_j be the exterior algebra part of* CTor_j*, then*

$$K_*\underline{BP\langle q\rangle}_r \simeq \mathrm{CTor}_*^{K_*[BP^*]}(K_*[BP\langle q\rangle^*], K_*\underline{BP}_{r+*})/(E_j,\, j>0).$$

We can calculate this by taking a Koszul resolution of $BP\langle q\rangle_*$ by BP_*, going to the 'ring-rings' $K_*[BP^*]$ and $K_*[BP\langle q\rangle^*]$, taking the generalized tensor product and computing the homology. Thus it is easy to see that

$$\mathrm{CTor}_*^{K_*[BP^*]}(K_*[BP\langle q\rangle^*], K_*\underline{BP}_*) \simeq H_*(\mathcal{K}^{BP\langle q\rangle_*}\mathcal{Z}_*^{K_*\underline{BP}_*}). \tag{1.21}$$

The notation here accurately suggests the use of a Koszul type resolution of $BP\langle q\rangle_*$ by BP_* and similar versions on spectra and spaces. We show it is equivalent to calculating the homology of $\mathcal{K}^{BP\langle q\rangle_*}\mathcal{Z}_*^{K_*\underline{BP\langle i\rangle}_*}$ for i big. For $q=0$ it is enough to do the homology of $\mathcal{K}\mathcal{Z}_*^{K_*BP\langle i\rangle_*}$, something we have already studied. We use this and the obvious exact sequences to compute the general case.

Such a result is not completely unexpected. Letting $F(j) = \mathcal{K}^{BP\langle q\rangle}\mathcal{Z}_j^{BP}$ and letting $G(0) = BP\langle q\rangle$ we can define $G(j+1)$ inductively (stably) where $G(j+1) \to F(j) \to G(j)$ is a stable triangle. If, and this is a big if since it doesn't happen, we were lucky enough to get a long exact sequence in Morava K-theory on the spaces in the Omega spectra

$$\cdots \to K_*\underline{G(j+1)}_i \to K_*\underline{F(j)}_i \to K_*\underline{G(j)}_i \to K_*\underline{G(j+1)}_{i+1} \to \cdots \tag{1.22}$$

then we would have a spectral squence:

$$\mathrm{CTor}_*^{K_*[BP^*]}(K_*[BP\langle q\rangle^*], K_*\underline{BP}_{r+*}) \Rightarrow K_*\underline{BP\langle q\rangle}_r. \tag{1.23}$$

Our theory says that if we ignore the exterior part of CTor_j, $j>0$, we get the correct answer.

The space $\underline{L_qBP\langle q\rangle}^{(q+1)}_r$, $r>q+1$, plays a crucial role in our study because we can show the map $\underline{L_qBP\langle q\rangle}_r \longrightarrow \underline{L_qBP\langle q\rangle}^{(q+1)}_r$ gives a surjection in ALL

Morava K-theories. The space $\underline{L_q BP\langle q\rangle}_r^{(q+1)}$ thus arises for the age old reason "because it works". After this paper was submitted, some communication with Pete Bousfield helped give us a partial, but not complete, step towards understanding where this space comes from.

Theorem 1.24 (with A. K. Bousfield). *When $r > q$, we have a homotopy equivalence*:

$$L_q \underline{BP\langle q\rangle}_r \simeq \underline{L_q BP\langle q\rangle}_r^{(q)}.$$

Pete Bousfield made a very general conjecture which included the above theorem. He had every step of the proof for this special case except for showing the map is an isomorphism on the q-th Morava K-theory. (The map, in general, is neither surjective nor injective for higher Morava K-theories.) This isomorphism is easy to see from our work here. We had hoped he could put this result in his paper, [Bou], so we could just quote it, but he claimed it didn't fit. This is not satisfactory for us because it still differs by one homotopy group from the space important to us. However, it is a lot closer than we were before.

A word about motivation is probably appropriate here. Several times while working on other projects, the question of the Morava K-theory of spaces in the Omega spectrum for $BP\langle q\rangle$ has come up as possibly useful. Since the answers were not known the questions have generally gone away. However, they did serve to get us somewhat interested. With the discovery that many spaces have computable Landweber flat Brown-Peterson cohomology, [RWY98], [Kas98] and [Kasb], the question naturally arose for the spaces $\underline{BP\langle q\rangle}_r$. In particular there are the spectra and maps:

(1.25)

$$S^0 \to T(1) \to \cdots \to T(q) \to \cdots BP \cdots \to BP\langle q\rangle \to \cdots \to BP\langle 1\rangle \to K(\mathbb{Z}_{(p)}).$$

The $T(q)$ are Ravenel's ring spectra, from [Rav84], [Rav85] and [Rav86], which were so important in the proof of his conjectures (from [Rav84]) in [DHS88]. The BP cohomology of the spaces in the Omega spectrum for BP has been known explicitly since [RW77]. In [RWY98] the Eilenberg-Mac Lane spaces (on the right) were done. Using [RWY98] and [Kas98] did all of the even spaces for S^0 and $T(q)$. His work in [Kas98], together with either [Wil99a] or [Kasb] gets all of the odd spaces as well. This paper completes our knowledge of the BP cohomology of the spaces in the above sequence.

Two examples motivated us further. The first was the simple case of the fibration (see [RWY98, Section 2.6])

$$F \longrightarrow BSU^{(2m-1)} \longrightarrow BSU.$$

Here F is a finite Postnikov system. The Morava K-theory of this fibration gives rise to a short exact sequence of Hopf algebras. As m gets bigger, the Morava K-theory continues to see all of the space BSU and also all of the missing homotopy groups! When we use $K(1)$ it doesn't see F at all. We found this intriguing and pursued its generalization. Turning this example into

$\underline{BP\langle 1\rangle}_k$ and replacing F with the fiber of the rationalization of F, this example is our $q = 1$ case. This example was almost more confusing than helpful since it was not at all clear how to generalize it. It is only after the fact that we see that the $E(1)$ localization of bu is the correct object to have here instead of BSU on the right, which works in this case but which does not generalize.

The Koszul complexes came about from the second example. In [RWY98, Proposition 1.16], the exact sequence

(1.26)

$$K_* \longrightarrow K_*K(\mathbb{Z}_{(p)}, q+2) \longrightarrow K_*\underline{BP\langle q\rangle}_{g(q)} \xrightarrow{v_{q*}} K_*\underline{BP\langle q\rangle}_{g(q)-2(p^q-1)} \longrightarrow \cdots$$

was proven as well as the more likely exact sequence:

(1.27)

$$K_* \longrightarrow K_*K(\mathbb{Z}_{(p)}, q+2) \longrightarrow K_*\underline{BP\langle q\rangle}_{g(q)} \xrightarrow{\otimes v_{i*}} \bigotimes_{0<i\leq q} K_*\underline{BP\langle q\rangle}_{g(q)-2(p^i-1)} \longrightarrow \cdots$$

The lack of the necessity of the v_i for $i < q$ mystified us and is now explained by the Koszul complexes; as we deloop, they become necessary to keep the sequence exact, Corollary 2.8. It is these exact sequences which give us the BP cohomology of Eilenberg-Mac Lane spaces in [RWY98, Theorem 1.14]. All of the $BP\langle q\rangle$ spaces in the above case have no torsion and so it is the "easy" case. The last sequence still holds when delooped and so still works for BP cohomology, Corollary 1.4. The sequence (1.27) is the beginning point for all of our calculations here and is used repeatedly in this paper.

Finally, we had proven many of the Morava K-theory exact sequences in this paper purely algebraically where the only spaces involved were $\underline{BP\langle q\rangle}_*$. Because of the appearance of the Morava K-theory of Eilenberg-Mac Lane spaces we had a strong feeling that there had to be topology underlying it and that the topology would be interesting. Furthermore, we needed the topology to go from Morava K-theory results to our results about Brown-Peterson cohomology. We came up with our own version of topology but with the help of Mark Mahowald, Douglas Ravenel, Charles Rezk and Hal Sadofsky, were able to get the more interesting L_n localizations and Brown-Comenetz duals involved. We wish to thank them for their help in this matter. In addition, the second author wishes to thank the Centre de Recerca Matemàtica in Barcelona, Spain and the Department of Mathematics at Kyoto University in Kyoto, Japan, for their ideal work environments during the writing of this paper.

The work of Richard Kramer, [Kra90] and [BKW99], and the short exact squence

$$K(n)_* \to K(n)_*K(\mathbb{Z}/(p), r-2p^q+1) \to \\ K(n)_*\underline{k(q)}_r \to K(n)_*\underline{k(q)}_{r-2(p^q-1)} \to K(n)_*$$

when $r \geq 2p^q + q$ suggests that similar things happen with the spectra $P(n, q)$ of [RWY98], [BWa] and [BWb]. We found ourselves easily discouraged at the thought of any more of this.

At one time potential bases were written down for much of this work. However, they are very difficult to prove. The present “coordinate free” version is much nicer anyway.

Our next section is a more detailed statement of results. Section 3 does the preliminary work we need for our proofs. Section 4 proves our core results from which almost everything else follows. We finish up many of the rest of the proofs in Section 5 except for the lifting of our results to BP. Those results are proven in Section 6 (where we also set up completed Hopf algebras) except for our isolation of the results on the E cohomology of the Brown-Comenetz dual and the Baas-Sullivan spectra in Section 7 which are put here for ease of access to readers only interested in them. Section 8 discusses our results on CTor and our final Section 9 proves Theorem 1.24.

2. Detailed statement of results

For $q < s < r-1$ we consider diagram (2.1) in which all rows and columns are fibration sequences.

(2.1)
$$\begin{array}{ccccc}
 & & & & \underline{IBP\langle q\rangle}^{(q,s]}_{r-g_\delta(q)} \\
 & & & & \downarrow \\
\underline{IBP\langle q\rangle}^{(s)}_{r-g_\delta(q)} & \longrightarrow & \underline{BP\langle q\rangle}_r & \longrightarrow & \underline{L_qBP\langle q\rangle}^{(s+1)}_r \\
\downarrow & & \simeq\downarrow & & \downarrow \\
\underline{IBP\langle q\rangle}^{(q)}_{r-g_\delta(q)} & \longrightarrow & \underline{BP\langle q\rangle}_r & \longrightarrow & \underline{L_qBP\langle q\rangle}^{(q+1)}_r \\
\downarrow & & \downarrow & & \downarrow \\
\underline{IBP\langle q\rangle}^{(q,s]}_{r-g_\delta(q)} & \longrightarrow & * & \longrightarrow & \underline{IBP\langle q\rangle}^{(q+1,s+1]}_{r+1-g_\delta(q)}
\end{array}$$

Theorem 2.2. *If we apply Morava K-theory, $K_*(-)$, to diagram* (2.1) *for $q < s < r-1$, then we get a series of short exact sequences of Hopf algebras.*

(i) *The following short exact sequence is split as Hopf algebras. Each space is a finite Postnikov system and the Morava K-theory is naturally isomorphic to that of a product of Eilenberg-Mac Lane spaces with the same homotopy and thus is completely algebraically determined and concentrated entirely* *in even degrees.*

$$K_* \longrightarrow K_* \underline{IBP\langle q\rangle}^{(s)}_{r-g_\delta(q)} \longrightarrow K_* \underline{IBP\langle q\rangle}^{(q)}_{r-g_\delta(q)} \longrightarrow K_* \underline{IBP\langle q\rangle}^{(q,s]}_{r-g_\delta(q)} \longrightarrow K_*$$

(ii) *The following short exact sequence is split as Hopf algebras.*

$$K_* \longrightarrow K_* \underline{IBP\langle q\rangle}^{(s)}_{r-g_\delta(q)} \longrightarrow K_* \underline{BP\langle q\rangle}_r \longrightarrow K_* \underline{L_q BP\langle q\rangle}^{(s+1)}_r \longrightarrow K_*.$$

When $s = q+1$ the last term is completely algebraically determined by the Goerss-Hunton-Turner generalized tensor product:

$$\begin{aligned} K_* \underline{L_q BP\langle q\rangle}^{(q+2)}_r &\simeq image\ \{K_* \underline{BP}_r \to K_* \underline{BP\langle q\rangle}_r\} \\ &\simeq K_*[BP\langle q\rangle^*] \overline{\otimes}_{K_*[BP^*]} K_* \underline{BP}_r \end{aligned}$$

and so is in even degrees if r is even. This splitting gives a completely algebraic determination for $K_* \underline{BP\langle q\rangle}_r$. When $n \le q+1$ (and $s = q+1$) the first term is trivial and we get the nice

$$K(n)_*[BP\langle q\rangle^*] \overline{\otimes}_{K(n)_*[BP^*]} K(n)_* \underline{BP}_r \simeq K(n)_* \underline{BP\langle q\rangle}_r.$$

(iii) *The following short exact sequence is split as algebras.*

$$K_* \longrightarrow K_* \underline{IBP\langle q\rangle}^{(q)}_{r-g_\delta(q)} \longrightarrow K_* \underline{BP\langle q\rangle}_r \longrightarrow K_* \underline{L_q BP\langle q\rangle}^{(q+1)}_r \longrightarrow K_*.$$

When r is even, the last term is a polynomial algebra on even degree generators and so $K_* \underline{BP\langle q\rangle}_r$ is also concentrated in even degrees. When r is odd, the last term is an exterior algebra on odd degree generators. This last term is completely algebraically determined.

(iv) *There is a short exact sequence of Hopf algebras which is split as algebras.*

$$K_* \to K_* \underline{IBP\langle q\rangle}^{(q,s]}_{r-g_\delta(q)} \to K_* \underline{L_q BP\langle q\rangle}^{(s+1)}_r \to K_* \underline{L_q BP\langle q\rangle}^{(q+1)}_r \to K_*.$$

When $s = q+1$ this is:

$$\begin{aligned} K_* \longrightarrow K_* K(\pi_{q+1} \underline{IBP\langle q\rangle}_{r-g_\delta(q)}, q+1) \longrightarrow \\ K_*[BP\langle q\rangle_*] \overline{\otimes}_{K_*[BP_*]} K_* \underline{BP}_r \longrightarrow K_* \underline{L_q BP\langle q\rangle}^{(q+1)}_r \to K_*. \end{aligned}$$

Part (ii) of this for $s = q+1$ is Theorem 1.14 in the Introduction.

We have another exact sequence which does not come from the diagram. We recall that there are maps $BP\langle q\rangle \to BP\langle q-1\rangle$ and $X \to L_q X$ and that $L_{q-1} L_q X \simeq L_{q-1} X$.

Theorem 2.3. *For $r > q+1 > 1$ there is a four term exact sequence of Hopf algebras,*

$$\begin{aligned} K_* \longrightarrow A(q,r) \longrightarrow K_* \underline{L_q BP\langle q\rangle}^{(q+1)}_{r+2(p^q-1)} \xrightarrow{v_{q*}} K_* \underline{L_q BP\langle q\rangle}^{(q+1)}_r \longrightarrow \\ K_* \underline{L_{q-1} BP\langle q-1\rangle}^{(q+1)}_r \simeq K_*[BP\langle q-1\rangle^*] \overline{\otimes}_{K_*[BP^*]} K_* \underline{BP}_r. \longrightarrow K_*, \end{aligned}$$

where $A(q,r)$ is trivial, thus giving us a short exact sequence, except when

$$r + 2(p^q - 1) = g(q) - 1 + 2(p-1)t, \quad t \geq 0.$$

In this case $A(q,r)$ is an exterior algebra with $\mathrm{Tor}^{A(q,r)}(K_*, K_*)$ an associated graded object for

$$K_*K\left(\pi_{q+1}\underline{IBP\langle q-1\rangle}_{r-g_\delta(q-1)}, q+1\right).$$

In particular, note that when r is even, v_{q_*} injects.

Theorem 1.19 is just one of many Koszul complex type theorems we have. It is the $s = q+1$ case of part (iii) below.

Theorem 2.4. (i) *There is a short exact sequence of long exact sequences of Hopf algebras with all maps split as Hopf algebra maps*:

$$K_* \to \mathcal{EKZ}_*^{K_*\underline{IBP\langle q\rangle}^{(s)}_*} \to \mathcal{EKZ}_*^{K_*\underline{IBP\langle q\rangle}_*} \to \mathcal{EKZ}_*^{K_*\underline{IBP\langle q\rangle}^{[s]}_*} \to K_*.$$

(ii) *There is a short exact sequence of long exact sequences of Hopf algebras*:

$$K_* \longrightarrow \mathcal{EK}^{\mathbb{Z}/(p^c)}\mathcal{Z}_*^{K_*\underline{IBP\langle q\rangle}^{(s)}_*} \longrightarrow \mathcal{EK}^{\mathbb{Z}/(p^c)}\mathcal{Z}_*^{K_*\underline{IBP\langle q\rangle}_*} \longrightarrow \mathcal{EK}^{\mathbb{Z}/(p^c)}\mathcal{Z}_*^{K_*\underline{IBP\langle q\rangle}^{[s]}_*} \longrightarrow K_*.$$

(iii) *For $2(q+1)+k = t$, $k \geq 0$, and $q \leq s < t-(q+1)$ there is a short exact sequence of long exact sequences with all maps split as algebra maps*:

$$K_* \longrightarrow \mathcal{EKZ}_*^{K_*\underline{IBP\langle q\rangle}^{(s)}_{t-g_\delta(q)}} \longrightarrow \mathcal{EKZ}_*^{K_*\underline{BP\langle q\rangle}_t} \longrightarrow \mathcal{KZ}_*^{K_*\underline{L_qBP\langle q\rangle}^{(s+1)}_t} \longrightarrow K_*.$$

The right side long exact sequence gives an inductive algebraic determination of the final term, i.e. $K_*\underline{L_qBP\langle q\rangle}^{(s)}_{g(q)+k}$.

(iv) *For $2(q+1)+k = t$, $k \geq 0$, and $q \leq s < t-(q+1)$ there is a short exact sequence of long exact sequences*:

$$K_* \longrightarrow \mathcal{EK}^{\mathbb{Z}/(p^c)}\mathcal{Z}_*^{K_*\underline{IBP\langle q\rangle_{t-g_\delta(q)}}^{(s)}_*} \longrightarrow \mathcal{EK}^{\mathbb{Z}/(p^c)}\mathcal{Z}_*^{K_*\underline{BP\langle q\rangle}_t} \longrightarrow \mathcal{K}^{\mathbb{Z}/(p^c)}\mathcal{Z}_*^{K_*\underline{L_qBP\langle q\rangle}^{(s+1)}_t} \longrightarrow K_*.$$

(v) *For $2(q+1)+k = t$, $k \geq 0$, and $q < s < t-(q+1)$ there is a short exact sequence of long exact sequences which is split as algebras*:

$$K_* \longrightarrow \mathcal{KZ}_*^{K_*\underline{IBP\langle q\rangle}^{(q,s]}_{t-g_\delta(q)}} \longrightarrow \mathcal{KZ}_*^{K_*\underline{L_qBP\langle q\rangle}^{(s+1)}_t} \longrightarrow \mathcal{KZ}_*^{K_*\underline{L_qBP\langle q\rangle}^{(q+1)}_t} \longrightarrow K_*.$$

The restrictions on s in the theorem are just to avoid degenerate cases. This, in (iii), calculates the homology of the complex $\mathcal{EKZ}_*^{K_*\underline{BP\langle q\rangle}_t}$ when $t \geq 2(q+1)$. We return to this in Corollary 8.10 where we compute the homology for $t < 2(q+1)$ as well. The homology of the other Koszul complexes follows from this.

For $r \leq g(q)$, the spaces $\underline{BP\langle q\rangle}_r$ split off of the BP spaces and have no torsion and so their Morava K-theory is described by [RW77]. We have given one algebraic determination of $K_*\underline{BP\langle q\rangle}_r$ for $r > g(q)$ above in Theorem 2.2 (ii) (the $s = q+1$ version). The Koszul complexes give us yet another way to do this (the $s = q+k$ version of Theorem 2.2 (ii)). For the extended complexes the only thing missed is the extension itself, which happens to split off in this range anyway. This splitting is part of the following corollary. Let $k > 0$ and recall the identifications and iterated boundary map

$$(2.5)\quad K_*\underline{BP\langle q\rangle}_{g(q)+k} = \mathcal{KZ}_q^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}} \xrightarrow{\delta} K_*\underline{BP\langle q+k\rangle}_{g(q+k)} = \mathcal{KZ}_{q+k}^{K_*\underline{BP\langle q+k\rangle}_{2(q+k+1)}}.$$

An algebraic determination of $K_*\underline{BP\langle q\rangle}_{g(q)+k}$ follows from the corollary which splits off the extension.

Corollary 2.6. *For $k > 0$ and $r = g(q)+k$, we have the Hopf algebra decomposition*

$$K_*K(\mathbb{Q}/\mathbb{Z}_{(p)}, q+1+k) \bigotimes K_*\underline{L_q BP\langle q\rangle}_r^{(q+k+1)} \simeq K_*\underline{BP\langle q\rangle}_r$$

coming from the exact sequence

$$K_* \longrightarrow \mathcal{KZ}_q^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}} \longrightarrow \mathcal{KZ}_{q+k}^{K_*\underline{BP\langle q+k\rangle}_{2(q+k+1)}} \bigotimes \mathcal{KZ}_{q-1}^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}} \longrightarrow \mathcal{KZ}_{q+k-1}^{K_*\underline{BP\langle q+k\rangle}_{2(q+k+1)}} \bigotimes \mathcal{KZ}_{q-2}^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}} \longrightarrow \cdots$$

which inductively algebraically determines $K_*\underline{BP\langle q\rangle}_{g(q)+k}$ as well as the maps v_{i*}.

Remark 2.7. This follows immediately from Theorem 2.4 (iii). However, we need a little more than this for future reference. We need that the composition of the two geometric maps in the corollary are trivial. The second one is easy since it is just the Koszul complex. In the first one, the maps can be taken in any order since they commute. We need to show that $v_i \circ \delta \simeq \delta \circ v_i$ is trivial, but the delta from the lower space on the right side of this equation must be trivial because it factors through the range of the splitting Theorem 3.29.

The final part of the extended Koszul complex (the middle term of Theorem 2.4 (iii)) gives another algebraic determination of the Morava K-theory of Eilenberg-Mac Lane spaces.

Corollary 2.8. *For $k \geq 0$, $v_0 = p^c$, $c \leq \infty$ the exact sequence of Hopf algebras,*

$$K_* \longrightarrow K_*K(\mathbb{Z}/(p^c), q+1+k) \xrightarrow{\delta_*} K_*\underline{BP\langle q\rangle}_{g(q)+k} \xrightarrow{\otimes v_{i*}} \bigotimes K_*\underline{BP\langle q\rangle}_{g(q)+k-|v_i|} \longrightarrow \cdots,$$

algebraically determines $K_*K(\mathbb{Z}/(p^c), q+1+k)$.

The $k = 0$ case of this was done in [RWY98] and we rely on it extensively in this paper.

The fact that the extension splits off $K_*\underline{BP\langle q\rangle}_r$ when $r > g(q)$ can be generalized significantly. The algebraic splitting of Theorem 2.2 (ii) comes from geometric maps. Let $v^I = v_1^{i_1} v_2^{i_2} \ldots v_q^{i_q}$ with $d(I) = |v^I| = \sum 2(p^j - 1)i_j$. Let $r = g(q) + k$, $k > 0$. The top homotopy group of $\underline{IBP\langle q\rangle}^{(s)}_{r-g_\delta(q)}$, $s > q$, is in degree $r + (q+1) - g(q) = (q+1) + k$ and the bottom is in degree $s+1$. The $\mathbb{Q}/\mathbb{Z}_{(p)}$ summands in homotopy can be indexed over $1/v^I$, $d(I) \leq k - (s-q)$ with the $1/v^I$ summand in degree $(q+1) + k - d(I) = (q+1) + k_I$ where $k_I > 0$. By this definition of k_I we have $k - k_I = d(I)$ and we see that $k_I > 0$ because $s > q$ and so the lowest homotopy group is always in degree greater than s, i.e. definitely greater than $q+1$. We can map all of our spaces with a v^I followed by an iterated δ to get:

$$\underline{IBP\langle q\rangle}^{(s)}_{r-g_\delta(q)} \longrightarrow \underline{BP\langle q\rangle}_{r=g(q)+k} \xrightarrow{v^I} \underline{BP\langle q\rangle}_{g(q)+k_I} \xrightarrow{\delta} \underline{BP\langle q+k_I\rangle}_{g(q+k_I)} \simeq \mathcal{K}\underline{\mathcal{Z}}^{BP\langle q+k_I\rangle}_{q+k_I}{}_{2(q+k_I+1)}.$$

Note that with this last space we are in the range of torsion free spaces and are at the first exact extended Koszul complex. The image of the Morava K-theory of our left hand space is just the image of the Morava K-theory of our chosen $\mathbb{Q}/\mathbb{Z}_{(p)}$ summand which is precisely the kernel of the Morava K-theory of the last space mapped to the $(q + k_I - 1)$-th term of its Koszul complex. Thus we get:

Theorem 2.9. *Let $r = g(q) + k$, $k > 0$, $k_I = k - d(I)$ and $q < s < (q+1) + k$. We have the Hopf algebra decomposition*

$$K_*\underline{BP\langle q\rangle}_r \simeq K_*\underline{L_q BP\langle q\rangle}^{(s+1)}_r \bigotimes_{d(I)\leq k-(s-q)} K_*K(\mathbb{Q}/\mathbb{Z}_{(p)}, (q+1)+k_I)$$

coming from the maps

$$\begin{aligned}&\underline{BP\langle q\rangle}_{r} \longrightarrow \\ &\qquad \underline{L_q BP\langle q\rangle}_{r}^{(s+1)} \prod_{d(I)\leq k-(s-q)} \mathcal{K}\mathcal{Z}_{q+k_I}^{\underline{BP\langle q+k_I\rangle}_{2(q+k_I+1)}} \\ &\qquad\qquad \longrightarrow * \prod_{d(I)\leq k-(s-q)} \mathcal{K}\mathcal{Z}_{q+k_I-1}^{\underline{BP\langle q+k_I\rangle}_{2(q+k_I+1)}}\end{aligned}$$

which gives the exact sequence

$$\begin{aligned}&K_* \longrightarrow K_*\underline{BP\langle q\rangle}_{r} \longrightarrow \\ &\qquad K_*\underline{L_q BP\langle q\rangle}_{r}^{(s+1)} \bigotimes_{d(I)\leq k-(s-q)} \mathcal{K}\mathcal{Z}_{q+k_I}^{K_*\underline{BP\langle q+k_I\rangle}_{2(q+k_I+1)}} \\ &\qquad\qquad \longrightarrow K_*(pt) \bigotimes_{d(I)\leq k-(s-q)} \mathcal{K}\mathcal{Z}_{q+K_I-1}^{K_*\underline{BP\langle q+k_I\rangle}_{2(q+k_I+1)}} \longrightarrow \cdots .\end{aligned}$$

This is a more geometric version of our Theorem 2.2 (ii) and gives us the Hopf algebra splitting there. The interesting cases are the two extremes. On the one extreme, for $s = q + 1$ this is the version of Theorem 2.2 (ii) which uses the Goerss-Hunton-Turner tensor product and which is particularly useful for the proof of Theorem 1.1, Description 1. The other extreme, with $s = k + q$ (i.e. Corollary 2.6), will give us Description 2.

When we started this project an obvious approach was to look at the spectral sequences which come from the fibration sequence

(2.10)
$$\begin{aligned}&\cdots \longrightarrow \underline{BP\langle q-1\rangle}_{r-1} \longrightarrow \underline{BP\langle q\rangle}_{r+2(p^q-1)} \\ &\longrightarrow \underline{BP\langle q\rangle}_{r} \longrightarrow \underline{BP\langle q-1\rangle}_{r} \longrightarrow \cdots .\end{aligned}$$

This approach turned out to not be productive. However, it is a sequence which should be understood now that we know more. The Morava K-theory does not give an exact sequence but we can measure how far it deviates from that by taking its homology. In fact, it turns out to be exact mostly and its homology is moderately well understood.

Theorem 2.11. *Taking the homology of the complex we obtain by taking the Morava K-theory of the sequence of fibrations* (2.10) *we find that it is an exact sequence everywhere but at*

$$K_*\underline{BP\langle q-1\rangle}_{r-1} \longrightarrow K_*\underline{BP\langle q\rangle}_{r+2(p^q-1)} \longrightarrow K_*\underline{BP\langle q\rangle}_{r}.$$

The homology at this point is trivial except when $r + 2(p^q - 1) = g(q) - 1 + 2(p-1)t$, $t \geq 0$. In this case the homology is just the $A(q,r)$ of Theorem 2.3.

Having come this far it is reasonable to ask what the Morava K-theory of the spaces in the Omega spectrum for $L_qBP\langle q\rangle$ are. This is easy for us to answer for the even spaces but not so easy, and not pursued, for the odd spaces.

Theorem 2.12. *There is a short exact sequence of Hopf algebras*:

$$K_* \longrightarrow K_*\underline{L_qBP\langle q\rangle}^{(q+1)}_{2r} \longrightarrow K_*\underline{L_qBP\langle q\rangle}_{2r} \longrightarrow K_*\underline{IBP\langle q\rangle}^{[q+1]}_{2r+1-g_\delta(q)} \longrightarrow K_*.$$

3. Preliminaries

Remark 3.1 (Morava homology Hopf algebras)**.** We will denote by $K_*(-)$ the Morava K-theory, $K(n)_*(-)$. It has a Künneth isomorphism which makes it particularly amenable to calculations. We use Hopf algebras over K_*, see [Bou96b], [Bou96a], [HRW98] and [SW98]. The Hopf algebras we use form an abelian category so a short exact sequence of complexes gives rise to a long exact sequence in homology. In particular, if two of them are long exact, i.e. have trivial homology, then so does the third. When we just have a map of two short exact sequences this degenerates into the snake lemma giving a six term exact sequence relating the kernels, $H_1(-)$, with the cokernels, $H_0(-)$.

Remark 3.2 (Finite Postnikov systems)**.** All of our spaces are infinite loop spaces and all of our maps are infinite loop maps. The spaces $\underline{IBP\langle q\rangle}_*$, as well as their $X^{(s)}$ and $X^{[s]}$ versions, are all finite Postinikov systems, i.e. they have only a finite number of non-trivial homotopy groups. The work in [HRW98] tells us that the Morava K-theory is the same as if it was a product of Eilenberg-Mac Lane spaces with the same homotopy. This splitting is natural and so when such spaces give a short (or long) exact sequence on homotopy we get a short exact (or long) exact sequence of Hopf algebras. If the maps on homotopy are split then the Hopf algebra maps are too.

Remark 3.3 (The Goerss-Hunton-Turner tensor product (GHT))**.** We will use the Goerss-Hunton-Turner generalized tensor product, [Goe99] and [HT98]

$$K_*[BP\langle q\rangle^*]\overline{\otimes}_{K_*[BP^*]}K_*\underline{BP}_*$$

and abuse the language slightly by writing

$$K_*[BP\langle q\rangle^*]\overline{\otimes}_{K_*[BP^*]}K_*\underline{BP}_r$$

when we mean the r-th part of it. In this case it amounts to setting the element $[v_s] = [0_{-2(p^s-1)}]$, $s > q$, where this last is the Hopf algebra unit in the same space as $[v_s]$.

We make frequent use of the bar spectral sequence. It will be helpful to write down some of the things we use over and over again.

Theorem 3.4 (Folk)**.** *Given a fibration of infinite loop spaces and maps,*

$$F \xrightarrow{i} E \longrightarrow B,$$

(i) *then there is a spectral squence of Hopf algebras*

$$\mathrm{Tor}^{K_*F}_{*,*}(K_*E, K_*) \Rightarrow K_*B$$

(ii) *with E^2 term isomorphic to*

$$\mathrm{Tor}_{*,*}^{\mathrm{Ker}\, i_*}(K_*, K_*) \otimes \mathrm{Coker}\, i_*$$

with $\mathrm{Coker}\, i_* \simeq \mathrm{Tor}_{0,*}^{K_*F}(K_*E, K_*) \simeq K_*E \otimes_{K_*F} K_*$.

(iii) *If i_* is injective we get a short exact sequence of Hopf algebras*:

$$K_* \longrightarrow K_*F \xrightarrow{i_*} K_*E \longrightarrow K_*B \longrightarrow K_*.$$

(iv) *$K_*K(\mathbb{Q}/\mathbb{Z}_{(p)}, i)$ is even degree and the spectral sequence*

$$\mathrm{Tor}^{K_*K(\mathbb{Q}/\mathbb{Z}_{(p)},i)}(K_*, K_*) \Rightarrow K_*K(\mathbb{Q}/\mathbb{Z}_{(p)}, i+1)$$

for the fibration

$$K(\mathbb{Q}/\mathbb{Z}_{(p)}, i) \longrightarrow * \longrightarrow K(\mathbb{Q}/\mathbb{Z}_{(p)}, i+1)$$

is all in even degrees and collapses.

(v) Tor *of an exterior algebra is a divided power algebra and* Tor *of a polynomial algebra is an exterior algebra.*

Proof. See [HRW98, pp. 144–5] and [RW80, pp. 704–705] for a discussion of this spectral sequence. Part (iii) follows from the previous part. Part (iv) follows from [RW80, Theorem 12.3]. Part (v) is standard. □

Remark 3.5 (The Brown-Comenetz dual)**.** In [Rav84, Section 5], Ravenel inductively constructs functors, N_n and M_n on the stable category with the stable cofibration:

$$N_nX \longrightarrow M_nX \longrightarrow N_{n+1}X \tag{3.6}$$

using $N_0X = X$ and $L_nN_nX = M_nX$ where L_n is Bousfield's localization with respect to the theory $E(n)$, [Bou79]. In Section 5 he defines C_n as the fibre of $X \to L_nX$ and in Theorem 5.10 he shows that $N_nX = \sum^n C_{n-1}X$ giving us the stable cofibration

$$\Sigma^{-n-1}N_{n+1}X \longrightarrow X \longrightarrow L_nX \tag{3.7}$$

for all X. He then goes on, Theorem 6.1, to calculate the homotopy groups of all of these functors when $X = BP$. We are interested in the same results for $X = BP\langle q\rangle$ and it appears that the same proof works in this case when $n \leq q$. The proof is a bit scanty and seriously nested. However, improvements in technology since that paper have made this much easier. Special thanks to Doug Ravenel and Hal Sadofsky for helping us work through this and to Mark Hovey who would have done it if they hadn't. As part of the calculation of the homotopy groups, Ravenel shows that N_nBP and M_nBP are BP module spectra, something we need for our case as well. This is now easy due to the smash product theorem, [Rav92, Theorem 7.5.6], which says $L_nX \simeq X \wedge L_nS^0$. If X is a BP module spectrum, then all one has to do is take the fibration

$$C_nS^0 \longrightarrow S^0 \longrightarrow L_nS^0 \tag{3.8}$$

for the sphere and smash it with $BP \wedge X \to X$. This gives the BP module structures we need.

Specializing to $BP\langle q\rangle$, we want our homotopy groups to be

$$\pi_*N_nBP\langle q\rangle \simeq N^nBP\langle q\rangle_* \qquad \pi_*M_nBP\langle q\rangle \simeq M^nBP\langle q\rangle_* \tag{3.9}$$

where we start with $N^0BP\langle q\rangle_* = BP\langle q\rangle_*$ and define $M^nBP\langle q\rangle_* = v_n^{-1}N^nBP\langle q\rangle_*$ and $N^{n+1}BP\langle q\rangle_*$ inductively using the short exact sequence:

$$0 \to N^nBP\langle q\rangle_* \to M^nBP\langle q\rangle_* \to N^{n+1}BP\langle q\rangle_* \to 0 \tag{3.10}$$

for $n \le q$. To do this it is enough to show:

Lemma 3.11. *If X is a BP module spectrum with π_*X all I_n torsion, then $\pi_*L_nX \simeq v_n^{-1}\pi_*X$.*

Proof. Theorem 1 of [Rav87] states that if BP_*X is all I_n torsion, then BP_*L_nX is just $v_n^{-1}BP_*X$. The fact that π_*X is I_n torsion implies BP_*X is also I_n torsion. Since we have maps $Y \to BP \wedge Y \to Y$ for $Y = X$ and L_nX this result follows from the BP homology result. □

We are assured that the identification of $N_{q+1}BP\langle q\rangle$ with the Brown-Comenetz dual of $BP\langle q\rangle$ can be done directly from here but since we have a concrete reference in the literature we will use it. In [MR, Corollary 9.3] Mahowald and Rezk construct a stable cofibration (using the p-adic completion of $BP\langle q\rangle$):

$$\Sigma^{-g_\delta(q)}IBP\langle q\rangle \longrightarrow BP\langle q\rangle \longrightarrow L_qBP\langle q\rangle \tag{3.12}$$

where $IBP\langle q\rangle$ is the Brown-Comenetz dual and the identification

$$\Sigma^{-g_v(q)}N_{q+1}BP\langle q\rangle \simeq IBP\langle q\rangle \tag{3.13}$$

follows.

When we look at the spaces in the Omega spectra for the spectra above we run into some problems, namely, the spaces for $IBP\langle q\rangle$ and $L_qBP\langle q\rangle$ are not of finite type. This presents no problems as long as we are working with Morava K-theory, but when we try to lift our results to the other theories such as BP, then we rely on the work of [RWY98] which requires finite type.

We present two ways of working around this problem. The first, and easiest came to us last because we did not know about the Brown-Comenetz dual identification. However, using it, we can take the dual of the sequence

$$BP\langle q\rangle \longrightarrow M_0BP\langle q\rangle\ (\,= p^{-1}BP\langle q\rangle)\ \longrightarrow N_1BP\langle q\rangle \tag{3.14}$$

to get another stable cofibration

$$IM_0BP\langle q\rangle \longrightarrow IBP\langle q\rangle \longrightarrow \Sigma IN_1BP\langle q\rangle. \tag{3.15}$$

The map

$$IM_0BP\langle q\rangle \longrightarrow IBP\langle q\rangle \longrightarrow \Sigma^{g_\delta(q)}BP\langle q\rangle \tag{3.16}$$

is trivial and so the map $IBP\langle q\rangle \longrightarrow \Sigma^{g_\delta(q)}BP\langle q\rangle$ factors through $\Sigma IN_1BP\langle q\rangle$. We can now define $L'_qBP\langle q\rangle$ and we have maps

$$\begin{array}{ccccc} \Sigma^{-g_\delta(q)}IBP\langle q\rangle & \longrightarrow & BP\langle q\rangle & \longrightarrow & L_qBP\langle q\rangle \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^{-g_\delta(q)+1}IN_1BP\langle q\rangle & \longrightarrow & BP\langle q\rangle & \longrightarrow & L'_qBP\langle q\rangle. \end{array} \tag{3.17}$$

The spaces in our new Omega spectra now have lots of p-adics in them but we can handle that since their homology is of finite type over the p-adics. Unstably, the above maps induce isomorphisms on the Morava K-theories.

We want to explore an alternative approach. This is what we had done before we learned about the Brown-Comenetz dual identification. It leads to a collection of finite Postnikov systems each with BP cohomology which splits as if it were just a product of Eilenberg-Mac Lane spaces. Furthermore, it allows us to identify the iterated boundary map we use with the map of the top homotopy group in (3.12). We doubt if this is original but we don't know of any reference to anyone else doing it this way. There is something similar in [MS95].

We use the Baas-Sullivan theory of manifolds with singularities, [Baa73], to construct spectra. Let $I_n = (i_0, i_1, \ldots, i_{n-1})$ with $i_k > 0$ for all k. The Baas-Sullivan theory gives us BP module spectra and maps of BP module spectra

$$\Sigma^{|v_n^{i_n}|}BP\langle q\rangle_{I_n} \xrightarrow{v_n^{i_n}} BP\langle q\rangle_{I_n} \longrightarrow BP\langle q\rangle_{I_{n+1}} \tag{3.18}$$

for $n < q$ and where $BP\langle q\rangle_{I_n *} \simeq BP\langle q\rangle_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$. Ravenel's spectrum $N_{n+1}BP\langle q\rangle$ is just the direct limit of this taken over the various I_{n+1} but we must prove that. We do this by first constructing maps of stable BP module cofibrations.

$$\begin{array}{ccccc} \Sigma^{|v_n^{i_n}|}BP\langle q\rangle_{I_n} & \xrightarrow{v_n^{i_n}} & BP\langle q\rangle_{I_n} & \longrightarrow & BP\langle q\rangle_{I_{n+1}} \\ \downarrow j_1 & & \downarrow j_2 & & \downarrow j_3 \\ N_nBP\langle q\rangle & \xrightarrow{f} & M_nBP\langle q\rangle & \longrightarrow & N_{n+1}BP\langle q\rangle. \end{array} \tag{3.19}$$

We understand that some large desuspension has to be applied to the top row to make sense of this. We can, by induction, define our maps. Since $M_nBP\langle q\rangle$ is known to be localization with respect to v_n we can define $j_2 = v_n^{-i_n} \circ f \circ j_1$. Since we know all of the homotopy groups the rest follows and we have an injection of homotopy from the top row to the bottom row. Taking an appropriate limit over the I_{q+1} we see

$$\lim_{I_{q+1}} BP\langle q\rangle_{I_{q+1}} \simeq N_{q+1}BP\langle q\rangle. \tag{3.20}$$

Iterated boundary maps give the analog of

$$\Sigma^{-(q+1)}N_{q+1}BP\langle q\rangle \longrightarrow BP\langle q\rangle \tag{3.21}$$

To see that this is the iterated boundary map defined in the introduction, just let $i_1 = i_2 = \ldots = 1$. We can take the cofibre of our map for use in all of our theorems.

However, we need something of finite type or we wouldn't bother with this at all. We modify our construction by using $I'_n = (i_1, i_2, \ldots, i_{n-1})$, $i_k > 0$ for all k. We construct $BP\langle q\rangle_{I'_n}$ just as above observing that we have a stable cofibration

$$BP\langle q\rangle_{I'_n} \xrightarrow{p^{i_0}} BP\langle q\rangle_{I'_n} \longrightarrow BP\langle q\rangle_{I_n} \tag{3.22}$$

which gives us the boundary map

$$\Sigma^{-1}BP\langle q\rangle_{I_n} \longrightarrow BP\langle q\rangle_{I'_n}. \tag{3.23}$$

This, in turn, has a boundary map to some suspension of $BP\langle q\rangle$. Take the limit of this for $n = q+1$ and we get

$$\Sigma^{-1}N_{q+1}BP\langle q\rangle \longrightarrow N'_{q+1}BP\langle q\rangle(\Sigma^{-g_v(q)}I'BP\langle q\rangle) \tag{3.24}$$

whose cofibre is rational. Consequently, we have a Morava K-theory isomorphism between the spaces in the Omega spectrum for these two. The stable cofibre of

$$\Sigma^{-q}N'_{q+1}BP\langle q\rangle \longrightarrow BP\langle q\rangle \tag{3.25}$$

is our replacement, $L'_qBP\langle q\rangle$, for $L_qBP\langle q\rangle$. In the Omega spectrum we have to replace s connectivity with $s+1$ connectivity of our new spaces. We get Morava K-theory isomorphisms of short exact sequences:

$$\begin{array}{ccccc}
\underline{IBP\langle q\rangle}^{(s)}_{r-g_\delta(q)} & \longrightarrow & \underline{BP\langle q\rangle}_r & \longrightarrow & \underline{L_qBP\langle q\rangle}^{(s+1)}_r \\
\downarrow & & \downarrow \simeq & & \downarrow \\
\underline{I'BP\langle q\rangle}^{(s+1)}_{r-g_\delta(q)+1} & \longrightarrow & \underline{BP\langle q\rangle}_r & \longrightarrow & \underline{L'_qBP\langle q\rangle}^{(s+2)}_r.
\end{array} \tag{3.26}$$

We can replace spaces accordingly with new spaces with finite type.

Remark 3.27 (Another useful fibration). We find it convenient, but not necessary, to have

$$\Sigma^{|v_q|} L_q BP\langle q\rangle \xrightarrow{v_q} L_q BP\langle q\rangle \longrightarrow L_{q-1}BP\langle q-1\rangle \tag{3.28}$$

be a fibration and it is. Ravenel points out that it is enough to show that the map $L_q BP\langle q-1\rangle \longrightarrow L_{q-1}BP\langle q-1\rangle$ is an equivalence and has kindly given us the proof of this fact. Since both are BP module spectra it is enough to show that the fibre, $M_q BP\langle q-1\rangle$ is BP acyclic. Since

$$BP \wedge M_q BP\langle q-1\rangle \simeq BP\langle q-1\rangle_\wedge M_q BP$$

which is Bousfield equivalent to $BP\langle q-1\rangle \wedge K(q)$ which is contractible because $K(q)_* BP\langle q-1\rangle$ is trivial.

We rely heavily on the theorem which splits the spaces in the BP Omega spectrum:

Theorem 3.29 ([Wil75], see also [BJW95] and [BWb]).
(i) *For $r \le g(q)$, $\underline{BP\langle q\rangle}_r$ splits off of $\underline{BP}_r$.*
(ii) *For $r \le g(q-1) = g(q) - |v_q| - 2$,*

$$\underline{BP\langle q\rangle}_r \simeq \underline{BP\langle q-1\rangle}_r \times \underline{BP\langle q\rangle}_{r+|v_q|}$$

Remark 3.30. We need to point out that although the Morava K-theory at $p = 2$ is not a communtative ring theory these results all still hold. For details, see [JW85, appendix] and [Wil84, pp. 1030–31]. When we go to lift the results to E cohomology, the same comments apply when we are working modulo 2. However, due to the work of Strickland, the theory $E(n)$ is a commutative ring spectrum if the higher v's which are killed are chosen carefully enough [Str]. Indeed, since these theories ($P(0)$ and $E(n)$) surject to all of the other theories, their commutativity is forced.

4. The main computation

We begin with the rather simple results on finite Postnikov systems.

Proof of Theorem 2.2 (i). This follows automatically from the discussion Remark 3.2 about [HRW98]. ☐

Proof of Theorem 2.4 (i) *and* (ii). If we can show that the maps which give rise to these Koszul complexes in (i) are split exact on homotopy, then the results of [HRW98] will give the exactness and splittings as Hopf algebras, Remark 3.2. This is not a hard result. One way to see it is to observe that the Koszul complex $\mathcal{K}\mathcal{Z}_*^{\pi_* IBP\langle q\rangle}$, is given by

$$\mathcal{K}\mathcal{Z}_{q-j}^{\pi_* \Sigma^{-g_v(q)} IBP\langle q\rangle} \simeq \mathrm{Hom}(\mathcal{K}\mathcal{Z}_j^{BP\langle q\rangle_*}, \mathbb{Q}/\mathbb{Z}_{(p)}).$$

Since $\mathbb{Q}/\mathbb{Z}_{(p)}$ is injective, our split exactness is preserved. There is a little worry about the extension but that is easy. Since this is split exact degree by degree all of the cases of interest follow. (ii) is done similarly just using exactness on homotopy. ☐

We prove several of our main results at once and get the others from them or using similar proofs. We start with the following double induction.

Theorem 4.1. (i) *For $k > q+1-g(q)$ there is a short exact sequence of K_*-Hopf algebras which is split as algebras. The right hand term is polynomial if k is even and exterior if k is odd.*

$$K_* \longrightarrow K_*\underline{IBP\langle q\rangle}^{(q)}_{k+q+1} \longrightarrow K_*\underline{BP\langle q\rangle}_{g(q)+k} \longrightarrow K_*\underline{L_qBP\langle q\rangle}^{(q+1)}_{g(q)+k} \longrightarrow K_*.$$

(ii) *For $k \geq 0$ the complex $\mathcal{EKZ}_*^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}}$ is exact in the category of K_*-Hopf algebras.*

(iii) *For $k \geq 0$ the complex $\mathcal{KZ}_*^{K_*\underline{L_qBP\langle q\rangle}^{(q+1)}_{2(q+1)+k}}$ is exact in the category of K_*-Hopf algebras and splits as algebras.*

(iv) *The bar spectral sequence for the Morava K-theory of the fibration*

$$\underline{BP\langle q\rangle}_{g(q)+k-1} \longrightarrow * \longrightarrow \underline{BP\langle q\rangle}_{g(q)+k}$$

collapses.

(v) *For $k \geq 0$ we have a short exact sequence of long exact sequences with all maps split as algebra maps:*

$$K_* \to \mathcal{EKZ}_*^{K_*\underline{IBP\langle q\rangle}^{(q)}_{k+2(q+1)-g_\delta(q)}} \to \mathcal{EKZ}_*^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}} \to \mathcal{KZ}_*^{K_*\underline{L_qBP\langle q\rangle}^{(q+1)}_{2(q+1)+k}} \to K_*.$$

Proof. The proof is by double induction. When $q = 0$ all of the statements are trivial or vacuous except for (iv) which was proven in [RW80, Theorem 12.3, p. 743].

For $q > 0$ and $k < 0$, (i) is true because the first space in the fibration is trivial. (iv) is true for $k \leq 0$ by [Wil75].

We prove (ii) for $k = 0$ assuming both (i) and (ii) for $q-1$. According to Theorem 3.29 we have homotopy equivalences

$$\underline{BP\langle q\rangle}_{g(q)-\alpha-|v_q|} \cong \underline{BP\langle q\rangle}_{g(q)-\alpha} \times \underline{BP\langle q-1\rangle}_{g(q)-\alpha-|v_q|}$$

if $\alpha \geq 2$. The map $\pm v_q$

$$\underline{BP\langle q\rangle}_{g(q)-\alpha} \longrightarrow \underline{BP\langle q\rangle}_{g(q)-\alpha-|v_q|}$$

induces a homotopy equivalence on the first factor and thus an isomorphism of Hopf algebras. Filtering out by these maps we are left with showing that the following sequence of spaces induces a long exact sequence of Hopf algebras:

$$* \to K(\mathbb{Z}_{(p)}, q+2) \to \underline{BP\langle q\rangle}_{g(q)} \xrightarrow{\pm v_q} \underline{BP\langle q\rangle}_{g(q)-|v_q|} \to$$
$$\prod_{i<q} \underline{BP\langle q-1\rangle}_{g(q)-|v_q|-|v_i|} \longrightarrow \prod_{0<i_1<i_2<q} \underline{BP\langle q-1\rangle}_{g(q)-|v_q|-|v_{i_1}|-|v_{i_2}|} \longrightarrow \cdots .$$

The terms on the right give an exact sequence by the $q-1$ version of (ii). According to [RWY98, sections 8.3.1–2] the sequence

$$* \to K(\mathbb{Z}_{(p)}, q+2) \to \underline{BP\langle q\rangle}_{g(q)} \xrightarrow{\pm v_q} \underline{BP\langle q\rangle}_{g(q)-|v_q|}$$

induces an exact sequence of Hopf algebras in Morava K-theory, (1.26). That leaves only exactness at

$$K_*\underline{BP\langle q\rangle}_{g(q)-|v_q|} \qquad \text{and} \qquad K_*\prod_i \underline{BP\langle q-1\rangle}_{g(q)-|v_q|-|v_i|}$$

to prove. The kernel at

$$K_*\prod_i \underline{BP\langle q-1\rangle}_{g(q)-|v_q|-|v_i|}$$

is known to be the cokernel of

$$K_*K(\mathbb{Z}_{(p)}, q+3) \longrightarrow K_*\underline{BP\langle q-1\rangle}_{g(q)-|v_q|}$$

by induction on q using (ii). To complete our proof we will show that this is precisely the cokernel of

$$K_*\underline{BP\langle q\rangle}_{g(q)} \xrightarrow{\pm v_{q_*}} K_*\underline{BP\langle q\rangle}_{g(q)-|v_q|}$$

as well.

Consider the bar spectral sequence associated to the fibration

$$\underline{BP\langle q\rangle}_{g(q)} \to \underline{BP\langle q\rangle}_{g(q)-|v_q|} \to \underline{BP\langle q-1\rangle}_{g(q)-|v_q|}.$$

By [RWY98, Section 8.3.2], the kernel of $K_*(\pm v_q)$ is $K_*K(\mathbb{Z}_{(p)}, q+2)$. Thus we have, by Theorem 3.4 (ii),

$$E_2 \cong E_\infty \cong Coker(K_*(\pm v_q)) \otimes \mathrm{Tor}^{K_*K(\mathbb{Z}_{(p)},q+2)}(K_*, K_*),$$

which collapses because it is even degree. This Tor part is the $K_*K(\mathbb{Z}_{(p)}, q+3)$ factor in $K_*\underline{BP\langle q-1\rangle}_{g(q)-|v_q|}$, and the rest is just the cokernel discussed above. This concludes our proof.

We prove (i) and (iii) for $k = 0$. The map

$$K_*K(\mathbb{Q}/\mathbb{Z}_{(p)}, q+1) \longrightarrow K_*K(\mathbb{Z}_{(p)}, q+2)$$

is an isomorphism. Recall that $\underline{IBP\langle q\rangle}^{(q)}_{q+1} = K(\mathbb{Q}/\mathbb{Z}_{(p)}, q+1)$. This, together with the injection of (1.26) and Theorem 3.4 (iii) give the short exact portion of (i). The exactness (ii) is just (iii) for $k = 0$ spliced together with the short exact sequence (i). The term $K_*\underline{L_q BP\langle q\rangle}^{(q+1)}_{g(q)}$ is the last term in the Koszul complex (iii) (for $k = 0$) and therefore injects into a polynomial algebra. By [Bou96b, Theorem B.7], which says a subHopf algebra of a polynomial Hopf algebra is polynomial too, it must be polynomial. Since $K_*\underline{L_q BP\langle q\rangle}^{(q+1)}_{g(q)}$ is polynomial, we get (i) with the splitting for $k = 0$ from [Bou96b, Theorem B.9], which says that a short exact sequence of Hopf algebras which ends with a polynomial algebra is split as algebras. Combining these two results of Bousfield's we see that all the maps in (iii) must split.

There is little content to (v) for $k = 0$.

We now do our induction on k. Assume all parts of the theorem for $(k-1) \geq 0$. We show (ii) and (iv) simultaneously. Taking the bar spectral sequenceon all of $\mathcal{EKZ}_*^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k-1}}$ we have a spectral sequence converging to $\mathcal{EKZ}_*^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}}$. By induction we know that all of it collapses except possibly the q-th term, i.e. (iv) for k. Furthermore, everything in sight which we use in our induction is split as algebras, see (v). Because all of the maps split as algebras before we take Tor, we have exactness for the E^2 term of the bar spectral sequence on $\mathcal{EKZ}_*^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k-1}}$ after we take Tor. Since we know that all but one of these collapses, we get exactness for $\mathcal{EKZ}_*^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}}$ except at the point

$$K_*K(\mathbb{Z}_{(p)}, q+2+k) \to K_*\underline{BP\langle q\rangle}_{g(q)+k} \to \mathcal{EKZ}_{q-1}^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}}$$

recalling that $K_*\underline{BP\langle q\rangle}_{g(q)+k} = \mathcal{EKZ}_q^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}}$. We have exactness as Hopf algebras on the Tor level of the spectral sequence so if we can show that the spectral sequence collapses, i.e. (iv), then we will also have our exactness (ii). We want to remind the reader that our induction is on k. We are assuming that we know everything for $k-1$. We could start this induction because we have already shown $k = 0$. In particular, we know the Morava K-theory of all spaces involved for $k-1$ as algebras, which is all it takes to determine Tor. We want to emphasize that we are not doing induction on degrees. If k is even then everything (all of the Tor groups) is in even degrees and the spectral sequence collapses. If k is odd then all generators must be either odd degree in the first filtration, in which case they have no differentials, or in even degrees, in which case their target must be an odd degree element in the first filtration. By the exactness of the spectral sequence, and the fact that the first term above is even degree, we see that all of these odd elements must map to the next term which is part of a collapsing spectral sequence and therefore these elements cannot be targets of differentials. This concludes the proof for (ii) and (iv).

We now do the induction for (i) and (v). The left hand term of (v) is already known to be long exact and to split as Hopf algebras by Theorem 2.4 (i). We have just proven the center term, (ii), is exact. By induction, we have the injection from the left to the center except for the q-th term of the complex. So, in the diagram (4.2), the vertical columns are exact, the top horizontal arrow is an isomorphism and the third horizontal arrow is an injection. This is enough to force an injection on the second horizontal arrow. By Theorem 3.4 (iii), we get our exact sequence (i). We now have a short exact sequence for (v) with the first two sequences long exact. This forces the third to be long exact and gives us (iii). Because all of the terms in (iii) are either polynomial or exterior, Bousfield's results tell us that all maps are split as algebra maps. Likewise the splitting for (i) follows too. All of the splittings together give us the splittings for (v). ☐

We have proven our basic results about short and long exact sequences now, namely, we have proven Theorem 2.2 (i) and (iii), and Theorem 2.4 (i), (ii), and the $s = q$ version of (iii). We still have a number of such sequences to verify. We do this mostly from the ones we know or with similar techniques.

(4.2)
$$\begin{array}{ccc}
K_* & & K_* \\
\downarrow & & \downarrow \\
K_*K(\mathbb{Q}/\mathbb{Z}_{(p)}, q+1+k) & \xrightarrow{\simeq} & K_*K(\mathbb{Z}_{(p)}, q+2+k) \\
\downarrow & & \downarrow \\
K_*\underline{IBP\langle q\rangle}^{(q)}_{q+1+k} & \longrightarrow & K_*\underline{BP\langle q\rangle}_{g(q)+k} \\
\downarrow & & \downarrow \\
\mathcal{EKZ}_{q-1}^{K_*\underline{IBP\langle q\rangle}^{(q)}_{q+1+k-g_v(q)}} & \hookrightarrow & \mathcal{EKZ}_{q-1}^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}} \\
\downarrow & & \downarrow
\end{array}$$

5. Proofs of odds and ends

Proof of the short exact sequence part of Theorem 2.2 (ii). We map the three term sequence of (ii) to the known short exact sequence (iii). We know that the left hand term of (ii) injects to the left hand (iii) and the middle term is an isomorphism. This forces the injection of the left hand term of (ii) into the middle. By Theorem 3.4 (iii) we have the short exact sequence of (ii). The splitting and identification with the Goerss-Hunton-Turner generalized tensor product will come later. □

Proof of Theorem 2.2 (iv) *and Theorem* 2.4 (v). Now we have a map of short exact sequences of Theorem 2.2 (ii) to (iii) and we get a six term exact sequence relating the kernels with the cokernels (the snake lemma, see Remark 3.1) except that since the middle term gave an isomorphism we have an isomorphism from the kernel of the right hand map to the cokernel of the left hand map. This gives an algebraic version of (iv) but not one that we know comes from the geometric map! When $r < g(q)$ the left hand term is trivial and the other two are isomorphic. For $r = g(q)$, (iv) is just the sequence (iii). When $s \geq r + g_\delta(q)$ this sequence is just (iii). To get the correct version coming from the geometric map we have to go back and use our previous little trick with the Koszul complexes. We can turn all three terms into the non-extended Koszul complexes of Theorem 2.4 (v). For $r > g(q)$ we want our short exact sequence to be the q-th term of our Koszul complexes. By induction we have short exact on all the lower terms of the complex. We also have split exact on the homotopy of the left terms so that Koszul complex is long exact. We have already covered the case above when we are in the range when the extension is there. The right hand side is already known to be long exact from Theorem 4.1 (v), $s = q$ (Theorem 4.1 (iii)). The left hand term of (iv) injects into the $(q-1)$-th term of its Koszul complex which in turn injects into the middle. This forces injection of (iv) and thus the short exact sequence (iv) coming from the geometry. Because we have a short exact sequence of complexes with both left and right long exact, the middle must be long exact too, Theorem 2.4 (v). □

Proof of Theorem 2.4 (iii). The $s = q$ case is done already. The short exact sequence of Theorem 2.2 (ii) gives Theorem 2.4 (iii) as a short exact sequence of complexes. However, all three terms are already known to be long exact. □

Proof of Theorem 2.4 (iv). We want to prove the $\mathbb{Z}/(p^c)$ version of the exactness of the Koszul complex now, the middle term of Theorem 2.4 (iv), i.e. for $\mathcal{EK}^{\mathbb{Z}/(p^c)}\mathcal{Z}_*^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}}$. There is an obvious injection of Koszul complexes

$$\mathcal{KZ}_*^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}} \longrightarrow \mathcal{K}^{\mathbb{Z}/(p^c)}\mathcal{Z}_*^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}}.$$

Note that these are the unextended versions. The quotient is just another copy of $\mathcal{KZ}_*^{K_*\underline{BP\langle q\rangle}_{2(q+1)+k}}$ shifted by one in the Koszul degree. We get a long exact sequence in the homology of the short exact sequence of Koszul complexes. We know that the homology of the first one is just $K_*K(\mathbb{Z}_{(p)}, q+2+k)$ in homological degree $q+1$. The homology of the quotient is just $K_*K(\mathbb{Z}_{(p)}, q+2+k)$ in homological degree $q+2$. The boundary homomorphism is easily seen to be p_*^c which is known to be surjective, [RW80, Corollary 13.1], with kernel $K_*K(\mathbb{Z}/(p^c), q+1+k)$ in homological degree $q+2$. Thus we have the proper Hopf algebra for the extension of $\mathcal{K}^{\mathbb{Z}/(p^c)}\mathcal{Z}_*^{\frac{K_*BP\langle q\rangle}{}}{}_{2(q+1)+k}$ which it takes to make it exact. We get the fact that it comes from the geometric map from the commutativity of the diagrams geometrically.

We get the short exactness from Theorem 2.2 (iii) plus the extension. We already know the first two are long exact so the short exactness gives us the third one is long exact too. □

Remark 5.1. We wish to mention here that for any of the Koszul complexes which are long exact and which are not extended, the last term is completely algebraically determined inductively. In particular this gives us part of the statement of Theorem 2.4 (iii).

Proof of the GHT tensor product identification of Theorem 2.2 (ii)*, Theorem* 2.3*, and Theorem* 2.11. Most of the remaining Morava K-theory results rely on an understanding of the various spectral sequences associated with the sequence of fibrations (2.10). We can analyze them one at a time using the fact that we already know all of the answers.

Since we need to use it soon we take a break to prove the v_q injection for r even in Theorem 2.3. When $r \leq g(q-1)$ (and even) we have the splitting Theorem 3.29 and get the injection easily. when $r > g(q)$ and even we use the Koszul long exact sequence on the right hand side of Theorem 4.1 (v). We know that the last term injects in the second to last term but we know by induction that all of the terms in that product inject into the preceeding one except the one which our last term maps to by v_q, so it must be an injection too. (This argument fails for r odd because when $r = g(q-1)+1$ we are neither in the range of the splitting nor in the range of the Koszul complexes.)

Let $r' = r + 2(p^q - 1)$ in the commuting diagram (5.2) of short exact sequences of Hopf algebras.

All of the horizontal and vertical maps are induced by the corresponding fibrations. The right one comes from Remark 3.27. We are interested in the bar spectral sequence of the middle vertical column and so we take three rows at a time. The bar spectral sequence of the left hand side is always even degree and collapses, Theorem 3.4 (iv). It maps to the bar spectral sequence for the middle and so there can be no differentials on any of these elements by naturality.

We start with the first three rows. We know the top left vertical map is surjective on homotopy and so it is surjective as Hopf algebras since the left side is all finite Postnikov systems. The kernel of the homotopy is just the homotopy of the delooping of the lower left space. The bar spectral sequence for the left vertical fibration is in even degrees and therefore collapses. Our concern of course is with the bar spectral sequence for the vertical fibration in the middle. We start with r even. We know from above that the right top

(5.2)

$$
\begin{array}{ccccccccc}
K_* \rightarrow & K_*\underline{IBP\langle q\rangle}^{(q)}_{r'-g_\delta(q)} & \rightarrow & K_*\underline{BP\langle q\rangle}_{r'} & \rightarrow & K_*\underline{L_qBP\langle q\rangle}^{(q+1)}_{r'} & \rightarrow K_* \\
& \Big\downarrow v_{q*} & & \Big\downarrow v_{q*} & & \Big\downarrow v_{q*} & \\
K_* \rightarrow & K_*\underline{IBP\langle q\rangle}^{(q)}_{r-g_\delta(q)} & \rightarrow & K_*\underline{BP\langle q\rangle}_{r} & \rightarrow & K_*\underline{L_qBP\langle q\rangle}^{(q+1)}_{r} & \rightarrow K_* \\
& \Big\downarrow & & \Big\downarrow & & \Big\downarrow & \\
K_* \rightarrow & K_*\underline{IBP\langle q-1\rangle}^{(q+1)}_{r-g_\delta(q-1)} & \rightarrow & K_*\underline{BP\langle q-1\rangle}_{r} & \rightarrow & K_*\underline{L_{q-1}BP\langle q-1\rangle}^{(q+2)}_{r} & \rightarrow K_* \\
& \Big\downarrow & & \Big\downarrow & & \Big\downarrow & \\
K_* \rightarrow & K_*\underline{IBP\langle q\rangle}^{(q+1)}_{r'-g_\delta(q)+1} & \rightarrow & K_*\underline{BP\langle q\rangle}_{r'+1} & \rightarrow & K_*\underline{L_qBP\langle q\rangle}^{(q+2)}_{r'+1} & \rightarrow K_* \\
& \Big\downarrow v_{q*} & & \Big\downarrow v_{q*} & & \Big\downarrow v_{q*} & \\
K_* \rightarrow & K_*\underline{IBP\langle q\rangle}^{(q+1)}_{r-g_\delta(q)+1} & \rightarrow & K_*\underline{BP\langle q\rangle}_{r+1} & \rightarrow & K_*\underline{L_qBP\langle q\rangle}^{(q+2)}_{r+1} & \rightarrow K_*
\end{array}
$$

vertical map is injective and that everything in sight is even degree. Thus, the E^2 term of the spectral sequence from Theorem 3.4 (iii) is all even degree and collapses. This gives exactness at $K_*\underline{BP\langle q\rangle}_r$ for Theorem 2.11 if r is even. If r is odd, then the top two on the right hand side are known to be exterior Hopf algebras and the short exact sequences they are part of split as Hopf algebras. If there is a kernel, and sometimes there is, then it too must be exterior as is the cokernel. The E^2 term of the spectral sequence has two parts as usual, see Theorem 3.4 (ii). One part is the Tor of the kernel. This comes in two parts, one part from the left hand side which we have already noticed has trivial differentials because the spectral sequence for the left hand side collapses. The second part will come from whatever kernel there is on the right hand side. Since it is exterior the Tor will be a divided power algebra all concentrated in even degrees. We have no obvious restrictions on differentials on this part. We also have the cokernal part of E^2. This will be exterior and if we can show that all of these elements exist in $K_*\underline{BP\langle q-1\rangle}_r$ then there will be no place for differentials to land. These exterior classes are in the zeroth filtration so they have no differentials and the even stuff can only land on an odd class and these are our only odd classes. So, our goal is to show that these exterior elements in the cokernel all survive. Then we must identify the exterior kernel part if there is one.

Our proof here uses some techniques which we have not yet had to invoke for this paper and we doubt the necessity of having to do this now but we need to prove the result. For r even we found the right vertical maps to give a short exact sequence. The Milnor-Moore theorem about exactness of indecomposables still holds in the category of Hopf algebras we are working in, namely we get an exact sequence:

$$(5.3)\quad \cdots \longrightarrow QK_*\underline{L_q BP\langle q\rangle}^{(q+1)}_{r'} \longrightarrow QK_*\underline{L_q BP\langle q\rangle}^{(q+1)}_{r} \longrightarrow QK_*\underline{L_{q-1} BP\langle q-1\rangle}^{(q+2)}_{r} \longrightarrow 0.$$

There are some dramatic differences however and for the presently perplexed we should point some of these out. The top two terms of our vertical short exact sequence on the right side are both polynomial algebras (recall r is temporarily even). The bottom one is split as algebras and is part polynomial and sometimes has the Morava K-theory of an Eilenberg-Mac Lane space in it (to be identified soon). So, we can have a non-trivial extension here. All iterated p-powers in the Morava K-theory of Eilenberg-Mac Lane spaces either go to zero or become cyclical, see [RW80]. One way to achieve this as a quotient of two polynomial algebras would be to take a sequence of generators mapped as follows:

$$x_0 \to y_0^p, \quad x_1 \to y_0 + y_1^p, \quad x_2 \to y_1 + y_2^p, \quad \text{etc.}$$

Note that there are no indecomposables for the Morava K-theory of $\mathbb{Q}/\mathbb{Z}_{(p)}$ Eilenberg-Mac Lane spaces! An easy example of this is to consider a little Hopf algebra generated by x with $x^p = x$. Its module of indecomposables is trivial. So, Milnor-Moore's exact sequence is not as useful for us as it is in the graded case. A serious failing is that a surjection of indecomposables does not imply a surjection of Hopf algebras (map the trivial Hopf algebra to the example just given). However, we are still going to extract the information we need from the exact sequence for the indecomposables. The polynomial part of the lower right hand part splits off, as algebras, from both the lower right and the middle right terms. Let us call this P_1. Thus the middle term is just the tensor product of two polynomial algebras, P_1 and P_2, one isomorphic to the polynomial part of the lower right term and the other in the middle of a short exact sequence starting with the upper right polynomial algebra, P_3, and ending with the Morava K-theory of an Eilenberg-Mac Lane space part of the lower right. Since the indecomposable module of the last one is trivial, the exact sequence gives a surjection from the indecomposables of the top right onto the indecomposables of the part of the middle right term involved in our short exact sequence, $QP_3 \to QP_2 \to 0$. We have some understanding of the behavior of the generators of the top two terms on the right row. We know that the bar spectral sequence for their delooping collapses. We know this because Tor of a polynomial algebra is just an exterior algebra with generators in filtration 1 and so there can be no differentials. For the bottom term we have to worry a little about the Eilenberg-Mac Lane part but we know that differentials on those elements are also trivial because they come from our finite Postnikov systems which always collapse. Thus, the Tor of the middle term is the tensor product of two exterior algebras. One maps isomorphically to the corresponding exterior algebra in the lower right hand term because of the splitting off of the polynomial algebra. Because of our surjection of indecomposables from P_3 to

P_2, we get that the exterior generators P_2 creates when we take Tor are all in the image of our map from the top. The point here is that we have shown that our exterior cokernel all survives and so the spectral sequence we care about collapses, and, we have also shown that we have exactness at the middle term, $K_*\underline{BP\langle q\rangle}_r$ for r odd.

Assuming we have a kernel for r odd, then it is easy to identify as having to have a Tor which gives the Eilenberg-Mac Lane part of the lower right hand term. At first glance, looking at all our results in Theorem 2.2, it appears that it should have all of the Morava K-theory of the q and $q+1$ homotopy of $\underline{IBP\langle q-1\rangle}^{(q-1)}_{r-g_\delta(q-1)}$, i.e. of a two stage Postnikov system. However, we know from sparseness that there are never two non-trivial homotopy groups in adjacent degrees. In fact there are other restrictions and we can now see that it must be in degree $q+1$ or not at all. The only degrees that the homotopy can be in are: $r-g_\delta(q-1)-2t(q+1)$. Recalling that r is odd, mod 2 this is $(1+q)$. Thus we can have $(q+1)$ degree homotopy but not q.

This particular spectral sequence argument, which shows that the cokernel of v_{q*} always injects, gives us the identification of the Goerss-Hunton-Turner tensor product in Theorem 2.2 (ii), which was the last remaining thing to prove in that theorem. We also have finished the proof of Theorem 2.3.

We pursue our calculations with these short exact sequences in order to prove Theorem 2.11.

We move now to the next spectral sequence, for the fibration:

$$\underline{BP\langle q\rangle}_r \longrightarrow \underline{BP\langle q-1\rangle}_r \longrightarrow \underline{BP\langle q\rangle}_{r'+1}.$$

Going back to our diagram (5.2) we consider now rows two through four. We have already analyzed the maps needed for the spectral sequence. The upper left map is zero and we see right off that the spectral sequence for it collapses. Let us first just look at the case for r even. The kernel of the right hand map is known to be $K_*\underline{L_qBP\langle q\rangle}^{(q+1)}_{r'}$ which is known to be polynomial. We also know the map is surjective. Thus we have the Tor part comes in two parts. One is from the finite Postnikov system and we know that it has no differentials in it by naturality from the left hand spectral sequence. The second is from the polynomial part and it gives an exterior part and so has no differentials and there is nothing which can be a source of differentials which can hit it. Thus the spectral sequence collapses and we see we have exactness at the middle term.

Now we let r be odd. We know that the kernel on the right is exterior so the E^2 is concentrated in even degrees and the spectral sequence collapses giving us exactness again at the middle space.

We are ready for our third and final spectral sequence. This comes from our middle vertical term of the last three rows of the big diagram and we only consider these last three rows. This time the upper left vertical map injects and so the left spectral sequence collapses and is even degree as usual. Let's start with r odd this time. The kernel of the right hand top map is exterior and perhaps some Eilenberg-Mac Lane stuff. In any case it has even degree Tor so the spectral sequence collapses and we get exactness at the middle term.

For r even, the upper right corner is two parts, (1) polynomial and (2) the $(q+1)$ Eilenberg-Mac Lane part if there is any and the map to the next term is trivial. Because we are now working with odd degree spaces in the second and third rows, we recognize from our previous calculation that these spaces are really equivalent to the q connected one on the left and $(q+1)$ connected one on the right. Tor of the kernel is all even degree and in E^2 we end up with too much odd degree stuff because we know we do not have injectivity of the middle right hand side to the lower right hand side. We also know the kernel as $A(q, r+1)$. This must all be hit by differentials then, i.e. this spectral sequence does not collapse when $A(q, r+1) \neq 0$. There can be no differentials coming from elements on the left because their spectral sequence collapses. We must then look at Tor of the kernel on the right. First, there is the polynomial part which gives rise to exterior elements. These must all survive. We know they survive in the spectral sequence for delooping and so by naturality any differential which hit them in the spectral sequence for $\underline{BP\langle q\rangle}_{r+1}$ would have to map to a differential which hit them in the spectral sequence for $\underline{BP\langle q-1\rangle}_{r+1}$. However, the Tor of the Eilenberg-Mac Lane part, $K_*K(\pi_{q+1}\underline{IBP\langle q-1\rangle}^{(q-1)}_{r-g_\delta(q-1)}, q+1)$ fits nowhere in our known answer. We know how this Tor looks from [RW80, Theorem 12.3]. It is a divided power algebra on transpotence elements in the second filtration. Since it must disappear the only possibility is a d^2 which takes these transpotence elements isomorphically to the exterior generators of $A(q, r+1)$, thus killing off all of $A(q, r+1)$ tensored with the Tor of this Eilenberg-Mac Lane part. Exactness fails here.

This finishes our description of the behavior of the spectral sequences and also of our proof of Theorem 2.11. It is now easy to see why this approach to the original problem failed us so badly. □

Proof of Theorem 2.12. We look at the bar spectral sequence for the fibration

$$\underline{IBP\langle q\rangle}^{[q]}_{r+1-g_\delta(q)} \longrightarrow \underline{L_qBP\langle q\rangle}^{(q+1)}_{r} \longrightarrow \underline{L_qBP\langle q\rangle}_{r}. \tag{5.4}$$

The first map is trivial in Morava K-theory. At first glance this is not so obvious. We know the target to be either an exterior algebra on odd generators or a polynomial algebra on even generators. The Hopf algebra splittings of [HRW98] tell us that if it is exterior then there are no maps. To do the even case you have to know something explicit about the Morava K-theory of the Eilenberg-Mac Lane spaces. The Dieudonné modules are written down in [SW98] and they are p-divisible groups. Polynomial algebras have no p-divisible elements in their Dieudonné modules so there can be no maps. Thus the E^2 term of the bar spectral sequence is

$$K_*\underline{L_qBP\langle q\rangle}_{r} \simeq K_*\underline{L_qBP\langle q\rangle}^{(q+1)}_{r} \bigotimes K_*\underline{IBP\langle q\rangle}^{[q+1]}_{r+1-g_\delta(q)}.$$

When r is even this is even degree and so collapses. It doesn't always collapse when r is odd. □

6. The BP results

Remark 6.1 (The Künneth isomorphism)**.** We use freely two previous results throughout. If $E^*(X)$ and $E^*(Y)$ are both Landweber flat, then

$$E^*(X \times Y) \simeq E^*(X)\widehat{\bigotimes}E^*(Y)$$

[RWY98, Theorem 1.11] and is also Landweber flat [Wil99a, Theorem 1.8].

In [RWY98, Theorem 1.19] it was proven that if you had maps of H-spaces

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \tag{6.2}$$

with the compositon trivial and all spaces having even Morava K-theory and giving an exact sequence of bicommutative Hopf algebras for all Morava K-theories

$$K_* \longrightarrow K_*X_1 \longrightarrow K_*X_2 \longrightarrow K_*X_3 \longrightarrow \tag{6.3}$$

then we got

$$E^*(X_1) \simeq E^*(X_2)/(f_2^*). \tag{6.4}$$

The statement in [RWY98] is only for $E = P(m)$ but the proof starts assuming the result for $E = K(n)$ and then proves it for all of our E on the way to $P(m)$.

Another theorem is that spaces with even Morava K-theory have Landweber flat E cohomology concentrated in even degrees, [RWY98, Theorem 1.8]. After this paper it was noticed that the theorem of (6.4) was true even if you drop the even Morava K-theory assumption and replaced it with the weaker Landweber flat assumption, [Kasb] and [Wil99a]. In fact one does not need even this assumption on the space X_1 to conclude the result and the fact that it too is Landweber flat. This is made explicit in [Wil99a] and is implicit in [Kasb]. What is really proven is that $E^*(-)$ for the sequence of spaces X_i is coexact in the category of algebras. This coexactness is just the definition of a cokernel, or, equivalently, (6.4).

In order to prove Theorem 1.1 we need a dual version of (6.4), something not to be found in any of [RWY98], [Wil99a] or [Kasb], caused principally, we suppose, by never having had a need for it before.

Remark 6.5 (Completed Hopf algebras)**.** We need to remind the reader that $E^*(X)$ always comes equipped with a topology on it which is an important part of the structure. If X and Y both have Landweber flat Brown-Peterson cohomology, then so does their product, and it is just the completed tensor product of the two, Remark 6.1. This means, in particular, that if X is an H-space, we get a "completed coalgebra" structure

$$E^*(X) \longrightarrow E^*(X)\widehat{\bigotimes}E^*(X). \tag{6.6}$$

The dual result would then say something about kernels in this category. However, if we combine the two structures, topologized algebra and completed coalgebra we get "completed Hopf algebras" and our cokernel of algebras and kernel of coalgebras become cokernel and kernel of completed Hopf algebras. If we begin with our completed coalgebras and note that the completed tensor product for this category is the product, then the Hopf algebra maps give us a group object in the category, i.e. a completed Hopf algebra. Our category of completed Hopf algebras is not abelian but we define a short exact sequence anyway. In our category we say

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{6.7}$$

is short exact if f is an injection, g is a surjection, A is the kernel of g, and C is the cokernel of f.

We need to state our theorems explicitly. We assume our spaces are all of finite type, i.e. $H_*(-;\mathbb{Z}_{(p)})$ is of finite type over $\mathbb{Z}_{(p)}$.

Theorem 6.8. *Let E be as in Theorem* 1.1. *Given a sequence of maps as in* (6.2) *with the composition trivial and an exact sequence of bicommutative Hopf algebras* (6.3) *for all Morava K-theories with the last two spaces having Landweber flat E cohomology, then the first does too and $E^*(X_1)$ is the completed Hopf algebra cokernel of f_2^*.*

As noted above, this was known to be an algebra cokernel. We will put the topology in it so it really is the cokernel in the correct category.

The dual result then is:

Theorem 6.9. *Let E be as in Theorem* 1.1. *Given a sequence of maps as in* (6.2) *with the composition trivial and an exact sequence of bicommutative Hopf algebras*

$$\longrightarrow K_*X_1 \longrightarrow K_*X_2 \longrightarrow K_*X_3 \longrightarrow K_*$$

for all Morava K-theories with the first two spaces having Landweber flat E cohomology, then the third does too and $E^*(X_3)$ is the completed Hopf algebra kernel of f_1^*.

Corollary 6.10. *Let E be as in Theorem* 1.1. *Given a sequence of maps as in* (6.2) *with the composition trivial and a short exact sequence of bicommutative Hopf algebras*

$$K_* \longrightarrow K_*X_1 \longrightarrow K_*X_2 \longrightarrow K_*X_3 \longrightarrow K_*$$

for all Morava K-theories with the middle space having Landweber flat E cohomology, then the other two spaces have Landweber flat E cohomology and we get a short exact sequence of completed Hopf algebras

$$E^* \longleftarrow E^*X_1 \longleftarrow E^*X_2 \longleftarrow E^*X_3 \longleftarrow E^*.$$

Proof. This follows at once from Theorems 6.8 and 6.9 after we get the flatness for the two ends. This flatness follows at once from the injection and the surjection, see [Wil99a] or [Kasb]. □

The proof of Theorem 1.10 now follows immediately from the Corollary and the corresponding result from [HRW98] on Morava K-theory.

It would be nice to be able to say that the proof of this is just dual to the proof of the dual theorem, but that isn't quite true and so we must delve into some detail here to show the difference and how to patch it up. The proof is dual once the following dual to [RWY98, Theorem 1.18] has been proven and so this is all we need to do.

Proposition 6.11. *Let E be as in Theorem* 1.1. *Given a sequence of maps as in* (6.2) *with the composition trivial and a short exact sequence of $K(n)^*$ modules for all Morava K-theories*:

$$\leftarrow K(n)^*(X_1) \leftarrow K(n)^*(X_2) \leftarrow K(n)^*(X_3) \leftarrow 0,$$

if the first two spaces have Landweber flat E cohomology then we get an exact sequence of E^* modules:

$$\leftarrow E^*(X_1) \leftarrow E^*(X_2) \leftarrow E^*(X_3) \leftarrow 0$$

and $E^*(X_3)$ is also Landweber flat.

Proof. The proof really differs little from being the dual proof of the related theorem. We start off with the cofibre of f_2, $C(f_2)$, but in order to get our short exact sequences on our cohomology theories, we have to suspend everything. The maps

$$C(f_2) \longrightarrow \Sigma X_2 \longrightarrow \Sigma X_3$$

now get a short exact sequence on all the cohomology theories used in the proof of the dual, in particular, all $E^*(-)$. If we stabilize, then we see that, by our assumptions, our map $X_1 \to X_2$ must factor through $\Sigma^{-1}C(f_2)$. The map $X_1 \to \Sigma^{-1}C(f_2)$ gives an injection on Morava cohomology and so on E cohomology. We find ourselves working with spectra using theorems which only work for spaces. However, these are suspension spectra and so the cohomology theories work the way they are supposed to. The only thing stable is the map and we use it just to get an algebraic factorization. All else, except this little trick making things stable, is the same (i.e. dual). □

Proof of Theorem 1.1. First we recall the maps and isomorphisms of (3.26). These are all BP module spectra and so we can replace our old spaces with our new spaces throughout the paper for all of our Morava K-theory results, in particular, our Theorem 2.6 (where we need to replace $K(\mathbb{Q}/\mathbb{Z}_{(p)}, q+1+k)$ with $K(\mathbb{Z}_{(p)}, q+2+k)$). Remark 2.7 gives us our final hypothesis for Theorem 6.8 and our Description 2 follows by induction and Remark 6.1.

Description 1 is a little different. The composition of the two maps in Theorem 2.9 is not zero so we have to do something different. We have the short exact sequence from Theorem 1.14 with the spaces replaced so we have finite type:

$$K_* \longrightarrow K_*\underline{I'BP\langle q\rangle}^{(q+2)}_{r+1-g_\delta(q)} \longrightarrow K_*\underline{BP\langle q\rangle}_r \longrightarrow K_*\underline{L'_qBP\langle q\rangle}^{(q+3)}_r \longrightarrow K_*.$$

This gives us, by Corollary 6.10, a short exact sequence of completed Hopf algebras:

$$E^* \longleftarrow E^*(\underline{I'BP\langle q\rangle}^{(q+2)}_{r+1-g_\delta(q)}) \longleftarrow E^*(\underline{BP\langle q\rangle}_r) \longleftarrow E^*(\underline{L'_qBP\langle q\rangle}^{(q+3)}_r) \longleftarrow E^*.$$

We have, by our identification of the Goerss-Hunton-Turner generalized tensor product, an exact sequence:

(6.12)

$$\cdots \longrightarrow \bigotimes_{i>q} K_*\underline{BP}_{r+|v_i|} \xrightarrow{\oplus v_{i*}} K_*\underline{BP}_r \longrightarrow K_*\underline{L'_qBP\langle q\rangle}^{(q+3)}_r \longrightarrow K_*.$$

This gives us the identification of the right hand term in Theorem 1.1, using Theorem 6.9.

We use Theorem 6.9 again to show

$$E^*(\underline{BP\langle q\rangle}_r) \simeq E^*(\underline{I'BP\langle q\rangle}^{(q+2)}_{r+1-g_\delta(q)})\widehat{\bigotimes}E^*(\underline{L'_qBP\langle q\rangle}^{(q+3)}_r). \tag{6.13}$$

We just use the maps

$$* \times \prod_{i>q}\underline{BP}_{r+|v_i|} \longrightarrow \underline{I'BP\langle q\rangle}^{(q+2)}_{r+1-g_\delta(q)} \times \underline{BP}_r \longrightarrow \underline{BP\langle q\rangle}_r. \tag{6.14}$$

This is right exact in $K_*(-)$ so our completed Hopf algebra splitting follows.

Thus we are only left with identifying $E^*(\underline{I'BP\langle q\rangle}^{(q+2)}_{r+1-g_\delta(q)})$ with the left side of Theorem 1.1 and Description 1. That will be done in the next section. □

Proof of Corollary 1.4. In [RWY98] we use the theorem of (6.3) with the sequences (1.26) and (1.27), [RWY98, Proposition 1.16], to calculate the Brown-Peterson cohomology of Eilenberg-Mac Lane spaces, [RWY98, Theorem 1.14]. This sequence shows up as the extension and the last two terms of the lowest case of the Koszul complexes in the middle terms of Theorem 2.4 (iii) and (iv). The exactness for the higher cases sets us up for the use of Theorem 6.8 if we have flatness. We have already calculated the E cohomology of the spaces in the Omega spectrum for $BP\langle q\rangle$ and we know they are flat so we are done. □

Remark 6.15 (The forgotten topology)**.** When you look back to [RWY98], [Wil99a] and [Kasb], the proof of (6.4) was just as algebras. Although there is nothing wrong with the statements and proofs in those papers, that is not really the category we are working in. We work with topologized algebras and so the cokernel of algebras must have a topology on it and it is possible to put a topology on it so that it is not the cokernel stated in our Theorem 6.8. Likewise for our Theorem 6.9. This is the sort of problem which prevents our category of completed Hopf algebras from being abelian. For an oversimplified example, let us look at the map $R \to R'$ where $R = \mathbb{F}_p[v_n]$ with the discrete topology and R' is $\mathbb{F}_p[v_n]$ where we use the ideals (v_n^k) as open sets. This map is continuous, injective and surjective, but not an isomorphism. If such things were to happen to us in our Theorems 6.8 and 6.9, then we really wouldn't have cokernels and kernels, something we want to use, for example in defining short exactness in our category of completed Hopf algebras. Injectivity and surjectivity would be unaffected. So, straightening out the problems with the topology is important for our results to work properly in the category of completed Hopf algebras. Thus, we have some nontrivial work yet to do on these theorems.

Below when we complete the proofs, we find that for the injection we don't really need our exact sequence but for our surjection we do. That is because our oversimplified example above almost does exist. In particular, let us look at the surjection

$$E(q,q+1)^*(\underline{BP\langle q\rangle}_{g(q)}) \longrightarrow E(q,q+1)^*(K(\mathbb{Z}/(p^c),q+1)).$$

By Remark 1.5 we know the right hand side is finitely generated and free. The left hand side has no torsion in standard homology and so any finitely generated submodule has the discrete topology on it. Thus every generator is also a generator of the q-th Morava K-theory whereas there is no q-th Morava K-theory for the Eilenberg-Mac Lane space. The v_q part of the topology of the Eilenberg-Mac Lane spaces is always nontrivial, but it is always trivial for the other space, much like our oversimplified example (but not the same). It is worth the trouble to make sense out of this since we claim our result produces the correct topology. We need the third term to do this so that we are really working with a cokernel. What happens is, roughly, there is a sequence of generators $\{x_i\}$ in $E(q,q+1)^*(\underline{BP\langle q\rangle}_{g(q)})$. x_1 may map to a generator of $E(q,q+1)^*(K(\mathbb{Z}/(p^c),q+1))$ which, although v_q acts freely on it, doesn't support a Morava K-theory generator. So, there must be a generator y_1 which comes and hits something like $v_q^{t_1}x_1 + x_2$. And then another y_2 must come and hit $v_q^{t_2}x_2 + x_3$. This must go on indefinitely with the x_i in higher and higher filtration. This produces the desired result of having our topology be both nontrivial and determined.

The rest of this section is devoted to dealing with the topology in the proofs of Theorems 6.8 and 6.9.

Completion of the proofs of Theorems 6.8 *and* 6.9*.* We first look at our new case, Theorem 6.9, as it is slightly easier but contains all of the ideas.

There is only one place the topology was ignored which matters. In [RWY98], (only for even degree Morava K-theories), [Wil99a] and [Kasb], it is proven that if $X \to Y$ gives an injection in all Morava cohomology K-theories then it does also for all of our E cohomology theories of Theorem 1.1. What we didn't concern ourselves with in those papers was to show that our injection gives a homeomorphism of $E^*(Y)$ to the image with the inherited topology. If we can show this then the our topological concerns for the health of Theorem 6.9 will be over. From [RWY98, Corollary 4.8] we know that there are never any phantom maps for any of the E which we use, so we always have the topology on $E^*(-)$ complete and Hausdorf. The topology comes from our open sets $F^{s+1}E^*(Z)$, the kernel of the map $E^*(Z) \to E^*(Z^s)$ where Z^s is the s-skeleton of Z and we have $E^*(Z) \simeq \lim^0 E^*(Z)/F^sE^*(Z)$. To prove our result we must see that the image of $F^sE^*(Y)$ contains some $F^tE^*(X)$ for some large t. An equivalent way of looking at this is that for every s there is a t such that if $x \in E^*(Y)/F^sE^*(Y)$ is nontrivial then there is a lift, $x' \in E^*(Y)$ such that its image in $E^*(X)/F^tE^*(X)$ is nontrivial. Another equivalent way to say it is that there is a bound on how high the map can raise filtration on elements in $E^*(Y)/F^sE^*(Y)$. We can use a couple of facts discussed in [RWY98]. First, $E^*(Z)/F^sE^*(Z)$ is always finitely presented [RWY98, Corollaries 3.13 and 4.8]. Second, it always has a finite Landweber filtration, so that the quotients look like $E^*/(v_m, \dots, v_k)$ where $k \le n$, see [RWY98, Theorem 3.10]. (We are using the E of Theorem 1.1 (and a few others in a minute) and when n isn't defined the last condition is k finite.) If n is defined then the Landweber filtration is just the tensor product of E^* with the Landweber filtration for $P(m)$ (because all of our coefficient rings are Landweber flat, [RWY98, Corollary 4.8]).

We begin by showing the result for Morava K-theory and anything which behaves like Morava K-theory, namely theories with coefficient rings $v_m^{-1}P(m)^*$, $v_m^{-1}v_n^{-1}P(m)^*$, or $v_m^{-1}E(m,n)^*$. In this case, $E^*(Y)/F^sE^*(Y)$ is always a finitely generated free module over E^* by our Landweber filtration. None of our generators for Morava K-theory get mapped to the infinite filtration since there are no phantom maps. Since we only have a finite number of generators it is easy to see we have our result. All of the other theories behave exactly the same, with the same generators.

We now do a downward induction on m for $E = E(m,n)$ and $v_n^{-1}P(m)$. Recall from our previous papers that we have a short exact sequence:

$$0 \longrightarrow E(m,n)^*(Y) \xrightarrow{v_m} E(m,n)^*(Y) \longrightarrow E(m+1,n)^*(Y) \longrightarrow 0 \tag{6.16}$$

which maps injectively into a similar short exact sequence for X. The same proofs work for the theory $v_n^{-1}P(m)$ as well. Inductively all of the elements in $E(m+1,n)^*(Y)/F^s$ are detected in $E(m+1,n)^*(X)/F^t$ for some big t. If elements in $E(m,n)^*(Y)/F^s$ are v_m torsion, then since it is finitely generated, there can only be a finite number of them to worry about and since nothing goes to the infinite filtration we can easily handle a finite situation. Our problems only come up if we have an element in $E(m,n)^*(Y)/F^s$ (the same proof works for $v_n^{-1}P(m)$) which is v_m torsion free, but which maps to an element in

$E(m,n)^*(X)/F^s$ which is v_m torsion and changes filtration indefinitely as you multiply more and more times by v_m. This is precisely the situation we had in our oversimplified example in the previous remark. In that example, if we invert v_n, one ring, R, becomes K^* and the other, R', goes away, and we lose our injection on Morava K-theory. We must use our Morava K-theories in the same way here. If we have our v_m torsion free element $x \in E(m,n)^*(Y)/F^s$ (where the topology is discrete) then it will still be nontrivial if we invert v_m and we have already handled the case of $v_m^{-1}E(m,n)$ and by naturality our x and the powers of v_m times it cannot raise filtration any higher than in $v_m^{-1}E(m,n)^*(X)$. Since we are finitely generated, there can only be a finite number of these torsion free elements to worry about so we can deal with them all at once.

To get our result up to $P(m)$ we need only use the fact that $P(m)^*(Y)/F^s$ injects into $v_n^{-1}P(m)^*(Y)/F^s$ for some high n. (The ring $v_n^{-1}P(m)^*$ is Landweber flat so we can tensor the Landweber filtration for $P(m)$ with it to get the Landweber filtration of it. Since the Landweber filtration is finite we can find an n big enough so there is no v_n torsion and our localization is injective.) Having already solved this case we are done.

We should make some noises about the $m = 0$ case here. When we are working over the p-adics, then there is more to the topology than just the skeletal filtration and in this topology everything is p torsion modulo an open set so our arguments still work ([RWY98, Just before Theorem 3.8]). When we do not need the p-adics, then we do need the injection on rational homology, usually considered Morava's zeroth K-theory, a hypothesis we have normally not needed.

Theorem 6.8 is a little more complicated because you wouldn't expect a surjection in Morava cohomology K-theory to imply the quotient topology is the same as the topology of what we are mapping onto. Our oversimplified example illustrates the point. For this we need extra information. We need the setup of (6.2) where we have not just a surjection but an exact sequence:

$$0 \longleftarrow K^*(X_1) \longleftarrow K^*(X_2) \longleftarrow K^*(X_3) \longleftarrow \cdots . \tag{6.17}$$

This will prevent us from having our oversimplified example show up. If it tries to, we can use X_3 to make it right. Otherwise the proof is quite similar. □

7. The Brown-Comenetz dual

We want this section to be as self contained as possible for readers interested in just this part of our work. The simplest case is the one we need the most, namely

$$E^*(\underline{I'BP\langle q\rangle}_r),$$

where this is just an "integral" version of the Brown-Comenetz dual of $BP\langle q\rangle$. The proof is the same for the highly connected case and needs only a minor modification for the $\mathbb{Z}/(p^c)$ case. The finite spectra of Theorem 1.13 will still need a little discussion.

Theorem 7.1. *As completed Hopf algebras,*

$$E^*(\underline{I'BP\langle q\rangle}_r) \simeq \widehat{\bigotimes_{d(I)\le r}} E^*(K(\mathbb{Z}_{(p)}, r-d(I)))$$

and is Landweber flat.

Proof. First, note that whenever we have $r < 2(p^q-1)$ we can replace our space with the $q-1$ version. We always do this when given a chance. Then, we study the map $\underline{I'BP\langle q\rangle}_r \longrightarrow \underline{BP\langle q\rangle}_{g(q)+r-q-2}$ from our section on preliminaries where we discuss the Brown-Comenetz dual, Remark 3.5. We know that $r \ge q+2$ by the previous remark. We can continue with iterated boundary maps (from the introduction) until we have

$$(7.2)\qquad \underline{I'BP\langle q\rangle}_r \longrightarrow \underline{BP\langle q\rangle}_{g(q)+r-q-2} \longrightarrow \underline{BP\langle r-2\rangle}_{g(r-2)}.$$

Note that this is independent of q! Mapping $K(\mathbb{Z}_{(p)}, r) \to \underline{I'BP\langle q\rangle}_r$ and composing, we get a sequence

$$(7.3)\qquad K(\mathbb{Z}_{(p)}, r) \longrightarrow \underline{BP\langle r-2\rangle}_{g(r-2)} \xrightarrow{v_{r-2}} \underline{BP\langle r-2\rangle}_{g(r-2)-2(p^{r-2}-1)}.$$

This is the sequence which gives rise to the exact sequence of Hopf algebras in Morava K-theory (1.26). For $d(I) < r$ we have a map

$$(7.4)\qquad \underline{I'BP\langle q\rangle}_r \xrightarrow{v^I} \underline{I'BP\langle q\rangle}_{r-d(I)} = \underline{I'BP\langle q'\rangle}_{r-d(I)} \longrightarrow \underline{BP\langle q'\rangle}_{g(q')+r-d(I)-q'-2} \longrightarrow \underline{BP\langle r-d(I)-2\rangle}_{g(r-d(I)-2)}.$$

Here we might have to replace $\underline{I'BP\langle q\rangle}_{r-d(I)}$ using a smaller q, i.e. q'. The maps

$$(7.5)\qquad \underline{I'BP\langle q\rangle}_r \xrightarrow{\times\delta v^I} \prod_{d(I)\le r} \underline{BP\langle r-d(I)-2\rangle}_{g(r-d(I)-2)} \xrightarrow{\times v_{r-d(I)-2}} \prod_{d(I)\le r} \underline{BP\langle r-d(I)-2\rangle}_{g(r-d(I)-2)-2(p^{r-d(I)-2}-1)}$$

give us an exact sequence

$$(7.6)\qquad K_* \longrightarrow K_*\underline{I'BP\langle q\rangle}_r \longrightarrow \bigotimes_{d(I)\le r} K_*\underline{BP\langle r-d(I)-2\rangle}_{g(r-d(I)-2)} \longrightarrow \bigotimes_{d(I)\le r} K_*\underline{BP\langle r-d(I)-2\rangle}_{g(r-d(I)-2)-2(p^{r-d(I)-2}-1)} \longrightarrow \cdots.$$

We are almost ready to use our Theorem 6.8. Since all of our spaces on the right are torsion free spaces with even homology, we only need the version from [RWY98]. There is one remaining thing to do. We must check that the composition of maps in (7.5) is trivial. Since the iterated boundary maps commute with multiplication by v_i it is enough to check our generic map (7.2) composed with multiplication by v_{r-2} is trivial. By the commutativity with the boundary map, it is enough to see that the map

$$\text{(7.7)} \qquad \underline{I'BP\langle q'\rangle}_{r-2(p^{r-2}-1)} \longrightarrow \underline{BP\langle r-2\rangle}_{g(r-2)-2(p^{r-2}-1)}$$

is trivial. From Theorem 3.29 we know that

$$\text{(7.8)} \qquad \underline{BP\langle q'\rangle}_{g(q')} \longrightarrow \underline{BP\langle q'+1\rangle}_{g(q'+1)-1}$$

is trivial and our map of concern factors through this. One has to worry a little about the low dimensional cases, but when $q'=0$ we don't have a composition because there is no v_0 in what we are doing. □

If we want to do the case of $E^*(\underline{I'BP\langle q\rangle}_r^{(s)})$ the same proof works, we just don't have to use as many maps. The only difference in the proof for Theorem 1.11 in the introduction and this is we must also use the maps p^c to get exactness from [RWY98]. Showing the composition is trivial is easy since p^c on our space is trivial. The connected case is similar.

Proof of Theorem 1.13. From Remark 3.5 we know we have a map of $BP\langle q\rangle_I$ to either $I'BP\langle q\rangle$ or $I^{\mathbb{Z}/(p^c)}BP\langle q\rangle$ which is split injective on homotopy. We need only restrict to the maps in the above proof which correspond to the homotopy here. □

8. CTor

We have given our motivation for looking at CTor in the introduction and so we can get to work on the calculations immediately. We can start assuming equation (1.21) in the introduction.

Proposition 8.1.

$$\operatorname{dir\,lim} \mathcal{K}^{BP\langle q\rangle_*}\mathcal{Z}_*^{K_*\underline{BP\langle m\rangle}_*} \simeq \mathcal{K}^{BP\langle q\rangle_*}\mathcal{Z}_*^{K_*\underline{BP}_*}.$$

Since homology respects direct limits it will be enough to compute

$$H_*\left(\mathcal{K}^{BP\langle q\rangle_*}\mathcal{Z}_*^{K_*\underline{BP\langle m\rangle}_*}\right)$$

for big m.

Proof. It isn't even clear that we have a map to begin with. We look at

$$\mathcal{K}^{BP\langle q\rangle_*}\mathcal{Z}_*^{K_*\underline{BP\langle m\rangle}_r}$$

where $r < g(q) + 2m - 2q$ and show that there is a map

$$\mathcal{K}^{BP\langle q\rangle_*}\mathcal{Z}_*^{K_*\underline{BP\langle m\rangle}_r} \to \mathcal{K}^{BP\langle q\rangle_*}\mathcal{Z}_*^{K_*\underline{BP}_r}.$$

Let $I = (i_{q+1}, i_{q+2}, \ldots, i_m)$ where each i_k is 0 or 1, $\ell(I) = \sum i_k$, and let $d(I) = \sum 2i_k(p^{i_k} - 1)$. Then

$$\mathcal{K}^{BP\langle q\rangle_*}\mathcal{Z}_j^{K_*\underline{BP\langle m\rangle}_r} \simeq \bigotimes_{\ell(I)=j} K_*\underline{BP\langle m\rangle}_{r+d(I)}. \tag{8.2}$$

We have $r + d(I) < g(q) + 2(m-q) + d(I) \le g(m)$. In this range we have an H-space splitting (Theorem 3.29) of all of the spaces and so we also get a Hopf algebra splitting and have maps $K_*\underline{BP\langle q\rangle}_{r+d(I)} \to K_*\underline{BP}_{r+d(I)}$. This is all we need. □

We can use the above proposition to calculate with because of the following. Note that we always have a map ($m \ge q$):

$$\mathcal{K}^{BP\langle q\rangle_*}\mathcal{Z}_*^{K_*\underline{BP\langle m+1\rangle}_r} \longrightarrow \mathcal{K}^{BP\langle q\rangle_*}\mathcal{Z}_*^{K_*\underline{BP\langle m\rangle}_r}. \tag{8.3}$$

Proposition 8.4. *For $r \le g(q) + 2m - 2q$, the map* (8.3) *induces an isomorphism on homology.*

Proof. Again, by the splitting Theorem 3.29 we can see what the kernel of the map (8.3) is. For each I with $i_{m+1} = 1$ we have a copy of $K_*\underline{BP\langle m+1\rangle}_{r+d(I)}$ and if $i_{m+1} = 0$ we have:

(8.5)

$$K_*\underline{BP\langle m+1\rangle}_{r+d(I)} \simeq K_*\underline{BP\langle m+1\rangle}_{r+d(I)+2(p^{i_{m+1}}-1)} \bigotimes K_*\underline{BP\langle m\rangle}_{r+d(I)}$$

and our kernel is $K_*\underline{BP\langle m+1\rangle}_{r+d(I)+2(p^{i_{m+1}}-1)}$. This makes it easy to calculate the homology since v_{m+1} maps the first kind of term with an $i_{m+1} = 1$ isomorphically to the second kind of term with the same I except with $i_{m+1} = 0$. Thus the homology is trivial and since we are taking the homology of a complex in a short exact sequence of complexes, the other two (i.e. the two in our proposition) must have isomorphic homologies. □

From Theorem 2.4 (iii), we have the homology of $\mathcal{K}\mathcal{Z}_*^{K_*\underline{BP\langle m\rangle}_{2m+2}}$ is given by the extension, i.e.

$$\begin{aligned} H_i\left(\mathcal{K}\mathcal{Z}_*^{K_*\underline{BP\langle m\rangle}_{2m+2}}\right) &\simeq K_* && i \ne m \\ &\simeq K_*K(\mathbb{Q}/\mathbb{Z}_{(p)}, m+1) && i = m. \end{aligned} \tag{8.6}$$

Before we proceed we must insert a previously unstated theorem. We would like to know the homology of the complex $\mathcal{K}\mathcal{Z}_*^{K_*\underline{BP\langle m\rangle}_{2m+1}}$.

Proposition 8.7.

$$H_i\left(\mathcal{KZ}_*^{K_*\underline{BP\langle m\rangle}_{2m+1}}\right) \simeq K_* \qquad i \neq m$$
$$\simeq A(m, g(m-1)+1) \qquad i = m$$

where $A(m, g(m-1)+1)$ *is an exterior algebra with* $\mathrm{Tor}^{A(m,g(m-1)+1)}(K_*, K_*)$ *an associated graded object for* $K_*K(\mathbb{Q}/\mathbb{Z}_{(p)}, m+1)$.

These A have already shown up in Theorems 2.3 and 2.11. To simplify notation, we will denote this special A, $A(m, g(m-1)+1)$ by A_m. It is the first nontrivial A in the sense that $A(m, i) = K_*$ when $i \leq g(m-1)$.

Proof. Most of the work has already been done, it is just a matter of reinterpreting it. In particular, we go back to the proof of Theorem 4.1 (ii) for $k = 0$. It works just as well for $k = -1$ if we ignore the extension and use the fact that the kernel of

(8.8) $$K_*\underline{BP\langle m\rangle}_{g(m)-1} \xrightarrow{v_{m*}} K_*\underline{BP\langle m\rangle}_{g(m-1)+1}$$

is just our A_m. This last fact we know from the proof of the identification of the GHT tensor product in Section 5. The proof is quite degenerate in the case we need since this is the first nontrivial A. $\square$

We have, using Propositions 8.1, 8.4, 8.7 and equation (8.6) completed the following calculation.

Corollary 8.9. $\mathrm{CTor}_m^{K_*[BP^*]}(K_*[BP\langle 0\rangle^*], K_*\underline{BP}_i) \simeq$

$$\begin{aligned} K_*K(\mathbb{Q}/\mathbb{Z}_{(p)}, m+1) &\qquad i = 2m+2 \\ A_m &\qquad i = 2m+1 \\ K_* &\qquad i \neq 2m+1 \textit{ or } 2m+2. \end{aligned}$$

This completes the answer to a previously unasked question. Namely, we have already calculated the homology of $\mathcal{KZ}_*^{K_*\underline{BP\langle m\rangle}_r}$ when $r \geq 2m+2$ in Theorem 2.4. We can compute the homology for lower r now. We have, using Propositions 8.1, 8.4 and Corollary 8.9 completed the following calculation.

Corollary 8.10.

$$\begin{aligned} H_j\left(\mathcal{KZ}_*^{K_*\underline{BP\langle m\rangle}_r}\right) &\simeq K_*K(\mathbb{Q}/\mathbb{Z}_{(p)}, r-m-1) &\quad& j = m \textit{ and } r \geq 2m+2 \\ &\simeq K_*K(\mathbb{Q}/\mathbb{Z}_{(p)}, j+1) && r = 2j+2 < 2m+2 \\ &\simeq A_j && r = 2j+1 < 2m+2 \\ &\simeq K_* && r \textit{ and } j \textit{ otherwise.} \end{aligned}$$

Our goal is to calculate $\mathrm{CTor}_*^{K_*[BP^*]}(K_*[BP\langle q\rangle^*], K_*\underline{BP}_*)$. We have done the $q = 0$ case now. By Propositions 8.1 and 8.4, and equation (1.21) in the introduction, we see that

$$\mathrm{CTor}_*^{K_*[BP^*]}(K_*[BP\langle q\rangle^*], K_*\underline{BP}_r) \simeq H_*\left(\mathcal{K}^{BP\langle q\rangle_*}\mathcal{Z}_*^{K_*\underline{BP\langle q\rangle}_r}\right) \tag{8.11}$$

when $r \le g(q)$. However, $\mathcal{K}^{BP\langle q\rangle_*}\mathcal{Z}_*^{K_*\underline{BP\langle q\rangle}_r}$ is just $K_*\underline{BP\langle q\rangle}_r$ which we know to be CTor_0, i.e. $K_*[BP\langle q\rangle^*]\overline{\otimes}_{K_*[BP^*]}K_*\underline{BP}_r$. So, we can begin our induction with the lemma:

Lemma 8.12. *For $r \le g(q)$, $\mathrm{CTor}_j^{K_*[BP^*]}(K_*[BP\langle q\rangle^*], K_*\underline{BP}_r)$*

$$\begin{aligned} &\simeq K_*\underline{BP\langle q\rangle}_r && j = 0 \\ &\simeq K_*[BP\langle q\rangle^*]\overline{\otimes}_{K_*[BP^*]}K_*\underline{BP}_r \end{aligned}$$

and

$$\simeq K_* \qquad\qquad j > 0.$$

The exterior algebras, A, which come into the calculations above are not of much interest to us. It makes calculations a lot easier to work modulo exterior algebras for the higher CTor groups. The Morava K-theory of Eilenberg-Mac Lane spaces, the part we are interested in, never contains exterior generators as it is concentrated in even degrees. These Hopf algebras are known to split into even degree parts and exterior Hopf algebras and there are never any maps between them, see [HRW98].

We are now ready to prove Theorem 1.20 from the Introduction. We always work modulo exterior algebras now in our CTor_j when $j > 0$. We denote this by CTorE. Working modulo the exterior part simplifies Corollary 8.9 and gives the $q = 0$ version of Theorem 1.20 which grounds our induction.

Theorem 8.13. *Let $r - g(q) - j > 0$, the only possible positive degree nontrivial* CTorE *groups are,*

$$\begin{aligned} &\mathrm{CTorE}_{r-g(q)-j}^{K_*[BP^*]}(K_*[BP\langle q\rangle^*], K_*\underline{BP}_{r+r-g(q)-j}) \\ &\qquad\qquad \simeq K_*K(\pi_{r-g_\delta(q)-j}\underline{IBP\langle q\rangle}_{r-g_\delta(q)}, r - g_\delta(q) - j). \end{aligned}$$

Proof of Theorem 1.20. We compare the answer in Theorem 1.14 to what we have here. First, we are not working modulo exterior algebras in CTor_0 and we have the right side of Theorem 1.20 is just the GHT tensor product, which is our CTor_0. The left side of Theorem 1.20 is precisely given by Theorem 8.13 above, which is written just as in Theorem 1.20. □

Proof of Theorem 8.13. We do this by induction on q having grounded our induction with the $q = 0$ case in Corollary 8.9. Our induction proceeds using a Böckstein spectral sequence which comes from the exact sequence

$$BP\langle q+1\rangle_* \xrightarrow{v_{q+1}} BP\langle q+1\rangle_* \longrightarrow BP\langle q\rangle_*.$$

This goes over to

$$K_* \longrightarrow K_*[BP\langle q+1\rangle^*] \xrightarrow{[v_{q+1}]} K_*[BP\langle q+1\rangle^*] \longrightarrow K_*[BP\langle q\rangle^*] \longrightarrow K_*.$$

From this we get a long exact sequence in CTorE. (Moding out by our exterior algebras does not destroy exactness because the category of Hopf algebras splits.) Assuming by induction that we know the q version of our CTorE we can attempt to use our Böckstein spectral sequence. First we must note that all elements of CTorE for the $q + 1$ case are $[v_{q+1}]$ torsion. This is because they must eventually land up in the groups of Lemma 8.12 which are all trivial. Now note that the j of CTorE_j for the q case is the same number used for the Eilenberg-Mac Lane space. Thus, any differential in the Böckstein spectral sequence must be a map of Morava K-theory of Eilenberg-Mac Lane spaces which are in different degrees. All such maps are trivial by [HRW98]. Thus our spectral sequence collapses. In principle there are two possibilities. The first is that an element in the q case comes from the $q + 1$ case and the element it comes from is $[v_{q+1}]$ torsion free. However, we have shown that there are no such elements. Thus, all elements must map nontrivially around the boundary. The image of each of these elements must then be infinitely $[v_{q+1}]$ divisible. Another way to say this is that the long exact sequence of CTorE relating the $q+1$ case to the q case is short exact in that the reduction of CTorE for $q+1$ to q is always trivial (remember that we are only dealing with CTorE_j for $j > 0$). The maps on CTorE precisely mimic the stable maps on homotopy which also give a short exact squence:

$$IBP\langle q\rangle \longrightarrow IBP\langle q+1\rangle \xrightarrow{v_{q+1}} \Sigma^{-2(p^{q+1}-1)} IBP\langle q+1\rangle.$$

The details are left to the reader. □

9. Unstable Bousfield localization

In this section we prove Theorem 1.24.

Proof of Theorem 1.24. We begin by showing that $\underline{L_q BP\langle q\rangle}_r^{(q)}$ is $E(q)$ local. First we note that $\underline{L_q BP\langle q\rangle}_r$ is $E(q)$ local by [Bou82, Proposition 1.3]. Next, we want $\underline{L_q BP\langle q\rangle}_r^{[q]}$ to be $E(q)$ local. This follows from Bousfield's most recent [Bou] where he shows that the $E(q)$ localization of a connected p-local Postnikov H-space preserves the j-th homotopy group for $j < q + 1$, divides the $q + 1$ group by its torsion subgroup and rationalizes the higher groups.

Then the fiber, the space we are concerned with, is also $E(q)$ local by [Bou75, Theorem 12.9].

All that remains to be shown is that our map $\underline{L_q BP\langle q\rangle}_r \longrightarrow \underline{L_q BP\langle q\rangle}_r^{(q)}$ is an $E(q)$ equivalence. This is true if it is a rational equivalence and an isomorphism on all Morava K-theories, $K(n)_*(-)$, $0 < n \leq q$, or, by [Bou99], a rational equivalence and an isomorphism on $K(q)_*(-)$ (we cannot use [Wil99b] because our spaces are not of finite type). Since we know the homotopy of the spaces, the rational equivalence is obvious.

From Theorem 2.2 (iii) we know $\underline{L_q BP\langle q\rangle}_r \longrightarrow \underline{L_q BP\langle q\rangle}_r^{(q+1)}$ gives an isomorphism on all $K(n)_*(-)$, $n \leq q$. All we need now is just to show there is a $K(n)_*(-)$ isomorphism for $n \leq q$ between the spaces: $\underline{L_q BP\langle q\rangle}_r^{(q+1)} \longrightarrow \underline{L_q BP\langle q\rangle}_r^{(q)}$. To see this we look at the bar spectral sequence for the fibration:

(9.1) $$K(G,q) \longrightarrow \underline{L_q BP\langle q\rangle}_r^{(q+1)} \longrightarrow \underline{L_q BP\langle q\rangle}_r^{(q)}.$$

Here, G is the missing homotopy group (which by sparseness is frequently zero). We use Theorem 3.4 to get our isomorphism. $K(n)_*K(G,q)$, $n < q$, is trivial, so we easily get an isomorphism of the other two spaces. For $n = q$ this is not trivial. However, if we can show the map from it is trivial then since Tor of $K(q)_*K(G,q)$ is trivial by [RW80], the result follows from Theorem 3.4. All we need to do is show that the map

$$K(q)_*K(G,q) \longrightarrow K(q)_*\underline{L_q BP\langle q\rangle}_r^{(q+1)}$$

is trivial. From Theorem 2.2 (iii) we know that this second Hopf algebra is either a polynomial algebra or an exterior algebra. Since our first Hopf algebra is even degree, we cannot have a nontrivial map to an exterior algebra. We also cannot have a map to a polynomial Hopf algebra but this is more difficult to see. The easy way for us is to note that the Dieudonné module ([SW98]) of a polynomial algebra has no elements which are p-divisible. However, all of the elements in the Dieudonné module for $K(q)_*K(G,q)$ are p-divisible ([SW98, Theorem 1.3]) (because G is a finite sum of $\mathbb{Q}/\mathbb{Z}_{(p)}$) and therefore all maps are trivial from the first Hopf algebra to the second. □

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Institut Fourier
Université de Grenoble I
U.M.R. au C.N.R.S., B. P. 74, 38402
Saint-Martin-d'Hères CEDEX France
e-mail: Takuji.Kashiwabara@ujf-grenoble.fr

Department of Mathematics
Johns Hopkins University
Baltimore, Maryland 21218
Department of Mathematics
Kyoto University
Kyoto 606-8502 Japan
e-mail: wsw@math.jhu.edu

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