AN UNSTABLE CHANGE OF RINGS FOR MORAVA E-THEORY

ROBERT THOMPSON

Abstract. The Bousfield-Kan (or unstable Adams) spectral sequence can be constructed for various homology theories such as Brown-Peterson homology theory BP, Johnson-Wilson theory $E(n)$, or Morava E-theory E_n. For nice spaces the E_2-term is given by Ext in a category of unstable comodules. We establish an unstable Morava change of rings isomorphism between $\mathrm{Ext}_{\mathcal{U}_{BP_*BP}}(BP_*, M)$ and $\mathrm{Ext}_{\mathcal{U}_{E_{n*}E_n}}(E_{n*}, E_{n*} \otimes_{BP_*} M)$ for unstable BP_*BP-comodules that are v_n-local and satisfy $I_nM = 0$. We show that the latter Ext groups can be interpreted as the continuous cohomology of the profinite monoid of endomorphisms of the Honda formal group law. By comparing this with the cohomology of the Morava stabilizer group we obtain an unstable Morava vanishing theorem when $p - 1 \nmid n$. This in turn has implications for the convergence of the Bousfield-Kan spectral sequence. When $p-1 > n$ and k is odd and sufficiently large we show that the $E(n)$-based spectral sequence for S^k has a horizontal vanishing line in E_2 and converges to the $E(n)$-completion of S^k.

1. Introduction

In [2] it is shown that the unstable Adams spectral sequence, as formulated by Bousfield and Kan [10], can be used with a generalized homology theory represented by a p-local ring spectrum E satisfying certain hypotheses, and for certain spaces X. In these cases the effectiveness of the spectral sequence is demonstrated by: 1) setting up the spectral sequence and proving convergence, 2) formulating a general framework for computing the E_2-term, and 3) computing the one and two line in the case where $E = BP$ and $X = S^{2n+1}$.

In [6] the present author and M. Bendersky showed that this framework can be extended to periodic homology theories such as the Johnson-Wilson spectra $E(n)$. However the approach to convergence in [6] is different from that in [2]. In the latter the Curtis convergence theorem is used to obtain a general convergence theorem based on the existence of a Thom map

$$H\mathbf{Z}_{(p)} \to E$$

and a tower over X. This necessitates that E be connective. Obviously this doesn't apply to periodic theories such as $E(n)$. In [6] we study a tower under X and define the E-completion of X to be the homotopy inverse limit of this tower. Convergence of the spectral sequence to the completion is guaranteed by, for example, a vanishing line in rth term of the spectral sequence. For the example of $E(1)$ and $X = S^{2n+1}$ we compute the E_2-term, for p odd, and obtain such a vanishing line.

It should be noted that the spectral sequence has been used to good effect in the work of Davis and Bendersky, in computing v_1-periodic homotopy groups of Lie Groups. It should also be noted that the construction of an E-completion given in [6] has been strongly generalized by Bousfield in [9]. Also, the framework for the construction of the spectral sequence and the computation of the E_2-term in [2] and [6] has been generalized by Bendersky and Hunton in [5] to the case of an arbitrary Landweber Exact ring spectrum E. This includes complete theories such as Morava E-theory.

In [5] the authors define an E-completion of X, and a corresponding Bousfield-Kan spectral sequence, for any space X and any ring spectrum E, generalizing the construction of [6]. If one further supposes that E is a Landweber exact spectrum then the authors show that one can define a category of unstable comodules over the Hopf algebroid $(E_*, E_*(E))$. This is accomplished by studying the primitives and indecomposables in the Hopf ring of E, extending the work of [2], [4]. Letting $\mathcal{U}$ denote this category of unstable comodules they show, for example, that if X is a space such that $E_*(X) \cong \Lambda(M)$, an exterior algebra on the E_*-module M of primitives, where M is a free E_*-module concentrated in odd degrees, then the E_2-term of the spectral sequence can be identified as

$$E_2^{s,t}(X) \cong \mathrm{Ext}^s_{\mathcal{U}}(E_*(S^t), M).$$

This is Theorem 4.1 of [5].

There remains the issue of convergence and the problem of actually computing the E_2-term. In this paper we establish bounds for the cohomological dimension of the unstable Ext groups of certain torsion unstable comodules. The means for doing this consist of first establishing an unstable version of the Morava change of rings theorem going from BP to Morava E-theory, and then identifying the unstable cohomology as the continuous cohomology of End_n, the profinite monoid of endomorphisms of Γ_n, the Honda formal group law, over $\mathbf{F}_{p^n}$. The multiplication in End_n is given by composition. The group of invertible endomorphisms is the well known Morava stabilizer group, and Morava theory tells us that the continuous cohomology of this group yields stable input into the chromatic machinery of stable homotopy theory. Unstable information is obtained by considering non-invertible endormorphisms of Γ_n as well.

In the next section we will recall the definition of the category of unstable comodules. In the following theorem, the cohomology on the right hand side is continuous monoid cohomology and Gal denotes the Galois group $\mathrm{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p)$ and E_{n*} is the coefficient ring of Morava E-theory.

Theorem 1.1. *Let M be an unstable BP_*BP-comodule, concentrated in odd degrees, on which v_n acts bijectively and satisfies $I_nM = 0$. Then there is an isomorphism*

$$\mathrm{Ext}^s_{\mathcal{U}_{BP_*BP}}(BP_*, M) \cong H^s_c(\mathrm{End}_n, (E_{n*} \otimes_{BP_*} M)_1)^{\mathrm{Gal}}.$$

In Section 6 we establish a relationship between the cohomology of End_n and the cohomology of S_n, the Morava stabilizer group. Using the cohomological dimension of S_n (see [21]) we obtain an unstable Morava vanishing theorem.

Theorem 1.2. *Let M be as in Theorem 1.1. Suppose $p-1 \nmid n$. Then*

$$\mathrm{Ext}^s_{\mathcal{U}_{BP_*BP}}(BP_*, M) = 0 \quad \textit{for} \quad s > n^2 + 1$$

Theorem 1.2 has implications for the convergence of the unstable $E(n)$-based Bousfield-Kan spectral sequence for an odd dimensional sphere. Martin Bendersky has observed that the dimension of a sphere S^k has to be sufficiently large relative to j and p in order for

$$BP_*(S^k)/I_j \xrightarrow{v_j} BP_*(S^k)/I_j$$

to be an unstable comodule map [3].

Theorem 1.3. *Let $p-1 > n$ and let be k be odd and large enough that*

$$E(n)_*(S^k)/I_j \xrightarrow{v_j} E(n)_*(S^k)/I_j$$

is an unstable comodule map for $0 \leq j \leq n-1$. Then

$$\mathrm{Ext}^s_{\mathcal{U}_{E(n)_*E(n)}}(E(n)_*, E(n)_*(S^k)) = 0 \quad \textit{for} \quad s > n^2 + n + 1.$$

Hence the unstable $E(n)$-based Bousfield-Kan spectral sequence for S^k converges to the $E(n)$-completion of S^k.

This theorem will be proved in Section 7, using a Bockstein argument and an unstable version of the Hovey-Sadofsky change of rings theorem [15].

This work came out of an extended discussion with Martin Bendersky about unstable chromatic homotopy theory and the author is very grateful for all the insight he has provided. The author also wishes to thank Mark Hovey for help in understanding the nature of faithfully flat extensions of Hopf algebroids and equivalences of categories of comodules. Also, thanks are due to Hal Sadofsky and Ethan Devinatz for several useful conversations.

2. Unstable Comodules

We begin by recalling some notions from [2] and [5]. Suppose that E is a spectrum representing a Landweber exact cohomology theory with coefficient ring concentrated in even degrees. Let $\underline{E}_*$ denote the corresponding Ω-spectrum. There are generators $\beta_i \in E_{2i}(CP^\infty)$ and under the complex orientation for complex cobordism $CP^\infty \to \underline{MU}_2$ these map to classes $E_{2i}(\underline{MU}_2)$. Localized at a prime p, denote the image of β_{p^i} by $b_{(i)} \in E_{2p^i}(\underline{E}_2)$. Let $b_i \in E_{2p^i-2}(E)$ denote the image under stabilization. Following [2] and [5], when $E = BP$, we replace the elements b_i with $h_i = c(t_i)$. For a finite sequence of integers $J = (j_1, j_2, \ldots, j_n)$ define the *length* of J to be $l(J) = j_1 + j_2 + \cdots j_n$ and define

$$b^J = b_1^{j_1} b_2^{j_2} \cdots b_n^{j_n}.$$

Definition 2.1. *Let (A, Γ) denote the Hopf algebroid (E_*, E_*E) for a Landweber exact spectrum E. Let M be a free, graded $A-$module. Define $U_\Gamma(M)$ to be sub-A-module of $\Gamma \otimes_A M$ spanned by all elements of the form $b^J \otimes m$ where $2l(J) < |m|$. Secondly, define $V_\Gamma(M)$ to be sub-A-module of $\Gamma \otimes_A M$ spanned by all elements of the form $b^J \otimes m$ where $2l(J) \leq |m|$.*

We will sometimes drop the subscript Γ from the notation if it will not cause confusion.

The following theorem was proved in [2] for $E = BP$ and in [5] for an arbitrary Landweber exact theory. Here M_s denotes a free A-module generated by one class i_s in dimension s.

Theorem 2.2. *In the Hopf ring for E the suspension homomorphism restricted to the primitives*

$$\sigma_* : PE_*(\underline{E}_s) \to U(M_s)$$

and the suspension homomorphism restricted to the indecomposables

$$\sigma_* : QE_*(\underline{E}_s) \to V(M_s)$$

are isomorphisms.

Definition 2.3. *The functors $U_\Gamma(-)$ and $V_\Gamma(-)$ are extended to arbitrary A-modules as follows. Let*

$$F_1 \to F_0 \to M \to 0$$

be exact with F_1 and F_0 free over A. Define $U_\Gamma(M)$ by

$$U_\Gamma(M) = \operatorname{coker}(U_\Gamma(F_1) \to U_\Gamma(F_0))$$

and $V_\Gamma(M)$ by

$$V_\Gamma(M) = \operatorname{coker}(v_\Gamma(F_1) \to V_\Gamma(F_0)).$$

It is shown in [2], [5] that U and V are each the functor of a comonad (U, δ, ϵ) and (V, δ, ϵ) on the category of A-modules.

Using some work from Dustin Mulcahey's thesis [19] we can extend the above to a much more general situation. Suppose (A, Γ) is a Landweber exact Hopf algebroid and $A \xrightarrow{f} B$ is a map of algebras. If we define $\Sigma = B \otimes_A \Gamma \otimes_A B$ then (B, Σ) becomes a Hopf algebroid and we have a map of Hopf Algebroids $(A, \Gamma) \to (B, \Sigma)$. The example that was treated in [19] was $A = BP_*$ and $B = K(n)_*$ but the following makes sense in general.

Definition 2.4. *Let $\mathcal{M}$ be the category of B-modules and define an endofunctor U_Σ on $\mathcal{M}$ by*

$$U_\Sigma(N) = B \otimes_A U_\Gamma(N).$$

Define a comultiplication by

$$\begin{array}{ccc} U_\Sigma(N) = B \otimes_A U_\Gamma(N) & \xrightarrow{B \otimes \Delta^\Gamma} & B \otimes_A U_\Gamma^2(N) \\ & \searrow{\scriptstyle \Delta^\Sigma} & \big\downarrow {\scriptstyle B \otimes U_\Gamma(f \otimes U_\Gamma(N))} \\ & & B \otimes_A U_\Gamma(B \otimes_A U_\Gamma(N)) \end{array}$$

and a counit

$$U_\Sigma(N) = B \otimes_A U_\Gamma(N) \xrightarrow{B \otimes \epsilon^\Gamma} B \otimes_A N \to N$$

Make an analogous definition for V_Σ.

Proposition 2.5 (See [19])**.** *The functors U_Σ and V_Σ are both comonads on the category of B-modules.*

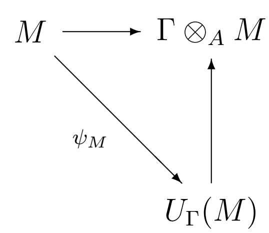
Proof. The proof is a straightforward diagram chase. □

By Corollary 2.12 of [5] this generalizes the definition of U and V in the Landweber exact case.

Definition 2.6. *Suppose* (A, Γ) *is the target of a map of Hopf algebroids from a Landweber exact Hopf algebroid as above. Let* $\mathcal{U}_\Gamma$ *denote the category of coalgebras over the comonad* U *and similarly let* $\mathcal{V}_\Gamma$ *denote the category of coalgebras over the comonad* V*. We call an object in* $\mathcal{U}_\Gamma$ *(or in* $\mathcal{V}_\Gamma$*, depending on the context) an unstable* Γ*-comodule.*

For now we will focus on the functor U but in everything that follows in this section there are analogous results for V. Keep in mind that if M is concentrated in odd dimensions, then $U(M)$ and $V(M)$ are the same.

Thus a Γ-comodule is unstable if the comodule structure map has a lifting:



The category $\mathcal{U}$ is an abelian category and the functor U restricted to $\mathcal{U}$ is the functor of a monad (U, μ, η), using the definitons $\mu = U\epsilon$ and $\eta = \psi$. The Ext groups in $\mathcal{U}$ can be computed as follows.

Definition 2.7. *Suppose* M *is an unstable comodule. Analogous to the stable case, the monad* (U, μ, η) *gives maps*

$$U^i \eta^U U^{n-i} : U^n(M) \to U^{n+1}(M),\ 0 \le i \le n,$$
$$U^i \mu^U U^{n-i} : U^{n+2}(M) \to U^{n+1}(M),\ 0 \le i \le n,$$

which define a cosimplicial object in $\mathcal{U}$ *called the cobar resolution. Apply the functor* $\hom_{\mathcal{U}}(A,\ )$ *to get a cosimplicial abelian group and hence a chain complex called the cobar complex*

$$M \xrightarrow{\partial} U(M) \xrightarrow{\partial} U^2(M) \xrightarrow{\partial} \cdots$$

with

$$\partial = \sum_{i=0}^{n} (-1)^i d^i : U^{n-1}(M) \to U^n(M).$$

Here $d^i = \hom_{\mathcal{U}}(A, U^i \eta^U U^{n-i})$ *and* $\hom_{\mathcal{U}}(A, U(N)) = N$*. Then the homology of this chain complex* $\mathrm{Ext}_{\mathcal{U}}(E_*, M)$*.*

Now consider a map of arbitrary Hopf algebroids $(A, \Gamma) \to (B, \Sigma)$. In [17] Miller and Ravenel define a pair of adjoint functors on the comodule categories

$$\Gamma\text{-comod} \underset{\pi^*}{\overset{\pi_*}{\rightleftarrows}} \Sigma\text{-comod}$$

defined by $\pi_*(M) = B \otimes_A M$ and $\pi^*(N) = (\Gamma \otimes_A B) \Box_\Sigma N$ for a Γ-comodule M and a Σ-comodule N. This adjunction is discussed in detail in several places, for example [14] and [19]. The functors π_* and π^* often define inverse equivalences of comodule categories. For example if $\Sigma = B \otimes_A \Gamma \otimes_A B$ and $A \to B$ is a faithfully flat extension of rings, then it is not difficult to see that this is the case.

Now suppose that $(A, \Gamma) \to (B, \Sigma)$ have suitably defined unstable comodule categories, as in Definition 2.6. Mulcahey defines unstable analogs of π_* and π_* in [19].

Definition 2.8. *Define functors*

$$\mathcal{U}_\Gamma \underset{\alpha^*}{\overset{\alpha_*}{\rightleftarrows}} \mathcal{U}_\Sigma$$

by $\alpha_*(M) = B \otimes_A M$ *for an unstable* Γ*-comodule* M*, and for an unstable* Σ*-comodule* N*, define* $\alpha^*(N)$ *to be the equalizer*

$$\alpha^*(N) \longrightarrow U_\Gamma(N) \underset{U_\Gamma(\beta) \circ \Delta_\Gamma}{\overset{U_\Gamma(\psi_N)}{\rightrightarrows}} U_\Gamma U_\Sigma(N)$$

where $\beta : U_\Gamma(N) \to U_\Sigma(N)$.

Proposition 2.9 (See [19])**.** *The functors* α_* *and* α^* *form an adjoint pair.*

Proof. This follows by considering the map

$$B \otimes_A U_\Gamma(M) \to B \otimes_A U_\Gamma(B \otimes_A M)$$

which is natural in the A-module M and gives a morphism of comonads $U_\Gamma \to U_\Sigma$ which leads to the adjoint pair on comodule categories. See [19] for the details. □

3. Faithfully Flat Extensions

The following theorem is an unstable version of a theorem due to Mike Hopkins, Mark Hovey, and Hal Sadofsky. See [12], [13], and [15]. Hovey's paper [13] has a detailed proof of the theorem in the form that we need, which is stated below as Theorem 3.2. The proof in [13] is based on a study of the category of quasi-coherent sheaves on a groupoid scheme. That theory has not yet been developed in an unstable setting but we don't need that for the present work. The author is very grateful to Mark Hovey for a detailed discussion of various aspects of Theorem 3.2 below.

Theorem 3.1. *Suppose $(A,\Gamma)\to(B,\Sigma)$ is a map of Hopf algebroids, both satisfying the conditions of Definition of 2.6 and thus possessing unstable comodule categories. Assume that $\Sigma = B\otimes_A\Gamma\otimes_A B$, and that there exists an algebra C along with an algebra map $B\otimes_A\Gamma\xrightarrow{g}C$ such that the composite*

$$A\xrightarrow{1\otimes\eta_R}B\otimes_A\Gamma\xrightarrow{g}C$$

is a faithfully flat extension of A-modules. To be explicit the first map is the one that takes a to $1\otimes\eta_R(a)$. Then α_* and α^* of 2.8 are adjoint inverse equivalences of categories.

The existence of the map g satisfying the stated condition generalizes the condition of $A\to B$ being faithfully flat.

Theorem 3.2 (M. Hovey [13])**.** *Let $(A,\Gamma)\to(B,\Sigma)$ be a map of Hopf algebroids such that $\Sigma = B\otimes_A\Gamma\otimes_A B$, and assume there exists an algebra C along with an algebra map $B\otimes_A\Gamma\xrightarrow{g}C$ such that the composite*

$$A\xrightarrow{1\otimes\eta_R}B\otimes_A\Gamma\xrightarrow{g}C$$

is a faithfully flat extension of A-modules. Then

$$\Gamma\text{-}comod\xrightarrow{\pi_*}\Sigma\text{-}comod$$

is an equivalence of categories.

This enables the following lemma.

Lemma 3.3. *Recall that $\alpha_*:\mathcal{U}_\Gamma\to\mathcal{U}_\Sigma$ is given by $\alpha_*(M)=B\otimes_A M$. Let $f:M\to N$ be a morphism in $\mathcal{U}_\Gamma$. Then $\alpha_*(f)$ is an isomorphism if and only if f is an isomorphism. Furthermore α_* is exact.*

Proof. By Theorem 3.2 the functor π_* is exact since an equivalence of abelian categories is an exact functor. An unstable Γ-comodule map is a stable Γ-comodule and a sequence in $\mathcal{U}_\Gamma$ is exact if and only if it's exact in Γ-comod, so α_* is exact on $\mathcal{U}_\Gamma$. A similarly argument gives the first statement. □

Proof of Theorem 3.1. For $N \in \mathcal{U}_\Sigma$ consider the counit of the adjunction

$$\alpha_* \alpha^* N \to N.$$

By Lemma 3.3 $\alpha_* \alpha^* N = B \otimes_A \alpha^*(N)$ sits in an equalizer diagram

$$B \otimes_A \alpha^*(N) \longrightarrow B \otimes_A U_\Gamma(N) \underset{B \otimes_A U_\Gamma(\beta) \circ \Delta_\Gamma}{\overset{B \otimes_A U_\Gamma(\psi_N)}{\rightrightarrows}} B \otimes_A U_\Gamma U_\Sigma(N)$$

which is the same thing as

$$B \otimes_A \alpha^*(N) \longrightarrow U_\Sigma(N) \underset{\Delta_\Sigma}{\overset{U_\Sigma(\psi_N)}{\rightrightarrows}} U_\Sigma U_\Sigma(N)$$

because $\Sigma = B \otimes_A \Gamma \otimes_A B$. It follows that $B \otimes_A \alpha^* N \cong N$.

For $M \in \mathcal{U}_\Gamma$ look at the unit of the adjunction

$$M \to \alpha^* \alpha_* M.$$

The target sits in an equalizer diagram

$$\alpha^* \alpha_* M \to U_\Gamma(B \otimes_A M) \rightrightarrows U_\Gamma U_\Sigma(B \otimes_A M)$$

Tensor this with B

$$B \otimes_A \alpha^* \alpha_* M \to B \otimes_A U_\Gamma(B \otimes_A M) \rightrightarrows B \otimes_A U_\Gamma U_\Sigma(B \otimes_A M)$$

which gives

$$B \otimes_A \alpha^* \alpha_* M \to U_\Sigma(B \otimes_A M) \rightrightarrows U_\Sigma U_\Sigma(B \otimes_A M)$$

So $B \otimes_A \alpha^* \alpha_* M \cong B \otimes_A M$. The unit of the adjunction is un unstable Γ-comodule map so Lemma 3.3 applies and we have $M \xrightarrow{\cong} \alpha^* \alpha_* M$. □

This equivalence of abelian categories induces a change of rings isomorphism of Ext groups. To be explicit we have

Theorem 3.4. *Assume the hypotheses of 3.1. Then for any unstable Γ-comodule M, there is an isomorphism*

$$\mathrm{Ext}_{\mathcal{U}_\Gamma}(A, M) \to \mathrm{Ext}_{\mathcal{U}_\Sigma}(B, B \otimes_A M).$$

First we make an observation.

Lemma 3.5. *For an unstable Σ-comodule N we have $\alpha^* U_\Sigma(N) = U_\Gamma(N)$.*

Proof. We have

$$U_\Gamma(N) \to \alpha^* U_\Sigma(N). \tag{3.6}$$

Tensor with B to get

$$B \otimes_A U_\Gamma(N) \to B \otimes_A \alpha^* U_\Sigma(N)$$

which is

$$U_\Sigma(N) \xrightarrow{\cong} U_\Sigma(N).$$

By Lemma 3.3 the map 3.6 is an isomorphism. □

Proof of 3.4. Let

$$N \to U_\Sigma(N) \to U_\Sigma^2(N) \to U_\Sigma^3(N) \to \dots$$

be the unstable cobar complex for N. Apply α^* to get

$$\alpha^* N \to \alpha^* U_\Sigma(N) \to \alpha^* U_\Sigma^2(N) \to \alpha^* U_\Sigma^3(N) \to \dots \tag{3.7}$$

Now tensor with B to get the unstable cobar resolution back again

$$N \to U_\Sigma(N) \to U_\Sigma^2(N) \to U_\Sigma^3(N) \to \dots$$

which is acyclic. By Lemma 3.3 the complex 3.7 is acyclic too.

By Lemma 3.5 $\alpha^* U_\Sigma(N) = U_\Gamma(N)$ so 3.7 is a resolution of $\alpha^* N$ by injective unstable Γ-comodules

$$\alpha^* N \to U_\Gamma(N) \to U_\Gamma U_\Sigma(N) \to U_\Gamma U_\Sigma^2(N) \to \dots.$$

Apply $\mathrm{Hom}_{\mathcal{U}_\Gamma}(A, \_)$ to get

$$N \to U_\Sigma(N) \to U_\Sigma^2(N) \dots$$

which is the Σ-cobar complex for N. This shows that

$$\mathrm{Ext}_{\mathcal{U}_\Gamma}(A, \alpha^* N) \to \mathrm{Ext}_{\mathcal{U}_\Sigma}(B, N).$$

Apply this to the case $N = \alpha_* M$ to get the result. □

4. Morava E-theory

This section is based on the work of Morava [18]. We will closely follow the exposition of Devinatz [11]. Let $W\mathbf{F}_{p^n}$ denote the Witt ring over $\mathbf{F}_{p^n}$, the complete local p-ring having $\mathbf{F}_{p^n}$ as its residue field. Let σ denote the generator of the Galois group $\mathrm{Gal} = \mathrm{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p)$ which is cyclic of order n. Note that Gal acts on $W\mathbf{F}_{p^n}$ by

$$(\sum_i w_i p^i)^\sigma = \sum_i w_i^p p^i$$

where the coefficients w_i are multiplicative representatives.

Let Γ_n be the height n Honda formal group law over a field k of characteristic p. The endomorphism ring of Γ_n over $k = \mathbf{F}_{p^n}$, denoted End_n, is known and is given by (see [21])

$$\mathrm{End}_n = W\mathbf{F}_{p^n}\langle S\rangle/(S^n = p, Sw = w^\sigma S).$$

Here one can think of S as a non-commuting indeterminant.

We will think of End_n as a monoid under multiplication. The submonoid consisting of invertible elements is the Morava Stabilizer Group $S_n = (\mathrm{End}_n)^\times$. Also, Gal acts on End_n and hence on S_n.

Morava E-theory, also referred to as Lubin-Tate theory, is a Landweber exact homology theory represented by a spectrum denoted E_n and corresponding to the completed Hopf algebroid

$$(E_{n*}, \mathrm{Map}_c(S_n, W\mathbf{F}_{p^n})^{\mathrm{Gal}}\hat{\otimes}E_{n*}).$$

Here Map_c refers to the set of continuous maps, and the coefficient ring has the following description:

$$E_{n*} = W\mathbf{F}_{p^n}[[u_1, \ldots, u_{n-1}]][u, u^{-1}].$$

The ring E_{n*} is graded by $|u_i| = 0$ and $|u| = -2$. There is a graded map of coefficients $BP \xrightarrow{r} E_{n*}$ given by

$$r(v_i) = \begin{cases} u_i u^{1-p^i} & i < n \\ u^{1-p^n} & i = n \\ 0 & i > n. \end{cases} \tag{4.1}$$

Theorem 4.2. *Let M be an unstable BP_*BP-comodule such that $I_nM = 0$ and v_n acts bijectively. Then*

$$\mathrm{Ext}^s_{\mathcal{U}_{BP_*BP}}(BP_*, M) \cong \mathrm{Ext}^s_{\mathcal{U}_{E_{n*}E_n}}(E_{n*}, E_{n*} \otimes_{BP_*} M).$$

Proof. We begin by considering the Johnson-Wilson spectrum $E(n)$, which is described in detail in [21]. The coefficients are given by

$$E(n) = Z_{(p)}[v_1, \ldots, v_{n-1}, v_n, v_n^{-1}].$$

There is an evident map of spectra inducing a ring map on coefficients given by Equation 4.1. The Johnson-Wilson spectrum is studied unstably by Roland Kargl in his thesis [16]. According to the main result of [16], for M satisfying the hypothesis above, the map $BP \to E(n)$, induces an isomorphism

$$\mathrm{Ext}^s_{\mathcal{U}_{BP_*BP}}(BP_*, M) \cong \mathrm{Ext}^s_{\mathcal{U}_{E(n)_*E(n)}}(E(n)_*, E(n)_* \otimes_{BP_*} M). \tag{4.3}$$

Note that Kargl's theorem is a special case of the unstable Hovey-Sadofsky change of rings theorem which is proved in section 7.

For the second step we complete Johnson-Wilson theory with respect to the ideal $I_n = (p, v_1, \dots, v_{n-1})$. The result has coefficients

$$\widehat{E(n)}_* = Z_p[[v_1, \dots, v_{n-1}]][v_n, v_n^{-1}]$$

and is sometimes referred to as Baker-Wurgler theory because this cohomology theory was studied in [1].

The completion map $E(n) \to \widehat{E(n)}$ induces a map of cobar complexes for M and because M is I_n torsion, it is I_n complete, so this map of cobar of complexes is an isomorphism and we get an isomorphism

(4.4)
$$\mathrm{Ext}^s_{\mathcal{U}_{E(n)_*E(n)}}(E(n)_*, E(n)_* \otimes_{BP_*} M) \cong \mathrm{Ext}^s_{\mathcal{U}_{\widehat{E(n)}_*\widehat{E(n)}}}(\widehat{E(n)}_*, \widehat{E(n)}_* \otimes_{BP_*} M).$$

For the final step consider the map of spectra from Baker-Wurgler to Lubin-Tate: $\widehat{E(n)} \to E_n$. This map of spectra is induced by the faithfully flat extension of coefficient rings

$$\mathbf{Z}_p[[v_1, \dots, v_{n-1}]][v_n, v_n^{-1}] \to W\mathbf{F}_{p^n}[[u_1, \dots, u_{n-1}]][u, u^{-1}].$$

induced by formula 4.1. Thus Theorem 3.4 applies and we get an isomorphism

(4.5)
$$\mathrm{Ext}^s_{\mathcal{U}_{\widehat{E(n)}_*\widehat{E(n)}}}(\widehat{E(n)}_*, \widehat{E(n)}_* \otimes_{BP_*} M) \cong \mathrm{Ext}^s_{\mathcal{U}_{E_{n*}E_n}}(E_{n*}, E_{n*} \otimes_{BP_*} M).$$

Putting 4.3, 4.4, and 4.5 together finishes the proof of Theorem 4.2. □

5. More on Unstable Comodules

Now we give the description of unstable comodules in Morava E-theory that we are after. Start by recalling from [11] that there is a Hopf algebroid (U, US) which is equivalent to (BP_*, BP_*BP) and lies between (BP_*, BP_*BP) and $(E_{n*}, E_{n*}E_n)$. The affine groupoid scheme $(\operatorname{Spec} U, \operatorname{Spec} US)$ is the scheme whose value on a ring A is the groupoid whose objects consist of the set of pairs (F, a) where F is a formal group law over A, a is a unit in A, and a morphism $(F, a) \to (G, b)$ is an isomorphism $f : F \to G$ with $a = f'(0)b$. As graded algebras,

$$U = \mathbf{Z}_{(p)}[u_1, u_2, \dots][u, u^{-1}]$$

and

$$US = U[s_0^{\pm 1}, s_1, s_2, \dots]$$

with $|u_i| = 0$, for $i \geq 1$, $|s_i| = 0$ for $i \geq 0$, and $|u| = -2$. The map

$$(\operatorname{Spec} U, \operatorname{Spec} US) \xrightarrow{\lambda} (\operatorname{Spec} BP_*, \operatorname{Spec} BP_*BP),$$

which sends (F, a) to the formal group law F^a given by $F^a(x, y) = a^{-1}F(ax, ay)$ is represented by the graded algebra map

$$\lambda(v_i) = u_i u^{-(p^i-1)}$$
$$\lambda(t_i) = s_i u^{-(p^i-1)} s_0^{-p^i}$$

The map $(BP_*, BP_*BP) \xrightarrow{\lambda} (U, US)$ is obtained by a faithfully flat extension of coefficient rings, hence by Hopkins' theorem induces an equivalence of comodule categories. We want to identify the unstable comodule category. Unstably it is preferable to use the generators for BP_*BP given by $h_i = c(t_i)$ where c is the canonical antiautomorphism. In US define $c_i = c(s_i)$ and note that

$$c_0 = c(s_0) = s_0^{-1}$$

and

$$\eta_R(u) = c(\eta_L(u)) = s_0 u = c_0^{-1} u.$$

Morava E-theory is obtained from (U, US) by killing off u_i for $i > n$, setting $u_n = 1$, completing with respect to the ideal $I = (u_1, u_2, \dots, u_{n-1})$, and tensoring with the Witt ring $W\mathbf{F}_{p^n}$. We have

$$(E_{n*}, E_{n*}E_n) = (E_{n*}, E_{n*}[c_0^{\pm 1}, c_1, \dots] \hat{\otimes}_U E_{n*}).$$

Applying the canonical anti-automorphism χ to the map λ we get

$$\begin{aligned}
\lambda(h_i) &= c_i (s_0 u)^{-(p^i-1)} c(s_0)^{-p^i} \\
&= c_i s_0^{-(p^i-1)} u^{-(p^i-1)} s_0^{p^i} \\
&= c_i u^{-(p^i-1)} s_0 \\
&= c_i u^{-(p^i-1)} c_0^{-1}
\end{aligned}$$

Let $K = (k_1, k_2, \dots)$ be a finite sequence of non-negative integers and denote $h_1^{k_1} h_2^{k_2} \dots$ by h^K and similiarly $c_1^{k_1} c_2^{k_2} \dots$ by c^K. Also denote

$$|K| = k_1(p-1) + k_2(p^2 -) + \dots$$

and

$$l(K) = k_1 + k_2 + \dots.$$

Then we have

$$\lambda(h^K) = c^K u^{-|K|} c_0^{-l(K)}.$$

If M is a (BP_*, BP_*BP)-comodule with coaction

$$M \xrightarrow{\psi} BP_*BP \otimes_{BP_*} M$$

then for each $x \in M$ we have

$$\psi(x) = \sum_K v_K h^K \otimes m_K$$

where the sum is indexed over sequences K. The coefficient v_K is just some element in BP_*. For each term in the sum we make the following calculation. Assume m_K is even.

$$\begin{aligned}
\lambda(v_K h^K) \otimes m_K &= u_K u^{-|v_K|/2} c^K u^{-|K|} c_0^{-l(K)} \otimes m_K u^{|m_K|/2} u^{-|m_K|/2} \\
&= u_K u^{-|v_K|/2} c^K u^{-|K|} c_0^{-l(K)} \eta_R(u^{-|m_K|/2}) \otimes m_K u^{|m_K|/2} \\
&= u_K u^{-|v_K|/2} c^K u^{-|K|} c_0^{-l(K)} (s_0 u)^{-|m_K|/2} \otimes m_K u^{|m_K|/2} \\
&= u_K u^{(-|v_K|/2 - |m_K|/2 - |K|)} c^K c_0^{-l(K)+|m_K|/2} \otimes m_K u^{|m_K|/2} \\
&= u_K u^{(-|v_K|/2 - |m_K|/2 - |K|)} c^K c_0^{-l(K)+|m_K|/2} \otimes y
\end{aligned}$$

where $|y| = 0$. In the case where $|m_k|$ is odd, multiply and divide on the right by $u^{(|m_K|-1)/2}$ resulting in y on the right with $|y| = 1$. In either case, by stipulating that the right hand tensor factor be dimension zero or one, the exponent of c_0 becomes well defined, and we make the following definition.

Definition 5.1. *A comodule M over (U, US), or over $(E_{n*}, E_{n*}E_n)$, is called non-negative if for each $x \in M$ and each term in the coaction of x, written so that the right hand tensor factor has dimension either zero or one, the exponent of c_0 is non-negative.*

Recall the category of unstable comodules $\mathcal{V}_\Gamma$ defined in 2.6.

Proposition 5.2. *The categories $\mathcal{V}_{BP_*BP}$ and $\mathcal{V}_{US}$ are both equivalent to the category of non-negative US-comodules. The category $\mathcal{V}_{E_{n*}E_n}$ is equivalent to the category of non-negative $E_{n*}E_n$-comodules.*

Proof. For BP, by definition a BP_*BP-comodule is unstable if the coaction of each element is in the BP_*-span of elements of the form $h^K \otimes m_K$ where $2l(K) \leq |m_K|$. For (U, US) and $E_{n*}(E_n)$ the same argument applies, using the generators $b_i = c_i u^{-(p^i-1)} c_0^{-1}$ which are the images under λ of h_i. (Refer to [2] or [5].) □

Note: This is an analog for height n of a height ∞ result which is described in [20] (Theorem 4.1.4) and [7] (Section 4 and Appendix B). It is classical that the dual Steenrod algebra is a group scheme which represents the automorphism group of the additive formal group law. If one considers endomorphisms of the additive formal group law, not necessarily invertible, the representing object of this monoid scheme is a bialgebra, i.e. a 'Hopf algebra without an antiautomorphism'. At the prime 2 this is described explicitly in [7] (see Section 4 and Appendix B) where this bialgebra is called the extended Milnor coalgebra. Whereas the classical dual Steenrod is expressed as $\mathcal{S}_* = \mathbf{Z}/2[\xi_1, \xi_2, \dots]$, the extended Milnor coalgebra is $\mathcal{A} = \mathbf{Z}/2[a_0^{\pm 1}, a_1, a_2, \dots]$. It is easy to see that there is an equivalence between the category of graded comodules over $\mathcal{S}_*$ and the category of comodules over $\mathcal{A}$. It is also easy to see that under this equivalence, the category of graded unstable comodules over $\mathcal{S}_*$ is equivalent to the category of 'positive' $\mathcal{A}$-comodules, i.e. comodules over the bialgebra $\mathcal{A}^+ = \mathbf{Z}/2[a_0, a_1, a_2, \dots]$. In [20] this result is extended to odd primes and generalized. Our Proposition 5.2 is a version for the Landweber-Novikov algebra. This goes back to [18].

We want to translate this to the $\mathbf{Z}/2$-graded case. Consider the Hopf Algebroid $((E_n)_0, (E_n)_0(E_n))$ of elements in degree 0. Let M be an $E_{n*}E_n$-comodule and let $\overline{M}$ consist of elements in M of degree zero or one. The functor $M \mapsto \overline{M}$ is an equivalence between the category of $E_{n*}E_n$-comodules and $(E_n)_0(E_n)$-comodules. There is an isomorphism

$$\mathrm{Ext}^s_{E_{n*}E_n}(A, M) \to \mathrm{Ext}^s_{(E_n)_0(E_n)}(A_0, \overline{M})$$

It is immediate that the non-negative comodules defined in 5.1 correspond to the comodules over the Hopf algebroid

$$((E_n)_0, (E_n)_0[c_0, c_1, \dots]\hat{\otimes}(E_n)_0)$$

and we have

Proposition 5.3. *For an unstable $E_{n*}(E_n)$-comodule M there is an isomorphism*

$$\mathrm{Ext}^s_{\mathcal{V}_{E_{n*}}}(E_{n*}, M) \to \mathrm{Ext}^s_{(E_n)_0[c_0, c_1, \dots]\otimes(E_n)_0}((E_n)_0, \overline{M})$$

The next step is to interpret an unstable $(E_n)_0(E_n)$-comodule in terms of a continuous action of the monoid End_n. According to Morava theory there is an isomorphism of Hopf algebroids

$$(E_{n*}, E_{n*}(E_{n*})) \cong (E_{n*}, \mathrm{Map}_c(S_n, W\mathbf{F}_{p^n})^{\mathrm{Gal}}\hat{\otimes}E_{n*}). \tag{5.4}$$

The category of graded, complete comodules over this Hopf algebroid is equivalent to the category of continuous, filtered, Galois equivariant twisted $S_n - E_{n*}$ modules. See [11], Section 4, for precise details.

We will be interested in $\mathbf{Z}/2$-graded comodules M which satisfy $I_n M = 0$. Mod I_n, in degree zero, the Hopf algebroid of equation 5.4 becomes

$$(\mathbf{F}_{p^n}, \mathrm{Map}_c(S_n, \mathbf{F}_{p^n})^{\mathrm{Gal}}\hat{\otimes}\mathbf{F}_{p^n}) = (\mathbf{F}_{p^n}, \mathbf{F}_{p^n}[c_0^{\pm 1}, c_1, \dots]/(c_i^{p^n} - c_i)).$$

The explicit description of the group scheme of automorphisms of Γ_n over an $\mathbf{F}_p$-algebra k is as follows. Let $D = \mathbf{F}_p[c_0^{\pm 1}, c_1, \dots]/(c_i^{p^n} - c_i)$.

In [21] it is shown that every automorphism of Γ_n has the form

$$f(x) = \sum_{i\geq 0}^{\Gamma_n} a_i x^{p^i}, \quad a_i \in k,\ a_0 \in k^\times$$

and

$$f(\Gamma_n(x,y)) = \Gamma_n(f(x), f(y)).$$

For a ring map $h : D \to k$ let h give the automorphism

$$f(x) = \sum_{i\geq 0}^{\Gamma_n} h(c_i) x^{p^i}.$$

If we do not require the cooefficient of x to be a unit, then it is apparent that $\mathrm{Spec}(\mathbf{F}_p[c_0, c_1, \ldots]/(c_i^{p^n} - c_i))$ is the monoid scheme whose value on k is the monoid of endomorphisms of Γ_n over k.

Proposition 5.5. *There is an isomorphism of bialgebras*

$$(\mathbf{F}_{p^n}, \mathrm{Map}_c(\mathrm{End}_n, \mathbf{F}_{p^n})^{\mathrm{Gal}} \hat{\otimes} \mathbf{F}_{p^n}) = (\mathbf{F}_{p^n}, \mathbf{F}_{p^n}[c_0, c_1, \ldots]/(c_i^{p^n} - c_i)).$$

Proof. The proof given in Section four of [11] applies to End_n as well. In particular equation (4.14) establishes the result one generator at a time. □

So we are studying left comodules over the bialgebra $\mathrm{Map}_c(\mathrm{End}_n, \mathbf{F}_{p^n})^{\mathrm{Gal}}$. Still following [11], given a left comodule M with coaction

$$M \xrightarrow{\psi_M} \mathrm{Map}_c(\mathrm{End}_n, \mathbf{F}_{p^n})^{\mathrm{Gal}} \hat{\otimes} M \cong \mathrm{Map}_c(\mathrm{End}_n, \mathbf{F}_{p^n} \otimes M)^{\mathrm{Gal}}$$

define a right action of End_n on $\mathbf{F}_{p^n} \otimes M$ by

$$(a \otimes m)g = a\psi_M(m)(g).$$

Note that this is a right action.

Proposition 5.6. *The functor $M \to \mathbf{F}_{p^n} \otimes M$ is an equivalence from the category of $\mathbf{Z}/2$-graded complete $\mathrm{Map}_c(\mathrm{End}_n, \mathbf{F}_{p^n})^{\mathrm{Gal}}$-comodules to the category of continuous filtered Galois equivariant right End_n-modules.*

Proof. This is the End_n-analog of proposition 5.3, $\mod I_n$, in [11], and the same proof applies. □

Finally, the change of rings theorem takes the form

Theorem 5.7. *Let M be an unstable BP_*BP-comodule such that $I_n M = 0$ and v_n acts bijectively. Then*

$$\mathrm{Ext}^s_{\mathcal{V}_{BP_*BP}}(BP_*, M) \cong H^s_c(\mathrm{End}_n; \overline{E_{n*} \otimes_{BP_*} M})^{\mathrm{Gal}}.$$

Proof. Again the proof is a straightforward adaptation of the proof given in [11]. The cohomology of End_n with coefficients in a right module N can be defined by the cochain complex

$$C^k(\mathrm{End}_n; N) = \mathrm{Map}_c(\mathrm{End}_n \times \cdots \times \mathrm{End}_n, N)$$

with differential

$$\begin{aligned} df(g_1, \dots, g_{k+1}) = f(g_2, \dots, g_{k+1}) \\ + \sum_{j=1}^{k} (-1)^j (g_1, \dots, g_j g_{j+1}, \dots, g_{k+1}) \\ + (-1)^{k+1} f(g_1, \dots, g_k) g_{k+1}. \end{aligned}$$

The cobar complex for $\mathrm{Map}_c(\mathrm{End}_n, \mathbf{F}_{p^n})^{\mathrm{Gal}}$ is isomorphic to $C^*(\mathrm{End}_n; N)^{\mathrm{Gal}}$, the only difference from [11] being that we are interpreting the action as a right action. □

Finally note that if M is concentrated in odd degrees, then

$$\mathrm{Ext}^s_{\mathcal{V}_{BP_*BP}}(BP_*, M) \cong \mathrm{Ext}^s_{\mathcal{U}_{BP_*BP}}(BP_*, M)$$

and we obtain Theorem 1.1 of the introduction.

6. Cohomological Dimension

This section is based on a construction used by Bousfield - see for example [8], Subsection 3.1. Here we carry out a version for the stabilizer group. Let $\mathcal{A}$ denote the category consisting of p-complete abelian groups with a continuous right action of S_n. Let $\mathcal{E}$ denote the category consisting of p-complete abelian groups with a continuous right action of End_n. There is an obvious forgetful functor $J : \mathcal{E} \to \mathcal{A}$.

Definition 6.1. *We define a functor $\tilde{F} : \mathcal{A} \to \mathcal{E}$ as follows: For an $\mathcal{A}$-module M, let $\tilde{F}(M)$ be $M \times M \times M \dots$ as an abelian group. For $g \in S_n$, $x = (x_1, x_2, \dots) \in M$, define*

$$xg = (x_1 g, x_2 g^{\sigma}, x_3 g^{\sigma^2}, \dots, x_n g^{\sigma^{(n-1)}}, x_{n+1} g, \dots).$$

For $S \in \mathrm{End}_n$, define $xS = (0, x_1, x_2, \dots)$. This defines an End_n action on $\tilde{F}(M)$ as one can check the relation $xSg = xg^{\sigma}S$.

Proposition 6.2. *The functor $\tilde{F}$ is left adjoint to J.*

Proof. The unit of the adjunction $M \to J\tilde{F}(M)$ is given by $x \mapsto (x, 0, 0, \dots)$. The counit of the adjunction $\tilde{F}J(M) \to M$ is given by

$$(x_1, x_2, x_3, \dots) \mapsto x_1 + x_2 S + x_3 S^2 + \dots.$$

which converges by p-completeness and the fact that $S^n = p$. □

Proposition 6.2 says $\mathrm{Hom}_{\mathcal{E}}(\tilde{F}(M), N) \cong \mathrm{Hom}_{\mathcal{A}}(M, JN)$. We would like a similar statement for Ext.

Proposition 6.3. *The functors $\tilde{F}$ and J are exact. It follows that for all s, $\mathrm{Ext}^s_{\mathcal{E}}(\tilde{F}(M), N) \cong \mathrm{Ext}^s_{\mathcal{A}}(M, JN)$.*

Proof. Straightforward. □

Now we need a fundamental sequence. For an object M in $\mathcal{A}$ define an object M' in $\mathcal{A}$ as follows. Let $M' = M$ as abelian groups and for each $g \in S_n$, $x' \in M'$, let $x'g = xg^{\sigma}$, where $x = x'$ and the expression on the right is the action on M. If N is an object in $\mathcal{E}$ there is a map $S : N \to N$. If we think of S as a map $S : (JN)' \to JN$ then it is easy to check that S is a morphism in $\mathcal{A}$. Thus we can define $\tilde{F}(S)$ which is a morphism in $\mathcal{E}$. Also it is readily checked that $S : \tilde{F}((JN)') \to \tilde{F}(JN)$ is a morphism in $\mathcal{E}$. This gives a morphism

$$\partial = \tilde{F}(S) - S : \tilde{F}((JN)') \to \tilde{F}(JN)$$

in $\mathcal{E}$ and the following proposition.

Proposition 6.4. *There is a SES in $\mathcal{E}$*

$$0 \to \tilde{F}((JN)') \xrightarrow{\partial} \tilde{F}(JN) \to N \to 0.$$

Corollary 6.5. *There is a LES for any End_n-module L.*

$$\cdots \to \mathrm{Ext}^s_{\mathcal{E}}(N, L) \to \mathrm{Ext}^s_{\mathcal{A}}(JN, JL) \to \mathrm{Ext}^s_{\mathcal{A}}((JN)', JL) \to \mathrm{Ext}^{s+1}_{\mathcal{E}}(N, L) \cdots \to$$

Corollary 6.6. *Suppose L is a continuos Galois equivariant $End_n - E_{n*}$-module satisfying $I_n L = 0$. Let $p-1 \nmid n$. Then $H^s_c(\mathrm{End}_n, L)^{\mathrm{Gal}} = 0$ for $s > n^2 + 1$.*

Proof. It is well known that for the stated values of n and p, the Morava stabilizer group has finite cohomological dimension equal to n^2. See for example [21]. We have that $H^s_c(\mathrm{End}_n, L)^{\mathrm{Gal}}$ is a subgroup of $H^s_c(\mathrm{End}_n, L)$. By Corollary 6.5 the result follows. □

This implies Theorem 1.2 of the introduction.

7. A vanishing line for the $E(n)$-Bousfield-Kan spectral sequence

In this section we prove Theorem 1.3. First we need an unstable version of the Hovey-Sadofsky change of rings Theorem 3.1 of [15].

Theorem 7.1. *Let $n \geq j$ and suppose M is an unstable BP_*BP-comodule on which v_j acts isomorphically and $I_jM = 0$. Then the map*

$$\mathrm{Ext}^s_{\mathcal{U}_{BP_*BP}}(BP_*, M) \to \mathrm{Ext}^s_{\mathcal{U}_{E(n)_*E(n)}}(E(n)_*, E(n)_* \otimes_{BP_*} M).$$

is an isomorphism.

Proof. Just as in the stable setting, the theorem reduces to the map of Hopf algebroids

$$(v_j^{-1}BP_*/I_j, v_j^{-1}BP_*BP/I_j) \to (v_j^{-1}E(n)_*/I_j, v_j^{-1}E(n)_*E(n)/I_j).$$

It is proved in [15] that the faithfully flat condition of Theorem 3.1 is satisfied for this map. The unstable comodule structure comes from the map

$$BP_* \to v_j^{-1}BP_*/I_j \to v_j^{-1}E(n)_*/I_j.$$

Thus Theorem 3.1 applies and the conclusion follows.

□

Recall from the introduction that we are assuming that k is odd and sufficiently large relative to n and p so that

$$BP_*(S^k)/I_j \xrightarrow{v_j} BP_*(S^k)/I_j$$

defines a map in the category $\mathcal{U}_{BP_*BP}$ of unstable comodules over BP_*BP for $0 \leq j \leq n-1$. For precise bounds on how large k needs to be see [3]. We will suppress S^k from the notation.

Proof of Theorem 1.3. Working in the category $\mathcal{U}_{E(n)_*E(n)}$, start with the Bockstein sequence in $\mathrm{Ext}_{\mathcal{U}}$ coming from the short exact sequences of comodules

$$0 \to E(n)_*/I_{n-1} \xrightarrow{v_{n-1}} E(n)_*/I_{n-1} \to E(n)_*/I_n \to 0.$$

If $x \in \mathrm{Ext}^s(E(n)_*, E(n)_*/I_{n-1})$ and $v_{n-1}^i x = 0$ for some i, then x pulls back to $\mathrm{Ext}^{s-1}(E(n)_*, E(n)_*/I_n)$ and so $s \leq n^2+2$ in order for x to be non-zero. If x is v_{n-1}-non-nilpotent, then

$$\begin{aligned} x \in\ & v_{n-1}^{-1}\mathrm{Ext}^s_{\mathcal{U}_{E(n)_*E(n)}}(E(n)_*, E(n)_*/I_{n-1}) \\ &\cong \mathrm{Ext}^s_{\mathcal{U}_{E(n)_*E(n)}}(E(n)_*, v_{n-1}^{-1}E(n)_*/I_{n-1}) \\ &\cong \mathrm{Ext}^s_{\mathcal{U}_{E(n-1)_*E(n-1)}}(E(n-1)_*, E(n-1)_*/I_{n-1}) \quad \text{by 7.1.} \end{aligned}$$

This last group is zero if $s > (n-1)^2+1$. So

$$\mathrm{Ext}^s(E(n)_*, E(n)_*/I_{n-1}) = 0 \quad \text{if } s > n^2+2.$$

Continuing downward through the Bockstein sequences

$$0 \to E(n)_*/I_{j-1} \xrightarrow{v_{j-1}} E(n)_*/I_{j-1} \to E(n)_*/I_j \to 0$$

for successively smaller j leads to

$$\mathrm{Ext}^s(E(n)_*, E(n)_*) = 0 \quad \text{for } s > n^2 + n + 1.$$

□

References

[1] A.J. Baker and U. Würgler. Liftings of formal groups and the Artinian completion of BP. *Math. Proc. Cambridge Philos. Soc.*, 106:511–530, 1989.

[2] M. Bendersky, E. B. Curtis, and H. R. Miller. The unstable Adams spectral sequence for generalized homology. *Topology*, 17:229–248, 1978.

[3] Martin Bendersky. Dissonance in the unstable chromatic spectral sequence. Notes.

[4] Martin Bendersky. Some calculuations in the unstable Adams-Novikov spectral sequence. In *Publ. RIMS Kyoto*, volume 16, pages 739–766, 1980.

[5] Martin Bendersky and John R. Hunton. On the coalgebraic ring and bousfield-kan spectral sequence for a landweber exact spectrum. *J. Edinburgh Math. Soc.*, 47:513–532, 2004.

[6] Martin Bendersky and Robert D. Thompson. The Bousfield-Kan spectral sequence for periodic homology theories. *Amer. J. Math.*, 122:599–635, 2000.

[7] Terrence P. Bisson and André Joyal. Q-rings and the homology of the symmetric groups. In *Contemporary Mathematics*, volume 202, pages 235–286. American Mathematical Society, 1997.

[8] A. K. Bousfield. The K-theory localizations and v_1-periodic homotopy groups of H-spaces. *Topology*, 38:1239–1264, 1999.

[9] A. K. Bousfield. Cosimplicial resolutions and homotopy spectral sequences in model categories. *Geom. Topol.*, 7:1001–1053, 2003.

[10] A. K. Bousfield and D. M. Kan. The homotopy spectral sequence of a space with coefficients in a ring. *Topology*, 11:79–106, 1972.

[11] Ethan Devinatz. Morava's change of rings theorem. In *Contemporary Mathematics, The Cech Centennial*, volume 181, pages 83–118. American Mathematical Society, 1995.

[12] M. J. Hopkins. Hopf-algebroids and a new proof of the morava-miller-ravenel change of rings theorem. preprint, 1995.

[13] Mark Hovey. Morita theory for hopf algeboids and presheaves of groupoids. *Amer. J. Math.*, 124:1289–1318, 2002.

[14] Mark Hovey. Homotopy theory of comodules over a Hopf algebroid. In *Homotopy theory: relations with algebraic geometry, group cohomology and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 261–304. Amer. Math. Soc., 2004.

[15] Mark Hovey and Hal Sadofsky. Invertible spectra in the $E(n)$-local stable homotopy category. *J. London Math. Soc.*, 60:284–302, 1999.

[16] R. Kargl. An unstable adams spectral sequence based on a generalized homology theory. Ph.D. thesis, City University of New York, 1998.

[17] Haynes R. Miller and Douglas C. Ravenel. Morava stabilizer algebras and the localization of Novikov's E_2-term. *Duke Math. J.*, 44(2):433–447, 1977.

[18] J. Morava. Noetherian localization of categories of cobordism comodules. *Ann. of Math.*, 121:1–39, 1985.

[19] Dustin Mulcahey. An unstable variant of the morava change of rings theorem for $K(n)$ theory. Ph.D. thesis, City University of New York, 2011.

[20] Geoffrey M. L. Powell. Unstable modules over the steenrod algebra revisited. *Geom.Topol. Monagr.*, 11:245–288, 2007.

[21] Douglas C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. Academic Press, 1986.

Department of Mathematics and Statistics, Hunter College and the Graduate Center, CUNY, New York, NY 10065

E-mail address: robert.thompson@hunter.cuny.edu