Kontsevich's Swiss Cheese Conjecture

Justin D. Thomas

We prove a conjecture of Kontsevich which states that if A is an E_{d-1} algebra then the Hochschild cochain object of A is the universal E_d algebra acting on A. The notion of an E_d algebra acting on an E_{d-1} algebra was defined by Kontsevich using the swiss cheese operad of Voronov. The degree 0 and 1 pieces of the swiss cheese operad can be used to build a cofibrant model for A as an $E_{d-1} - A$ module. The theorem amounts to the fact that the swiss cheese operad is generated up to homotopy by its degree 0 and 1 pieces.

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1 Introduction

In [9] Gerstenhaber showed that the Hochschild cohomology $\mathsf{HH}^*(A)$ of an associative algebra A is a graded Lie algebra and a graded commutative algebra, and the two structures are compatible. Any graded vector space with this algebraic structure is now called a Gerstenhaber algebra. In [6] Cohen showed that the homology of the little disks operad, $H_*(E_2)$, is the Gerstenhaber operad. Sinha also has shown this in [20]. Deligne later asked if the action of $H_*(E_2)$ on $\mathsf{HH}^*(A)$ descends from a natural action at the level of chains. In other words, is there a natural algebra structure on $\mathsf{CH}^*(A)$ of $\mathsf{Chains}(E_2)$ which recovers the structure discovered by Gerstenhaber after passing to (co)homology?

Already, this question is evidently in the realm of homotopy theory. So let us replace the associative algebras by E_1 algebras. This makes it clear that the question is fundamentally one about the relationship between the operads E_1 and E_2. Indeed, we can generalize to consider the relationship between E_d and E_{d-1} algebras. For any E_{d-1} algebra in a sufficiently rich homotopical category $\mathcal{C}$ we can make sense of its Hochschild cochains as an object of $\mathcal{C}$. The Hochschild cochain object of A is denoted $\mathsf{Hoch}(A)$ and is an object of the same category to which A belongs. This terminology is based on the case where $\mathcal{C}$ is the category of differential graded vector spaces and A is an associative algebra. In that case, $\mathsf{Hoch}(A)$ is the usual Hochschild cochain complex of A.

The original Deligne conjecture where A is an E_1 algebra in the category of chain complexes has been solved by Tamarkin [22], Kontsevich-Soibleman [17], Voronov [26], McClure-Smith [19], Berger-Fresse [2], and Kaufmann-Schwell [15]. A cyclic version is also due to Kaufmann [14]. Vallette [24] generalized the theorem to include certain other Koszul operads. The generalized version where A is an E_d algebra in a general category like $\mathcal{C}$ has been proven by Hu-Kriz-Voronov [13] and in the ∞-operad setting by Lurie [18]. We show here that $\mathsf{Hoch}(A)$ is not just an E_d algebra, but comes equipped with a universal property. It is the universal E_d algebra acting on the E_{d-1} algebra A. This universal property is shown in the case $d = 2$ in a paper of Dolgushev-Tamarkin-Tsygan [8].

The notion of an E_d algebra acting on an E_{d-1} algebra was introduced by Kontsevich in [16]. This notion uses the swiss cheese operad SC_d of Voronov [25]. This is a two-colored operad which interpolates between E_d and E_{d-1}. A swiss cheese algebra is a pair (B, A) where B is an E_d algebra, A is an E_{d-1} algebra, and there is some extra structure compatible with these as seen in definition 2.1.9. We refer to this extra structure as an *action* of B on A.

The case $d = 1$ is enlightening. For simplicity, let us work in the category of vector spaces. A (non-unital) E_0 algebra A in vector spaces is just a vector space with no extra data. The Hochschild object in this case is $\hom(A, A)$. If B is an associative algebra, it is in particular an E_1 algebra. An SC_1 structure on the pair (B, A) then amounts to the choice of a B-module structure on A.

In this case, the swiss cheese conjecture merely states that $\hom(A, A)$ is an associative algebra, and giving an SC_1 structure on (B, A) is equivalent to giving a map of associative algebras $B \to \mathsf{Hoch}(A) = \hom(A, A)$. We prove the analog of this when B is an E_d algebra and A is an E_{d-1} algebra, $d \geq 1$.

Joseph Hirsh brought to the attention of the author the following helpful characterization of the results of this paper. Given a bifibrant E_{d-1} algebra A the functor from E_d algebras to spaces,

$$B \mapsto \{\text{the space of swiss cheese actions of } B \text{ on } A\}$$

is represented by the Hochschild cochain object of A.

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1.2 Outline of the paper

In section 2 we define the E_d and SC_d operads, and give both an imprecise and precise statement of the theorem we will prove. We also outline the idea of the proof. In section 3 we define the Hochschild cochain object for E_{d-1} algebras and show that we can use the swiss cheese operad to construct a model for the Hochschild cochain object. We use this model in section 4 to prove a "universal cheese" theorem which applies to an arbitrary operad acting on Hochschild cochains. In section 5 we show that E_d does indeed act, up to homotopy, on Hochschild cochains, and the "universal cheese" theorem specializes to the main theorem of this paper. Finally, in section 6 we prove the main theorem which allows the homotopy E_d action: the swiss cheese operad is freely generated up to homotopy by its degree 0 and 1 pieces.

2 The swiss cheese operad

We will define K-colored operads in general, and the swiss cheese operad in particular. We also describe algebras over the swiss cheese operad and state the main theorem of this paper.

2.1 The colored operad swiss cheese

Fix a set K. A K-colored set is a pair (I, col) where I is a set and $\mathsf{col} : I \to K$ is a map of sets, called the *coloring*. We will often denote such a colored set simply by I, leaving the coloring implicit. Let $\mathsf{aut}(I)$ be the group of bijections on the set I which preserve its coloring. Since we have left the coloring implicit, we use the notation $I^{\#}$ to refer to the underlying uncolored set.

Let $(\mathcal{S}, \otimes)$ be a symmetric monoidal category. We can speak of categories enriched over $\mathcal{S}$. In particular, suppose $\mathcal{O}$ is a category enriched over $\mathcal{S}$ and suppose the objects of $\mathcal{O}$ are finite K-colored sets. We let $\mathcal{O}(I; J)$ denote the $\mathcal{S}$

object of morphisms in $\mathcal{O}$ from I to J. If we further suppose that disjoint union of finite K-colored sets extends to an $\mathcal{S}$-enriched symmetric monoidal structure on $\mathcal{O}$, then each $\mathcal{O}(I;J)$ is a right $\mathsf{aut}(I)$ and left $\mathsf{aut}(J)$ module in a natural way. In addition, if $\mathcal{S}$ contains finite coproducts, the symmetric monoidal structure on morphisms in $\mathcal{O}$ is specified by equivariant maps

$$\coprod_{f\colon I^\# \to J^\#} \bigotimes_{j \in J} \mathcal{O}(f^{-1}\{j\};\{j\}) \to \mathcal{O}(I;J). \tag{1}$$

The following definition follows Boardman-Vogt [5].

2.1.1 Definition Let $(\mathcal{S}, \otimes)$ be a symmetric monoidal category containing all finite coproducts where $\otimes$ distributes over finite coproducts. Let K be a set. The data of a *K-colored operad $\mathcal{O}$ in the symmetric monoidal category $\mathcal{S}$* is a symmetric monoidal category, denoted $\mathcal{O}$, which is enriched over $\mathcal{S}$ and whose objects are K-colored finite sets, $I \to K$. This data must satisfy the following conditions. First, on objects, the symmetric monoidal structure of $\mathcal{O}$ is the disjoint union of sets over K. Second, the map in (1) must be an isomorphism for every I and J.

2.1.2 Remark We will use the unmodified noun *operad* to mean K-colored operad when the coloring set K is clear from context. The reader should note that this differs from an equally plausible convention where *operad* is always used to denote $\{*\}$-colored operads.

2.1.3 Notation Let $\underline{n}$ to denote the finite set $\{1, \dots, n\}$. Typically K will be $K = \{\mathsf{f}, \mathsf{h}\}$, where f stands for *full disk* and h stands for *half disk*. In this case we use (n, m) to denote the K-colored set which is the disjoint union of

$$\underline{n} \to \{\mathsf{f}\} \text{ and } \underline{m} \to \{\mathsf{h}\}.$$

If $K \simeq \{*\}$, then a K-colored operad will simply be called a 1-colored operad. Any 1-colored operad $\mathcal{E}$ gives for each $m, m' \in \mathbb{Z}_{\geq 0}$, objects $\mathcal{E}(\underline{m'};\underline{m}) \in \mathcal{S}$. We denote $\mathcal{E}(\underline{m'};\underline{m})$ simply by $\mathcal{E}(m';m)$ and $\mathcal{E}(m;1)$ simply by $\mathcal{E}(m)$.

Any $\{\mathsf{f}, \mathsf{h}\}$-colored operad $\mathcal{O}$ gives for each $n, m, n', m' \in \mathbb{Z}_{\geq 0}$, objects

$$\mathcal{O}(n', m'; n, m) := \mathcal{O}((n', m'); (n, m)) \in \mathcal{S}.$$

We denote $\mathcal{O}(n, m; 0, 1)$ by $\mathcal{O}^{\mathsf{h}}(n, m)$ and we denote $\mathcal{O}(n, m; 1, 0)$ by $\mathcal{O}^{\mathsf{f}}(n, m)$.

2.1.4 Definition Let $(\mathsf{Top}, \times)$ denote the symmetric monoidal category of compactly generated Hausdorff topological spaces with the cartesian product.

2.1.5 Example Let K be the one-point set $\{\mathsf{f}\}$ and let $(\mathcal{S}, \otimes) = (\mathsf{Top}, \times)$. The operad E_d is an $\{\mathsf{f}\}$-colored operad in the category Top. Let $\bar{D}^d$ be the closed unit disk inside $\mathbb{R}^d$. Call a map $f : \bar{D}^d \to \bar{D}^d$ a *little full-disk* (or *little d disk* or simply *little disk*) if f is of the form $f(x) = rx + c$ for some $0 < r \leq 1$ and $c \in \mathbb{R}^d$. Given a finite set I, the underlying set of $E_d(I;\mathsf{f})$ is the set of embeddings

$$f\colon \coprod_{i\in I} \bar{D}^d \to \bar{D}^d,$$

where each restriction $f_i\colon \bar{D}^d \to \bar{D}^d$ is a little full-disk. Using notation 2.1.3, any isomorphism $I \to \underline{n}$ induces an isomorphism $E_d(I;\mathsf{f}) \simeq E_d(n)$, and the latter can naturally be considered as a subset of $\mathbb{R}^{n+dn}$. This gives each $E_d(I;\mathsf{f})$ a topology. The operadic structure is given by composing little d disks as maps $\bar{D}^d \to \bar{D}^d$. The identity of E_d is the little full-disk $\mathsf{id}\colon \bar{D}^d \to \bar{D}^d$. This is the unital version of E_d, so $E_d(0) = *$ and $E_d(1)$ consists of more than just the identity.

2.1.6 Example Let $K = \{\mathsf{f}, \mathsf{h}\}$. The K-colored operad SC_d is called the (d-dimensional) swiss cheese operad and is the principal subject of this paper. Like example 2.1.5 it is an operad in $(\mathsf{Top}, \times)$.

Let $\{\mathsf{f}\}$ and $\{\mathsf{h}\}$ denote the evident singleton K-colored sets. By definition 2.1.1 and formula (1), we only need to define the spaces $SC_d(I;\mathsf{f})$ and $SC_d(I;\mathsf{h})$ for every K-colored set $(I, \mathsf{col}\colon I \to K)$. First, we define the "full-disk output" part of SC_d,

$$SC_d(I;\mathsf{f}) = \begin{cases} E_d(I;\mathsf{f}) & \mathsf{col}^{-1}(\mathsf{f}) = I \\ \emptyset & \text{else.} \end{cases}$$

To define the "half-disk output" part of SC_d, that is $SC_d(I;\mathsf{h})$, we first need the notion of *little half-disks*. Let $\bar{D}^d_+$ be the closed d-dimensional half-disk,

$$\bar{D}^d_+ = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid |x| \leq 1 \text{ and } x_d \geq 0\}.$$

A *little half-disk* is defined to be a map $f\colon \bar{D}^d_+ \to \bar{D}^d_+$ of the form $f(x) = rx + c$ for some $0 < r \leq 1$ and $c \in \mathbb{R}^{d-1} \times \{0\}$. As a set, we define $SC_d(I;\mathsf{h})$ to consist of embeddings

$$f\colon \coprod_{i\in I} \bar{D}^d_i \to \bar{D}^d_+$$

where each restriction $f_i\colon \bar{D}^d_i \to \bar{D}^d_+$ is a little full disk (example 2.1.5) if $\mathsf{col}(i) = \mathsf{f}$ or a little half-disk if $\mathsf{col}(i) = \mathsf{h}$. It is clear that if $|\mathsf{col}^{-1}(\mathsf{f})| = n$ and $|\mathsf{col}^{-1}(\mathsf{h})| = m$, then $SC_d(I;\mathsf{h})$ can be naturally embedded inside $\mathbb{R}^N$ where

$N = (d+1)n + dm$. We give $SC_d(I;\mathsf{h})$ the subspace topology inherited from such an embedding.

Following notation 2.1.3, a point in $SC_d^{\mathsf{h}}(n,m)$ is given by n labeled full-disks and m labeled half-disks in the unit half-disk where none of the disks intersect and the half-disks all lie on the bottom. We allow the degenerate configuration when $(n,m) = (0,1)$ which is the unit half-disk contained in itself. Note that we have $SC_d^{\mathsf{h}}(0,0) = *$ and $SC_d^{\mathsf{h}}(1,0)$ contains more than one point. Thus we are using the unital swiss cheese operad. This differs from Kontsevich [16] and Voronov [25].

Composition in SC_d is given by substituting full-disks and half-disks into each other . More precisely, we have maps

$$E_d(n) \times E_d(k_1) \times \cdots \times E_d(k_n) \to E_d(k_1 + \cdots + k_n) \tag{2}$$

and

$$\begin{aligned} SC_d^{\mathsf{h}}(n,m) \times E_d(k_1) \times \cdots \times E_d(k_n) \times SC_d^{\mathsf{h}}(k_{n+1},\ell_1) \times \cdots \times SC_d^{\mathsf{h}}(k_{n+m},\ell_m) \\ \longrightarrow SC_d^{\mathsf{h}}(k_1 + \cdots + k_{n+m}, \ell_1 + \cdots + \ell_m). \end{aligned} \tag{3}$$

Notice that we can identify $SC_d^{\mathsf{h}}(0,m)$ with $E_{d-1}(m)$ so that the restriction of SC_d to the spaces $SC_d^{\mathsf{h}}(0,\bullet)$ is isomorphic to the operad E_{d-1}. We say that E_{d-1} is the h color of SC_d and E_d is the f color of SC_d. We think of SC_d as interpolating between E_d and E_{d-1}.

2.1.7 Definition Suppose $\mathcal{O}$ is a K-colored operad in $\mathcal{S}$ and $\mathcal{C}$ is a symmetric monoidal category enriched over $\mathcal{S}$. An *algebra* over $\mathcal{O}$ in the category $\mathcal{C}$ is a strong symmetric monoidal functor $\mathcal{O} \to \mathcal{C}$. A morphism of $\mathcal{O}$ algebras is a monoidal natural transformation. The category of $\mathcal{O}$ algebras in $\mathcal{C}$ will be denoted $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$.

2.1.8 Example If $\mathcal{C}$ is a symmetric monoidal category enriched over Top, we can consider algebras over SC_d in $\mathcal{C}$. Such an algebra gives the data of a pair (B,A) of objects in $\mathcal{C}$ together with maps of topological spaces

$$E_d(n) \to \mathrm{map}_{\mathcal{C}}(B^{\otimes n}, B)$$

and

$$SC_d^{\mathsf{h}}(n,m) \to \mathrm{map}_{\mathcal{C}}(B^{\otimes n} \otimes A^{\otimes m}, A),$$

where $\mathrm{map}_{\mathcal{C}}(C, C')$ is the topological space of maps between two objects C, C' in $\mathcal{C}$. These data must satisfy conditions guaranteeing they assemble into a strong symmetric monoidal functor $SC_d \to \mathcal{C}$.

The object B corresponds to the object $\{\mathsf{f}\}$ of SC_d and the object A corresponds to $\{\mathsf{h}\}$. Together these form a K-colored operad $\mathsf{End}(B,A)$ in Top where, using notation 2.1.3,

$$\mathsf{End}^{\mathsf{f}}(B,A)(n,m) = \mathrm{map}(B^{\otimes n} \otimes A^{\otimes m}, B) \tag{4}$$

$$\mathsf{End}^{\mathsf{h}}(B,A)(n,m) = \mathrm{map}(B^{\otimes n} \otimes A^{\otimes m}, A). \tag{5}$$

The collection of SC_d algebra structures on a fixed pair (B,A) is the collection of strong symmetric monoidal functors

$$SC_d \to \mathsf{End}(B,A),$$

which restrict to the identity on the set of objects. Simply put, a swiss cheese algebra (B,A) is an E_d algebra B, an E_{d-1} algebra A, and some chosen mixing of these structures. We refer to this mixing as an action of B on A. The following definition is due to Kontsevich [16].

2.1.9 Definition Let B be an E_d algebra and A an E_{d-1} algebra. A *swiss cheese action of B on A* is the structure of a swiss cheese algebra on the pair (B,A) extending the given E_d and E_{d-1} structures. We may also simply call this "an action of B on A".

2.2 Statement of main theorem

Now we can informally state the conjecture proven in this paper.

Theorem (Informal statement of Swiss Cheese Conjecture) *The Hochschild cochain object* $\mathsf{Hoch}(A)$ *of an E_{d-1} algebra A is the universal E_d algebra acting on A. In other words, for any E_{d-1} algebra A, there is an E_d algebra structure on* $\mathsf{Hoch}(A)$ *such that for any E_d algebra B, giving a map of E_d algebras* $B \to \mathsf{Hoch}(A)$ *is equivalent to giving the structure of an SC_d algebra on the pair (B,A) extending the given E_d and E_{d-1} structures.*

The basic structure of the proof is outlined in diagram (6). The categories of operads shown in the diagram are defined precisely in section 2.3. The category Top^{Σ} consists of symmetric sequences of topological spaces. Informally,

$\mathsf{Op}(\mathsf{Top}^{\Sigma})$ consists of K-colored operads whose f-colored output is trivial, and $\mathsf{Op}(\mathsf{Top}^{\Sigma_{\leq 1}})$ further restricts to those pieces whose f-colored inputs total 0 or 1. The forgetful functors are presented below as straight arrows; there is a left adjoint shown as a bent arrow.

Boardman and Vogt's W construction [5] is an explicit cofibrant replacement functor, which we apply to SC_d, to get an equivalent cofibrant operad $\mathsf{SC_d} := WSC_d$. The W construction does not strictly commute with the forgetful functors in (6). In particular $W(SC_d^{\mathsf{h}})$ is not isomorphic to $(WSC_d)^{\mathsf{h}}$, but they are homotopy equivalent. We will let $\mathsf{SC_d^h}$ denote $W(SC_d^{\mathsf{h}})$ and $\mathsf{SC_d^{h1}} = W(SC_d^{\mathsf{h1}})$.

$$
(6)\qquad
\begin{array}{ccccc}
\mathsf{Op}_K & \longrightarrow & \mathsf{Op}(\mathsf{Top}^{\Sigma}) & \underset{\curvearrowleft}{\longrightarrow} & \mathsf{Op}(\mathsf{Top}^{\Sigma_{\leq 1}}) \\
\mathsf{SC_d} & \longmapsto & \mathsf{SC_d^h} & \longmapsto & \mathsf{SC_d^{h1}} \\
\mathsf{SC_d^{h\infty}} \rtimes \mathsf{E} & \xleftarrow[\text{structure}]{\text{transfer of}} & \mathsf{SC_d^{h\infty}} & \longleftarrow & \mathsf{SC_d^{h1}}
\end{array}
$$

Proposition 3.2.7 shows that we can use $\mathsf{SC_d^{h1}}$ to construct a model for the Hochschild cochain object. This allows us to prove a weak version of the swiss cheese theorem in proposition 4.1.9 taking place in the context of $\mathsf{Op}(\mathsf{Top}^{\Sigma_{\leq 1}})$. Next, we will take the free extension of $\mathsf{SC_d^{h1}} \in \mathsf{Op}(\mathsf{Top}^{\Sigma_{\leq 1}})$ to an operad in $\mathsf{Op}(\mathsf{Top}^{\Sigma})$, to get $\mathsf{SC_d^{h\infty}}$. This immediately gives a version of the swiss cheese theorem in the context of $\mathsf{Op}(\mathsf{Top}^{\Sigma})$, see corollary 4.1.11. Then we use the fact that $\mathsf{SC_d^{h\infty}}$ is freely generated by its degree 0 and 1 pieces to prove a version of the swiss cheese theorem in the context of Op_K. None of these three versions of the swiss cheese theorem make any use of E_d. One can think of this last "universal cheese" theorem (proposition 4.1.16) as a construction of the universal K-colored operad built from $\mathsf{E_{d-1}}$ and controlling $\mathsf{E_{d-1}}$-linear actions on A.

To bring E_d back in to the story, we use a technical result, 5.1.17, which shows that the canonical map $\mathsf{SC_d^{h\infty}} \to \mathsf{SC_d^h}$ is an equivalence. Observe that one can view $\mathsf{SC_d}$ as $\mathsf{SC_d^h}$ equipped with the extra structure of a right action of E_d. Now use a transfer of structure argument to construct an operad E which is equivalent to E_d and which acts on the right on $\mathsf{SC_d^{h\infty}}$. This allows us to define $\mathsf{SC_d^{h\infty}} \rtimes \mathsf{E}$, which we show is equivalent to SC_d in 5.1.22. The universal property of $\mathsf{SC_d^{h\infty}} \rtimes \mathsf{E}$ with respect to the Hochschild cochain object is stated in theorem 2.2.1 and follows from proposition 4.1.16.

2.2.1 Theorem (Precise version of Kontsevich's Swiss Cheese Conjecture) *Let $\mathcal{C}$ be a symmetric monoidal model category tensored over* Top *and satisfying* *the conditions in notation 3.1.9. Let* $A \in \mathrm{Alg}_{\mathsf{E_{d-1}}}(\mathcal{C})$ *be cofibrant and fibrant using the projective model structure (3.1.8). There is a model of* SC_d*, called* $\mathsf{SC_d^{h\infty}} \rtimes \mathsf{E}$ *where* $\mathsf{E} \simeq E_d$*. There is also a model for the Hochschild cochain object of* A*, called* $\mathrm{Hoch}(A)$*, such that* $\mathrm{Hoch}(A)$ *is the universal* E *algebra acting on* A *through* $\mathsf{SC_d^{h\infty}} \rtimes \mathsf{E}$*. That is,* $\mathrm{Hoch}(A)$ *is an* E *algebra and this structure, together with the* E_{d-1} *algebra structure on* A*, can be extended to an* $\mathsf{SC_d^{h\infty}} \rtimes \mathsf{E}$ *algebra structure on* $(\mathrm{Hoch}(A), A)$ *in such a way that there is an isomorphism of categories*

$$\mathrm{Alg}^{A}_{(\mathsf{SC_d^{h\infty}} \rtimes \mathsf{E})}(\mathcal{C}) \cong \mathrm{Alg}_{\mathsf{E}}(\mathcal{C})_{/\mathrm{Hoch}(A)}.$$

The category on the left consists of E *algebras* B *together with an action of* B *on* A*. The category on the right consists of* E *algebras* B *together with an* E *algebra map* $B \to \mathrm{Hoch}(A)$*.*

Lemma 5.1.22 shows that $\mathsf{SC_d^{h\infty}} \rtimes \mathsf{E}$ is equivalent to SC_d. Lemma 5.1.21 shows that E is equivalent to E_d. Proposition 3.2.7 sets up our choice of model of $\mathrm{Hoch}(A)$. Proposition 4.1.16, together with the construction of the operad $\mathsf{SC_d^{h\infty}} \rtimes \mathsf{E}$ in section 5 shows the desired isomorphism of categories.

2.3 Defining SC_d^{h} and SC_d^{h1}

Recall that $\mathcal{C}$ is a symmetric monoidal category enriched over $\mathcal{S}$, our basic category in which our operads live. We will assume that both $\mathcal{C}$ and $\mathcal{S}$ have all coproducts and that tensor products distribute over finite coproducts. In the case of operads from Op_K we have $(\mathcal{S}, \otimes) = (\mathsf{Top}, \times)$, the symmetric monoidal category of compactly generated topological spaces with cartesian product. In the case of operads such as SC_d^{h} and $\mathsf{SC_d^h}$ from $\mathsf{Op}(\mathsf{Top}^{\Sigma})$, we have $(\mathcal{S}, \otimes) = (\mathsf{Top}^{\Sigma}, \otimes)$ from definition 2.3.1. Finally, for $\mathsf{SC_d^{h1}}$ and SC_d^{h1} we use $\mathcal{S} = \mathsf{Top}^{\Sigma_{\leq 1}}$ as in definition 2.3.2.

2.3.1 Definition Let Σ denote the *opposite* of the category of finite sets with morphisms given by bijections, and let $(\mathcal{D}, \otimes_{\mathcal{D}})$ be any symmetric monoidal category. The category of functors $\Sigma \to \mathcal{D}$, denoted $\mathcal{D}^{\Sigma}$, is usually called the category of *symmetric sequences* in $\mathcal{D}$. We endow $\mathcal{D}^{\Sigma}$ with the usual symmetric monoidal structure given by left Kan extension of $\Sigma \times \Sigma \to \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ along disjoint union of sets $\Sigma \times \Sigma \to \Sigma$. Specifically, if $X, Y \in \mathcal{D}^{\Sigma}$, then $X \otimes Y \in \mathcal{D}^{\Sigma}$ satisfies

$$(X \otimes Y)(\underline{n}) = \coprod_{n=n_1+n_2} \mathrm{Ind}^{\mathrm{aut}(\underline{n})}_{\mathrm{aut}(\underline{n}_1)\times\mathrm{aut}(\underline{n}_2)}(X(\underline{n}_1) \otimes_{\mathcal{D}} Y(\underline{n}_2)),$$

See Harper [11, definition 3.3] for more details.

2.3.2 Definition Let $\Sigma^{\leq 1}$ denote the full subcategory of Σ consisting of finite sets of size 0 or 1 together with bijections as morphisms. Let $\mathcal{D}^{\Sigma_{\leq 1}}$ denote the symmetric monoidal category of functors $\Sigma^{\leq 1} \to \mathcal{D}$, with monoidal structure inherited from $\mathcal{D}^{\Sigma}$. Call these the *degree 0-1 symmetric sequences in* $\mathcal{D}$. Concretely, $\mathcal{D}^{\Sigma_{\leq 1}}$ is just the category $\mathcal{D} \times \mathcal{D}$ endowed with the symmetric monoidal structure

$$(C_0, C_1) \otimes (D_0, D_1) = (C_0 \otimes_{\mathcal{D}} D_0, (C_0 \otimes_{\mathcal{D}} D_1) \coprod (C_1 \otimes_{\mathcal{D}} D_0)). \tag{7}$$

The braiding isomorphism $(C_0, C_1) \otimes (D_0, D_1) \to (D_0, D_1) \otimes (C_0, C_1)$ is induced from the braiding isomorphism on $\mathcal{D}$.

2.3.3 Definition Let Op_K denote the category of $K = \{\mathsf{f}, \mathsf{h}\}$-colored operads in Top. There is a forgetful functor $\mathsf{Op}_K \to \mathsf{Op}(\mathsf{Top}^{\Sigma})$. This functor takes $\mathcal{O} \in \mathsf{Op}_K$ to the operad $\mathcal{O}^{\mathsf{h}}$ whose arity m component is the symmetric sequence $n \mapsto \mathcal{O}^{\mathsf{h}}(n, m)$ (see notation 2.1.3). We forget the spaces $\mathcal{O}^{e}(n, m)$, and think of elements of $\mathcal{O}^{h}(n, m)$ as degree n, arity m elements of $\mathcal{O}^{h}$.

The functor $\mathsf{Op}(\mathsf{Top}^{\Sigma}) \to \mathsf{Op}(\mathsf{Top}^{\Sigma_{\leq 1}})$ is induced by the symmetric monoidal forgetful functor $\mathsf{Top}^{\Sigma} \to \mathsf{Top}^{\Sigma_{\leq 1}}$. Denote the image of $\mathcal{O}$ in $\mathsf{Op}(\mathsf{Top}^{\Sigma_{\leq 1}})$ as $\mathcal{O}^{\mathsf{h}1}$.

2.3.4 Example We outline the structure of $\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}$ as an operad in $\mathsf{Op}(\mathsf{Top}^{\Sigma})$. Think of $\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}(\bullet, m)$ as the symmetric sequence $n \mapsto \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}(n, m)$. In (8), $\otimes$ is the tensor product of symmetric sequences. The operad composition law is

$$\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}(\bullet, m) \otimes \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}(\bullet, \ell_1) \otimes \cdots \otimes \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}(\bullet, \ell_m) \to \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}(\bullet, \ell), \tag{8}$$

where $\ell = \sum_i \ell_i$. The degree n component of the right hand symmetric sequence is $\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}(n, \ell)$. The degree n component of the left hand symmetric sequence is

$$\coprod_{n_0 + \cdots + n_m = n} \operatorname{Ind} \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}(n_0, m) \times \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}(n_1, \ell_1) \times \cdots \times \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}(n_m, \ell_m),$$

where Ind is the induction functor giving the correct symmetric group action. The point is that if we delete all appearances of E_d from (3) then it provides exactly the data of (8).

2.3.5 Example We outline the structure of $SC_d^{\mathsf{h}1}$ as a 1-colored operad in $\mathsf{Top}^{\Sigma_{\leq 1}}$. For each m', m, define an object $SC_d^{\mathsf{h}1}(m'; m) \in \mathsf{Top}^{\Sigma_{\leq 1}}$,

$$\begin{aligned} SC_d^{\mathsf{h}1}(m'; m) &= (SC_d^{\mathsf{h}}(0, m'; 0, m), SC_d^{\mathsf{h}}(1, m'; 0, m)) \\ &\cong (E_{d-1}(m'; m), SC_d^{\mathsf{h}}(1, m'; 0, m)). \end{aligned} \tag{9}$$

Using the symmetric monoidal structure from definition 2.3.2 and the identification in equation (9) we can write the operad structure maps on $SC_d^{\mathsf{h}1}$ as a triple of morphisms. The map of degree 0 pieces,

$$E_{d-1}(m'';m) \times E_{d-1}(m';m'') \to E_{d-1}(m';m),$$

and the maps of degree 1 pieces,

$$E_{d-1}(m'';m) \times SC_d^{\mathsf{h}}(1,m';0,m'') \to SC_d^{\mathsf{h}}(1,m';0,m),$$

and

$$SC_d^{\mathsf{h}}(1,m'';0,m) \times E_{d-1}(m';m'') \to SC_d^{\mathsf{h}}(1,m';0,m).$$

3 Hochschild cohomology from swiss cheese

For the remainder of the paper we replace E_{d-1}, E_d, and SC_d by cofibrant models given by the Boardman-Vogt W construction [5]. A proof that this gives a cofibrant replacement for certain operads can be found in Berger-Moerdijk [1]. We will denote these cofibrant replacements by $\mathsf{E}_{\mathsf{d}-1}, \mathsf{E}_{\mathsf{d}}$, and SC_{d}. We also want to restrict our attention to swiss cheese algebras in categories where we can do homotopy theory. In the proper context the Hochschild cochain object of an $\mathsf{E}_{\mathsf{d}-1}$ algebra A has a natural model constructed from A and the degree 0-1 parts of $\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}$. This is the content of proposition 3.2.7.

3.1 Homotopy theoretic context

3.1.1 Definition From [12, definition 4.2.6], a *symmetric monoidal model category* $\mathcal{S}$ is a closed symmetric monoidal category whose monoidal structure $\otimes\colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is a Quillen bifunctor, and where the cofibrant replacement $Q1 \to 1$ of the monoidal unit induces weak equivalences $Q1 \otimes X \to X$ for every cofibrant X.

3.1.2 Example The category $(\mathsf{Top}, \times)$ of compactly generated spaces with the cartesian product and Serre model structure is a symmetric monoidal model category.

3.1.3 Definition Let $\mathcal{S}$ be a symmetric monoidal model category. A *symmetric monoidal model category tensored over* $\mathcal{S}$ is a closed symmetric monoidal model category $\mathcal{C}$, together with a symmetric monoidal Quillen functor $\mathcal{S} \to \mathcal{C}$. For more details see Hovey [12, definition 4.2.20].

In particular, $\mathcal{C}$ comes equipped with functors

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \quad \otimes : \mathcal{S} \times \mathcal{C} \to \mathcal{C}.$$
$$\underline{\mathrm{hom}}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C} \quad \mathrm{map}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}$$

The mapping spaces $\mathrm{map}_{\mathcal{C}}(A, B)$ give $\mathcal{C}$ the structure of a category enriched over $\mathcal{S}$, so we can speak of $\mathsf{E_{d-1}}, \mathsf{E_d}$ and $\mathsf{SC_d}$ algebras in $\mathcal{C}$.

For any object A of $\mathcal{C}$, the functor $-\otimes A$ has right adjoints $\underline{\mathrm{hom}}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathcal{C}$ and $\mathrm{map}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathcal{S}$. This data satisfies Quillen's SM7 axiom (Hovey [12, section 4.2]).

3.1.4 Example The category $\mathcal{C}^{\Sigma_{\leq 1}}$ from definition 2.3.2 is tensored over the symmetric monoidal category $\mathsf{Top}^{\Sigma_{\leq 1}}$ with

$$\mathsf{Top}^{\Sigma_{\leq 1}} \otimes \mathcal{C}^{\Sigma_{\leq 1}} \to \mathcal{C}^{\Sigma_{\leq 1}}$$

given by the analogue of equation (7),

$$(X_0, X_1) \otimes (C_0, C_1) = (X_0 \otimes C_0, X_0 \otimes C_1 \coprod X_1 \otimes C_0),$$

where $X_i \in \mathsf{Top}$ and $C_i \in \mathcal{C}$.

3.1.5 Example The symmetric monoidal functor $\mathsf{Top} \to \mathsf{Top}^{\Sigma_{\leq 1}}$ sending X to $(X, \emptyset)$ makes both $\mathsf{Top}^{\Sigma_{\leq 1}}$ and $\mathcal{C}^{\Sigma_{\leq 1}}$ into symmetric monoidal model categories tensored over Top. Note that for a topological 1-colored operad $\mathcal{O}$, we can consider algebras over $\mathcal{O}$ in $\mathcal{C}$ as well as algebras over $\mathcal{O}$ in $\mathcal{C}^{\Sigma_{\leq 1}}$.

The category of degree 0-1 symmetric sequences is naturally home to $\mathcal{O}$-algebras A and $\mathcal{O}$-A modules M.

3.1.6 Definition Suppose $\mathcal{O}$ is a 1-colored operad in Top, and let (A, M) be an object of $\mathcal{C}^{\Sigma_{\leq 1}}$. The structure of an $\mathcal{O}$ algebra on the degree 0-1 symmetric sequence (A, M) is the structure of an $\mathcal{O}$ algebra on A together with the data of maps

$$\mathcal{O}(m) \otimes M \otimes A^{\otimes m-1} \to M,$$

satisfying certain conditions (see diagram (16)). We call this data the structure of an $\mathcal{O}$-A *module* on M. Given a fixed $\mathcal{O}$ algebra $A \in \mathcal{C}$, the category $\mathsf{Mod}^A_{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-A modules has objects $M \in \mathcal{C}$ together with the structure of an $\mathcal{O}$ algebra on $(A, M) \in \mathcal{C}^{\Sigma_{\leq 1}}$ extending the given $\mathcal{O}$ algebra structure on A.

We can enrich $\mathrm{Mod}^A_{\mathcal{O}}(\mathcal{C})$ over $\mathcal{C}$ and over Top. Indeed, given $\mathcal{O}$-A modules M' and M, we can define the hom-object of $\mathcal{O}$-A module morphisms from M' to M as the equalizer

$$\underline{\mathrm{hom}}^A_{\mathcal{O}}(M', M) \xrightarrow{\mathrm{eq}} \underline{\mathrm{hom}}_{\mathcal{C}}(M', M) \rightrightarrows \underline{\mathrm{hom}}_{\mathcal{C}}(F^A_{\mathcal{O}}(M'), M).$$

Here $F^A_{\mathcal{O}} \colon \mathcal{C} \to \mathcal{C}$ is the free $\mathcal{O}$-A module monad on $\mathcal{C}$. One can define this using the monad $F_{\mathcal{O}}$ on $\mathcal{C} \times \mathcal{C}$ which sends (A, M) to $(F^0_{\mathcal{O}}(A), F^1_{\mathcal{O}}(A, M))$, where $F^0_{\mathcal{O}}$ is the free $\mathcal{O}$-algebra monad and

$$F^1_{\mathcal{O}}(A, M) = \coprod_m \mathcal{O}(m) \otimes_{S_m} \Big(\coprod_{i=1}^m A^{\otimes i-1} \otimes M \otimes A^{\otimes m-i} \Big).$$

The natural transformation $F^1_{\mathcal{O}}(F^0_{\mathcal{O}}(-), F^1_{\mathcal{O}}(-,-)) \to F^1_{\mathcal{O}}(-,-)$ is given by composition in $\mathcal{O}$. This, along with $F^0_{\mathcal{O}} F^0_{\mathcal{O}} \to F^0_{\mathcal{O}}$, defines the monad structure $F_{\mathcal{O}} F_{\mathcal{O}} \to F_{\mathcal{O}}$. Finally, we define the free $\mathcal{O}$-A algebra monad via the coequalizer

$$F^1_{\mathcal{O}}(F^0_{\mathcal{O}}(A), F^1_{\mathcal{O}}(A, M)) \rightrightarrows F^1_{\mathcal{O}}(A, M) \xrightarrow{\mathrm{coeq}} F^A_{\mathcal{O}}(M).$$

It is clear that every $\mathcal{O}$-A module M is equipped with a canonical map $F^A_{\mathcal{O}}(M) \to M$. The two parallel arrows in the equalizer are given by the two maps $F^A_{\mathcal{O}}(M') \to M'$ and $F^A_{\mathcal{O}}(M) \to M$. The topological space $\mathrm{map}^A_{\mathcal{O}}(M', M)$ is defined as an equalizer in exactly the same manner.

3.1.7 Example Let $A \in \mathcal{C}$ be an $\mathsf{E_{d-1}}$ algebra, then the degree 0-1 symmetric sequence (A, A) is naturally an $\mathsf{E_{d-1}}$ algebra. That is, A is naturally an $\mathsf{E_{d-1}}$-A module.

3.1.8 Definition Recall from example 2.3.5 that $SC^{\mathsf{h}1}_d$ is an operad in $\mathsf{Top}^{\Sigma_{\leq 1}}$. Let $\mathsf{SC}^{\mathsf{h}1}_{\mathsf{d}} \in \mathsf{Top}^{\Sigma_{\leq 1}}$ be $W(SC^{\mathsf{h}1}_d)$. Since $\mathcal{C}^{\Sigma_{\leq 1}}$ is enriched over $\mathsf{Top}^{\Sigma_{\leq 1}}$ by example 3.1.4, we can consider $\mathsf{SC}^{\mathsf{h}1}_{\mathsf{d}}$ algebras in $\mathcal{C}^{\Sigma_{\leq 1}}$. In addition, by example 3.1.5 we can consider $\mathsf{E_{d-1}}$ algebras in $\mathcal{C}^{\Sigma_{\leq 1}}$. There are adjunctions

$$\mathcal{C} \leftrightarrows \mathrm{Alg}_{\mathsf{E_{d-1}}}(\mathcal{C}) \quad \mathcal{C} \leftrightarrows \mathrm{Mod}^A_{\mathsf{E_{d-1}}}(\mathcal{C})$$

$$\mathcal{C} \times \mathcal{C} \leftrightarrows \mathrm{Alg}_{\mathsf{E_{d-1}}}(\mathcal{C}^{\Sigma_{\leq 1}}) \leftrightarrows \mathrm{Alg}_{\mathsf{SC}^{\mathsf{h}1}_{\mathsf{d}}}(\mathcal{C}^{\Sigma_{\leq 1}})$$

We describe the right adjoints only. On the top left the $\mathsf{E_{d-1}}$ algebra A is sent to the underlying object A of $\mathcal{C}$. The top right functor sends the $\mathsf{E_{d-1}}$-A module M to the underlying object M of $\mathcal{C}$. In the pair of composable adjunctions, an $\mathsf{SC}^{\mathsf{h}1}_{\mathsf{d}}$ algebra (A, M) can be considered as an $\mathsf{E_{d-1}}$ algebra by forgetting the structure maps in equation (13). For the final adjunction, any $\mathsf{E_{d-1}}$ algebra in $\mathcal{C}^{\Sigma_{\leq 1}}$ has an underlying pair of $\mathcal{C}$-objects (A, M).

3.1.9 Notation Throughout the remainder of this paper $\mathcal{C}$ will be a cofibrantly generated symmetric monoidal model category tensored over Top such that the adjunctions in definition 3.1.8 are Quillen adjunctions.

3.1.10 Remark In [21, theorem 6], Spitzweck shows that $\mathrm{Mod}^A_{\mathcal{O}}(\mathcal{C}) \leftrightarrows \mathcal{C}$ is a Quillen adjunction if A is cofibrant in $\mathcal{C}$. In addition Berger-Moerdijk show in [3, proposition 4.1] that if $\mathcal{D}$ is a symmetric monoidal model category which is cofibrantly generated, has a cofibrant unit, and a symmetric monoidal fibrant replacement functor, then the category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})$ has the projective model structure induced from the forgetful functor to $\mathcal{D}$. The operad $\mathcal{O}$ in this theorem is an operad in $\mathcal{D}$. However, their result is more general, as seen in remark 4.6.4 of the same article. Their argument extends without change to show that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})$ has the desired model structure in the case that $\mathcal{O}$ is an operad in Top and $\mathcal{D}$ is tensored over Top as in definition 3.1.1, and the generating trivial cofibrations of $\mathcal{D}$ are cofibrant. The condition that the monoidal unit of $\mathcal{D}$ is cofibrant is not necessary in this situation. The cofibrance of the monoidal unit in Top is enough. Taking $\mathcal{O} = \mathsf{E_{d-1}}$ or $\mathcal{O} = \mathsf{SC_d^{h1}}$ and $\mathcal{D} = \mathcal{C}$ or $\mathcal{D} = \mathcal{C}^{\Sigma_{\leq 1}}$, we conclude that the assumptions in notation 3.1.9, and in the main theorem of this paper, hold when $\mathcal{C}$ is the category of compactly generated Hausdorff spaces (Top), or spectra, or chain complexes.

We use the model structure on the category of $\mathsf{E_{d-1}}$-A modules to define the Hochschild cochain object.

3.1.11 Definition Given an $\mathsf{E_{d-1}}$ algebra $A \in \mathcal{C}$, let the *Hochschild cochain object* of A be

$$\mathrm{Hoch}(A) = \underline{\mathrm{hom}}^A_{\mathsf{E_{d-1}}}(A^c, A^f),$$

where $\underline{\mathrm{hom}}^A_{\mathsf{E_{d-1}}}$ is given by the equalizer in definition 3.1.6. The $\mathsf{E_{d-1}}$-A modules A^c and A^f are cofibrant and fibrant replacements for A respectively. Note that $\mathrm{Hoch}(A)$ is an object of $\mathcal{C}$.

We will use the degree 0 and 1 pieces of the swiss cheese operad to build a cofibrant replacement for A as an $\mathsf{E_{d-1}}$-A module.

3.2 Swiss cheese in degrees zero and one

An $\mathsf{SC_d^{h1}}$ algebra (see definition 3.1.8 and example 2.3.5) is a pair (A, M) of objects of $\mathcal{C}$ together with maps in $\mathcal{C}^{\Sigma_{\leq 1}}$ for every m,

$$(\mathsf{E_{d-1}}(m), \mathsf{SC_d^h}(1, m)) \otimes (A, M)^{\otimes m} \to (A, M), \tag{10}$$

where we have used the isomorphism $\mathsf{SC}_\mathsf{d}^\mathsf{h}(0,m) \simeq \mathsf{E}_{\mathsf{d}-1}(m)$. Alternatively, we can view the morphism in equation (10) as three separate maps in $\mathcal{C}$.

$$\mathsf{E}_{\mathsf{d}-1}(m) \otimes A^{\otimes m} \to A \tag{11}$$

$$\mathsf{E}_{\mathsf{d}-1}(m) \otimes M \otimes A^{\otimes m-1} \to M \tag{12}$$

$$\mathsf{SC}_\mathsf{d}^\mathsf{h}(1,m) \otimes A^{\otimes m} \to M \tag{13}$$

The condition that the maps in equation (10) define an $\mathsf{SC}_\mathsf{d}^{\mathsf{h}1}$ structure on the pair (A,M) is the condition that the diagram (14) commutes in the $\mathsf{Top}^{\Sigma_{\leq 1}}$-enriched category $\mathcal{C}^{\Sigma_{\leq 1}}$.

$$\begin{array}{ccc} \mathsf{SC}_\mathsf{d}^{\mathsf{h}1}(m') \otimes \mathsf{SC}_\mathsf{d}^{\mathsf{h}1}(m;m') \otimes (A,M)^{\otimes m} & \longrightarrow & \mathsf{SC}_\mathsf{d}^{\mathsf{h}1}(m) \otimes (A,M)^{\otimes m} \\ \downarrow & & \downarrow \\ \mathsf{SC}_\mathsf{d}^{\mathsf{h}1}(m') \otimes (A,M)^{\otimes m'} & \longrightarrow & (A,M) \end{array} \tag{14}$$

In terms of equations (11), (12), and (13), diagram (14) splits into four diagrams. Each diagram is determined by the degrees of the three tensor factors in the upper left hand corner of diagram (14). In the first the degrees are $0,0,0$; in the second the degrees are $0,0,1$; in the third, $1,0,0$; and in the fourth, $0,1,0$.

$$\begin{array}{ccc} \mathsf{E}_{\mathsf{d}-1}(m') \otimes \mathsf{E}_{\mathsf{d}-1}(m;m') \otimes A^{\otimes m} & \longrightarrow & \mathsf{E}_{\mathsf{d}-1}(m) \otimes A^{\otimes m} \\ \downarrow & & \downarrow \\ \mathsf{E}_{\mathsf{d}-1}(m') \otimes A^{\otimes m'} & \longrightarrow & A \end{array} \tag{15}$$

The above diagram, (15), commutes for all m', m if and only if A is an $\mathsf{E}_{\mathsf{d}-1}$ algebra. Diagram (16) below commutes if and only if M is an $\mathsf{E}_{\mathsf{d}-1}$-$A$ module.

$$\begin{array}{ccc} \mathsf{E}_{\mathsf{d}-1}(m') \otimes \mathsf{E}_{\mathsf{d}-1}(m;m') \otimes M \otimes A^{\otimes m-1} & \longrightarrow & \mathsf{E}_{\mathsf{d}-1}(m) \otimes M \otimes A^{\otimes m'-1} \\ \downarrow & & \downarrow \\ \mathsf{E}_{\mathsf{d}-1}(m) \otimes M \otimes A^{\otimes m-1} & \longrightarrow & M \end{array} \tag{16}$$

Diagram (17) shows a compatibility condition between the degree 0 and degree 1 structures.

$$\begin{array}{ccc} \mathsf{SC}_\mathsf{d}^\mathsf{h}(1,m') \otimes \mathsf{E}_{\mathsf{d}-1}(m;m') \otimes A^{\otimes m} & \longrightarrow & \mathsf{SC}_\mathsf{d}^\mathsf{h}(1,m) \otimes A^{\otimes m} \\ \downarrow & & \downarrow \\ \mathsf{SC}_\mathsf{d}^\mathsf{h}(1,m') \otimes A^{\otimes m'} & \longrightarrow & M \end{array} \tag{17}$$

Diagram (18) presents another compatibility condition between the degree 0 and degree 1 structures.

$$\begin{array}{ccc} \mathsf{E_{d-1}}(m') \otimes \mathsf{SC^h_d}(1,m;0,m') \otimes A^{\otimes m} & \longrightarrow & \mathsf{SC^h_d}(1,m) \otimes A^{\otimes m} \\ \downarrow & & \downarrow \\ \mathsf{E_{d-1}}(m') \otimes M \otimes A^{\otimes m'-1} & \longrightarrow & M \end{array} \tag{18}$$

3.2.1 Example In this example we construct the universal extension of an $\mathsf{E_{d-1}}$ algebra A to an $\mathsf{SC^{h1}_d}$ algebra. We denote this universal pair by (A, A^{sc}). The composite forgetful functor

$$\mathrm{Alg}_{\mathsf{SC^{h1}_d}}(\mathcal{C}^{\Sigma_{\le 1}}) \to \mathrm{Alg}_{\mathsf{E_{d-1}}}(\mathcal{C}^{\Sigma_{\le 1}}) \to \mathrm{Alg}_{\mathsf{E_{d-1}}}(\mathcal{C}),$$

has a left adjoint which sends the $\mathsf{E_{d-1}}$ algebra A to the pair (A, A^{sc}) where A^{sc}, which may be read as "A swiss cheese", is a quotient of

$$\bar{A}^{sc} = \coprod_{m \ge 0} \mathsf{SC^h_d}(1,m) \otimes_{S_m} A^{\otimes m}, \tag{19}$$

where $S_m = \mathsf{aut}(\underline{m})$. We can think of $\bar{A}^{sc}$ heuristically as $\mathsf{SC^h_d}(1,\bullet) \otimes A^{\otimes \bullet}$. Now both A and $\mathsf{SC^h_d}(1,\bullet)$ carry an action of $\mathsf{E_{d-1}}$, so we can form the quotient $A^{sc} := \mathsf{SC^h_d}(1,\bullet) \otimes_{\mathsf{E_{d-1}}} A^{\otimes \bullet}$. More precisely, A^{sc} is defined as the coequalizer

$$\coprod_{m,m'} \mathsf{SC^h_d}(1,m) \otimes \mathsf{E_{d-1}}(m';m) \otimes A^{\otimes m'} \rightrightarrows \bar{A}^{sc} \xrightarrow{\text{coeq}} A^{sc}, \tag{20}$$

where one of the arrows is given by the operadic composition on swiss cheese and the other by the $\mathsf{E_{d-1}}$ structure on A. Figure 1 shows the relation $\sim$ such that $A^{sc} = \bar{A}^{sc}/\sim$.

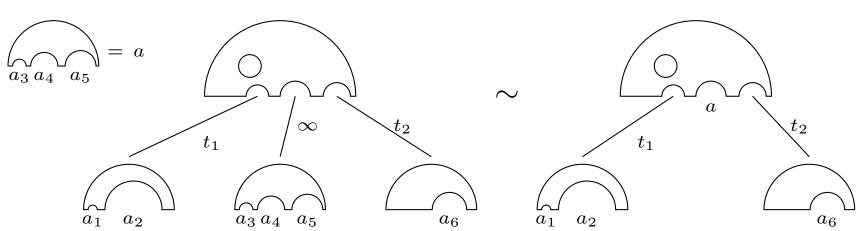


Figure 1: The relations in A^{sc} come from the $\mathsf{E_{d-1}}$ algebra structure of A. If m is the map $A^{\otimes 3} \to A$ given by the swiss cheese element in $\mathsf{SC^h_d}(0,3) \simeq \mathsf{E_{d-1}}(3)$ in the figure, set $a = m(a_1, a_2, a_3)$. The edges t_1 and t_2 are less than ∞, so the relation does not apply to the vertices on the left and right.

Verifying that (A, A^{sc}) is an $\mathsf{SC}_\mathsf{d}^{\mathsf{h}1}$ algebra is a matter of using the commuting left and right actions of $\mathsf{E}_{\mathsf{d}-1}$ on $\mathsf{SC}_\mathsf{d}^{\mathsf{h}1}$. By this we mean the morphism below uses both left and right actions, and can be obtained by performing the left action first, then the right, or vice versa,

$$\mathsf{E}_{\mathsf{d}-1}(m'') \times \mathsf{SC}_\mathsf{d}^{\mathsf{h}}(1, m'; 0, m'') \times \mathsf{E}_{\mathsf{d}-1}(m; m') \to \mathsf{SC}_\mathsf{d}^{\mathsf{h}}(1, m).$$

The left action defines a map

$$\mathsf{E}_{\mathsf{d}-1}(m) \otimes \bar{A}^{sc} \otimes A^{\otimes m-1} \to \bar{A}^{sc}, \tag{21}$$

using the the $\circ_1$ operad composition. Since the left and right $\mathsf{E}_{\mathsf{d}-1}$ actions on $\mathsf{SC}_\mathsf{d}^{\mathsf{h}1}$ commute, the arrow in (21) descends to give the data of a $\mathsf{E}_{\mathsf{d}-1}$-$A$ module structure on A^{sc}, i.e. equation (12) for $M = A^{sc}$. Of course, the maps from equation (13) with $M = A^{sc}$ are simply given by $\bar{A}^{sc} \to A^{sc}$.

Now let us observe that the four diagrams (15)-(18) commute for (A, A^{sc}). The first diagram, (15), is trivial since A is an $\mathsf{E}_{\mathsf{d}-1}$ algebra. The second diagram, (16), commutes since the left action of $\mathsf{E}_{\mathsf{d}-1}$ on $\mathsf{SC}_\mathsf{d}^{\mathsf{h}1}$ is indeed an action. That is, it is compatible with composition in $\mathsf{E}_{\mathsf{d}-1}$. The third diagram, (17), clearly commutes. Indeed, this diagram is the reason the coequalizer $\bar{A}^{sc} \to A^{sc}$ in equation (20) was defined in the first place. Finally, the fourth diagram, (18), certainly commutes if $M = \bar{A}^{sc}$. In this case, note that the bottom map in diagram (18) corresponds to equation (21). Thus if we pass from $\bar{A}^{sc}$ to A^{sc} this last diagram still commutes since, by definition, the $\mathsf{E}_{\mathsf{d}-1}$-$A$ module structure on A^{sc} is defined using the quotient map $\bar{A}^{sc} \to A^{sc}$ together with equation (21).

3.2.2 Lemma *Fix an $\mathsf{E}_{\mathsf{d}-1}$ algebra A and consider A^{sc} as an $\mathsf{E}_{\mathsf{d}-1}$-$A$ module, then there is an isomorphism of categories*

$$\mathrm{Mod}^A_{\mathsf{SC}_\mathsf{d}^{\mathsf{h}1}}(\mathcal{C}) \cong \mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C})_{A^{sc}/},$$

where $\mathrm{Mod}^A_{\mathsf{SC}_\mathsf{d}^{\mathsf{h}1}}(\mathcal{C})$ is the fiber over A of the forgetful functor $\mathrm{Alg}_{\mathsf{SC}_\mathsf{d}^{\mathsf{h}1}}(\mathcal{C}^{\Sigma_{\leq 1}}) \to \mathrm{Alg}_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C})$

Proof Let (A, M) be an $\mathsf{SC}_\mathsf{d}^{\mathsf{h}1}$ algebra extending the existing $\mathsf{E}_{\mathsf{d}-1}$ algebra structure on A. Then $M \in \mathrm{Mod}^A_{\mathsf{E}_{\mathsf{d}-1}}(\mathcal{C})$ and the structure maps in equation (13), when combined for all m, give a map $\bar{A}^{sc} \to M$. This descends to an $\mathsf{E}_{\mathsf{d}-1}$-$A$ module map $A^{sc} \to M$.

On the other hand if M is an $\mathsf{E}_{\mathsf{d}-1}$-$A$ module, then M is already equipped with the data of equation (12). If $A^{sc} \to M$ is a morphism of $\mathsf{E}_{\mathsf{d}-1}$-$A$ modules, then

M is equipped with the data of equation (13). The diagram (14) commutes because of relation defining A^{sc} in equation (20) and because $A^{sc} \to M$ is a morphism of $\mathsf{E_{d-1}}$-A modules. $\square$

3.2.3 Corollary *Let A be a cofibrant $\mathsf{E_{d-1}}$ algebra, then A^{sc} is a cofibrant $\mathsf{E_{d-1}}$-A module.*

Proof The forgetful functor $\mathrm{Alg}_{\mathsf{SC_d^{h1}}}(\mathcal{C}) \to \mathrm{Alg}_{\mathsf{E_{d-1}}}(\mathcal{C})$ preserves fibrations (see definition 3.1.8). Thus the left adjoint of this forgetful functor, applied to the cofibrant object A, gives a cofibrant $\mathsf{SC_d^{h1}}$ algebra (A, A^{sc}). Thus A^{sc} is cofibrant in $\mathrm{Mod}^A_{\mathsf{SC_d^{h1}}}(\mathcal{C})$. Lemma 3.2.2 shows that the forgetful functor $\mathrm{Mod}^A_{\mathsf{SC_d^{h1}}}(\mathcal{C}) \to \mathrm{Mod}^A_{\mathsf{E_{d-1}}}(\mathcal{C})$ preserves pushouts. The model structures here are cofibrantly generated, so A^{sc} is also cofibrant as an object of $\mathrm{Mod}^A_{\mathsf{E_{d-1}}}(\mathcal{C})$. $\square$

3.2.4 Definition Let $p(m) : \mathsf{SC_d^h}(1, m) \to \mathsf{SC_d^h}(0, m) \simeq \mathsf{E_{d-1}}(m)$ be the projection which forgets the single full disk. We can make $p = (p(m))_{m \geq 0}$ into a morphism of operads in the following way. For each m consider the degree 0-1 symmetric sequence $(\mathsf{E_{d-1}})^{\leq 1}(m) := (\mathsf{E_{d-1}}(m), \mathsf{E_{d-1}}(m))$. The structure of $\mathsf{E_{d-1}}$ as an operad in Top can be used to make $(\mathsf{E_{d-1}})^{\leq 1}$ an operad in $\mathsf{Top}^{\Sigma_{\leq 1}}$. This makes $(\mathsf{id}, p)\colon \mathsf{SC_d^{h1}} \to (\mathsf{E_{d-1}})^{\leq 1}$ into a morphism of operads.

If A is an $\mathsf{E_{d-1}}$ algebra we can define a morphism in $\mathcal{C}$,

$$\bar{A}^{sc} = \coprod_{m \geq 0} \mathsf{SC_d^h}(1, m) \otimes_{S_m} A^{\otimes m} \to \coprod_{m \geq 0} \mathsf{E_{d-1}}(m) \otimes_{S_m} A^{\otimes m} \to A,$$

where the first arrow uses p and the second arrow uses the $\mathsf{E_{d-1}}$ algebra structure on A. This map factors to give a morphism of $\mathsf{E_{d-1}}$-A modules $p_A\colon A^{sc} \to A$.

3.2.5 Remark By [21, section 5] we can conclude that $\mathsf{Op}(\mathsf{Top}^{\Sigma_{\leq 1}})$ is tensored over Top. If $\mathcal{O} = (\mathcal{O}^0, \mathcal{O}^1)$ is an operad in degree 0-1 symmetric sequences of topological spaces, and $K \in \mathsf{Top}$, then $K \otimes \mathcal{O}$ is defined to be the coequalizer

$$F(F(K \otimes \mathcal{O})) \rightrightarrows F(K \otimes \mathcal{O}) \xrightarrow{\text{coeq}} K \otimes \mathcal{O},$$

where $K \otimes \mathcal{O}$, is the symmetric sequence of degree 0-1 symmetric sequences whose arity m, degree i component is $K \times \mathcal{O}^i(m)$, and F is the free operad functor.

If $\mathcal{O}$ is a cofibrant operad in $\mathsf{Top}^{\Sigma_{\leq 1}}$, then $[0, 1] \otimes \mathcal{O}$ is a cylinder object, and a homotopy $h\colon [0, 1] \otimes \mathcal{O} \to \mathcal{P}$ gives the data of maps $h^i(m)\colon [0, 1] \times \mathcal{O}^i(m) \to \mathcal{P}^i(m)$, which assemble into $h^i(m';m)\colon [0,1]\times \mathcal{O}^i(m';m)\to \mathcal{P}(m';m)$ for $i=0,1$ and $m',m\geq 0$. The homotopy h is compatible with operad composition in the sense that if $\alpha\in\mathcal{O}^i(m)$ and $\beta\in\mathcal{O}^j(m';m)$, $i+j\leq 1$, and $t\in[0,1]$, then

$$h^i(m)(t,\alpha)\circ h^j(m';m)(t,\beta)=h^{i+j}(m')(t,\alpha\circ\beta).$$

3.2.6 Lemma *For any* $\mathsf{E_{d-1}}$ *algebra* $A\in\mathcal{C}$*, the map* $A^{sc}\to A$ *is a weak equivalence of* $\mathsf{E_{d-1}}$-A *modules.*

Proof Abusing notation we write $p:=(\mathsf{id},p)$ for the morphism of operads from definition 3.2.4. One can show that p is a weak equivalence of fibrant and cofibrant operads in degree 0-1 symmetric sequences of topological spaces. Therefore there is a map of operads $\iota\colon(\mathsf{E_{d-1}})^{\leq 1}\to\mathsf{SC_d^{h1}}$ and there are homotopies $h\colon \mathsf{id}_{\mathsf{SC_d^{h1}}}\simeq \iota p$, and $g\colon \mathsf{id}_{(\mathsf{E_{d-1}})^{\leq 1}}\simeq p\iota$. Since $[0,1]\otimes -$ distributes over coequalizers, we can use h to define a homotopy $h_A\colon[0,1]\otimes A^{sc}\to A^{sc}$,

$$\begin{array}{ccc}
\coprod_m [0,1]\otimes \mathsf{SC_d^h}(1,m)\otimes A^{\otimes m} & \longrightarrow & [0,1]\otimes A^{sc}\\
\downarrow{\scriptstyle \sqcup h^1(m)\otimes \mathsf{id}_A^{\otimes m}} & & \downarrow{\scriptstyle h_A}\\
\coprod_m \mathsf{SC_d^h}(1,m)\otimes A^{\otimes m} & \longrightarrow & A^{sc},
\end{array}$$

where $h^1(m)$ is defined from h as in remark 3.2.5. When $t=1$, $h_A(1,-)$ factors as

$$\begin{array}{ccc}
\coprod_m \mathsf{SC_d^h}(1,m)\otimes A^{\otimes m} & \longrightarrow & A^{sc}\\
\downarrow & \overset{s}{\dashleftarrow} & \downarrow{\scriptstyle p_A}\\
\coprod_m \mathsf{E_{d-1}}(m)\otimes A^{\otimes m} & \longrightarrow & A\\
\downarrow & & \downarrow{\scriptstyle \iota_A}\\
\coprod_m \mathsf{SC_d^h}(1,m)\otimes A^{\otimes m} & \longrightarrow & A^{sc},
\end{array}$$

where p_A is the map from definition 3.2.4, and ι_A is the evident composite in the diagram using s. The map s is a section of the middle horizontal arrow, defined using the identity of the operad $\mathsf{E_{d-1}}$,

$$A\simeq\{1_{\mathsf{E_{d-1}}}\}\otimes A\to \mathsf{E_{d-1}}(1)\otimes A\to\coprod_m \mathsf{E_{d-1}}(m)\otimes A^{\otimes m}.$$

The map ι_A is a map of $\mathsf{E_{d-1}}$-A modules, and h_A is a homotopy $\mathsf{id}_{A^{sc}}\simeq\iota_A p_A$. Similarly, g defines a homotopy $g_A\colon \mathsf{id}_A\to p_A\iota_A$. $\square$

The precise sense in which Hochschild cohomology can be obtained from the degree 0-1 pieces of the swiss cheese operad is contained in the following proposition.

3.2.7 Proposition *Let A be a fibrant and cofibrant $\mathsf{E_{d-1}}$ algebra. Then the Hochschild cochain object of A can be computed as*

$$\mathsf{Hoch}(A) \simeq \underline{\hom}^{A}_{\mathsf{E_{d-1}}}(A^{sc}, A).$$

Proof We are using the projective model structure from definition 3.1.8, so A is fibrant as an object of $\mathcal{C}$ and thus as a $\mathsf{E_{d-1}}$-A module. By corollary 3.2.3 and lemma 3.2.6, A^{sc} is a cofibrant replacement for A as an $\mathsf{E_{d-1}}$-A module. By definition 3.1.11, this proves the proposition. □

4 The universal cheese theorem

In the one-colored operad $\mathsf{SC_d^{h1}}$ the single full disk was never considered as input, only as a marker of degree one. Allowing the single disk to be considered as giving an input means viewing $\mathsf{SC_d^{h1}}$ as a partially defined 2-colored operad. Rather than making the notion *partially defined* precise, we simply set up the notion of a 2-colored algebra over $\mathsf{SC_d^{h1}}$ in definition 4.1.8. Proposition 4.1.9 is a version of the swiss cheese theorem for the operad $\mathsf{SC_d^{h1}}$. That is, a 2-colored $\mathsf{SC_d^{h1}}$ structure on the pair (B,A) is equivalent to a $\mathcal{C}$-morphism, $B \to \mathsf{Hoch}(A)$. In other words, $\mathsf{Hoch}(A)$ is the universal object of $\mathcal{C}$ acting on the $\mathsf{E_{d-1}}$ algebra A through $\mathsf{SC_d^{h1}}$. This result is generalized twice, first in corollary 4.1.11, then in proposition 4.1.16. We refer to proposition 4.1.16 as the universal cheese theorem since it replaces E_d in the swiss cheese theorem with an arbitrary operad.

4.1.8 Definition Let A be an $\mathsf{E_{d-1}}$ algebra. Let $\mathcal{C}_{/\mathsf{Hoch}(A)}$ denote the over category of $\mathsf{Hoch}(A) \in \mathcal{C}$. More precisely, the objects are $\mathcal{C}$-morphisms $B \to \mathsf{Hoch}(A)$ and the morphisms are $\mathcal{C}$ morphisms $B \to B'$ commuting with the maps to $\mathsf{Hoch}(A)$. In addition, let $\mathrm{Alg}^{A}_{\mathsf{SC_d^{h1}}}(\mathcal{C})$ denote the category of $\mathsf{SC_d^{h1}}$ algebras of the form (B,A) where the induced $\mathsf{E_{d-1}}$ structure on A is the one given. Morphisms are maps of $\mathsf{SC_d^{h1}}$ algebras which are identity on A.

Given a pair of objects (B,A) of $\mathcal{C}$, we let $\mathsf{End^{h1}}(B,A)$ denote the operad obtained by applying the forgetful functor $\mathsf{Op}_K \to \mathsf{Op}(\mathsf{Top}^{\Sigma_{\leq 1}})$ to $\mathsf{End}(B,A)$ from (4) and (5).

Let $\mathrm{Alg}^{A}_{\mathsf{SC}^{\mathsf{h1}}_{\mathsf{d}}}(\mathcal{C})$ denote the category of objects $B \in \mathcal{C}$ together with a morphism in $\mathsf{Op}(\mathsf{Top}^{\Sigma_{\leq 1}})$, $\mathsf{SC}^{\mathsf{h1}}_{\mathsf{d}} \to \mathsf{End}^{\mathsf{h1}}(B, A)$ extending the $\mathsf{E}_{\mathsf{d-1}}$ structure on A. A morphism $B \to B'$ in $\mathcal{C}$ induces a map of operads $\mathsf{End}^{\mathsf{h1}}(B', A) \to \mathsf{End}^{\mathsf{h1}}(B, A)$. Such a morphism gives a map in $\mathrm{Alg}^{A}_{\mathsf{SC}^{\mathsf{h1}}_{\mathsf{d}}}(\mathcal{C})$ if this induced map respects to maps from $\mathsf{SC}^{\mathsf{h1}}_{\mathsf{d}}$.

4.1.9 Proposition *There is an isomorphism of categories*

$$\mathrm{Alg}^{A}_{\mathsf{SC}^{\mathsf{h1}}_{\mathsf{d}}}(\mathcal{C}) \cong \mathcal{C}_{/\mathrm{Hoch}(A)}.$$

Proof The data of an algebra on the left hand side is an object $B \in \mathcal{C}$ together with maps

$$\mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(1, m) \otimes B \otimes A^{\otimes m} \to A, \tag{22}$$

for each $m \geq 0$. The conditions on (22) are that diagrams (23) and (24) commute.

$$\begin{array}{ccc} \mathsf{E}_{\mathsf{d-1}}(m') \otimes \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(1, m; 0, m') \otimes B \otimes A^{\otimes m} & \longrightarrow & \mathsf{E}_{\mathsf{d-1}}(m') \otimes A^{\otimes m'} \\ \downarrow & & \downarrow \\ \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(1, m) \otimes B \otimes A^{\otimes m} & \longrightarrow & A, \end{array} \tag{23}$$

$$\begin{array}{ccc} \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(1, m') \otimes \mathsf{E}_{\mathsf{d-1}}(m; m') \otimes B \otimes A^{\otimes m} & \longrightarrow & \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(1, m) \otimes B \otimes A^{\otimes m} \\ \downarrow & & \downarrow \\ \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(1, m') \otimes B \otimes A^{\otimes m'} & \longrightarrow & A. \end{array} \tag{24}$$

Equivalently, we can use the hom-tensor adjunction and assemble the maps in (22) to a single map $B \to \underline{\mathrm{hom}}_{\mathcal{C}}(\bar{A}^{sc}, A)$ (see (19)). The commutativity of diagram (24) is equivalent to this map lifting to

$$B \to \underline{\mathrm{hom}}_{\mathcal{C}}(A^{sc}, A). \tag{25}$$

Note that, dual to (20), $\underline{\mathrm{hom}}_{\mathcal{C}}(A^{sc}, A)$ is given by the equalizer

$$\underline{\mathrm{hom}}_{\mathcal{C}}(A^{sc}, A) \xrightarrow{\mathrm{eq}} \underline{\mathrm{hom}}_{\mathcal{C}}(\bar{A}^{sc}, A) \rightrightarrows \underline{\mathrm{hom}}_{\mathcal{C}}(\coprod_{m,m'} \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(1, m') \otimes \mathsf{E}_{\mathsf{d-1}}(m; m') \otimes A^{\otimes m}, A).$$

With this observation we can now rewrite diagram (23) as

$$\begin{array}{ccc} B \otimes F^{A}_{\mathsf{E}_{\mathsf{d-1}}}(A^{sc}) & \longrightarrow & F^{A}_{\mathsf{E}_{\mathsf{d-1}}}(A) \\ \downarrow & & \downarrow \\ B \otimes A^{sc} & \longrightarrow & A. \end{array} \tag{26}$$

Recall that $F^{A}_{\mathsf{E_{d-1}}}\colon \mathcal{C}\to\mathcal{C}$ is the free $\mathsf{E_{d-1}}$-A module functor from definition 3.1.6. Clearly, diagram (23) commutes if and only if diagram (26) commutes and if and only if the map (25) factors through $\mathrm{Hoch}(A) = \underline{\mathrm{hom}}^{A}_{\mathsf{E_{d-1}}}(A^{sc}, A)$. We conclude that the data of an $\mathsf{SC}^{\mathsf{h1}}_{\mathsf{d}}$ algebra structure on (B, A) is the data of an $\mathsf{E_{d-1}}$ algebra structure on A together with a $\mathcal{C}$-morphism $B \to \mathrm{Hoch}(A)$.

It is clear that a map $(B, A) \to (B', A)$ which is identity on A gives an $\mathsf{SC}^{\mathsf{h1}}_{\mathsf{d}}$ algebra morphism if and only if the map $B \to B'$ commutes with the corresponding morphisms to $\mathrm{Hoch}(A)$. □

4.1.10 Definition Define $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}} \in \mathsf{Op}(\mathsf{Top}^{\Sigma})$ as the image of the left adjoint of $\mathsf{Op}(\mathsf{Top}^{\Sigma}) \to \mathsf{Op}(\mathsf{Top}^{\Sigma_{\le 1}})$ applied to $\mathsf{SC}^{\mathsf{h1}}_{\mathsf{d}}$. Following definition 4.1.8, define $\mathrm{Alg}^{A}_{\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}}(\mathcal{C})$ as the category of objects $B \in \mathcal{C}$ together with a morphism in $\mathsf{Op}(\mathsf{Top}^{\Sigma})$, $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}} \to \mathrm{End}^{\mathsf{h}}(B, A)$ extending the $\mathsf{E_{d-1}}$ structure on A.

4.1.11 Corollary *There is an isomorphism of categories*

$$\mathrm{Alg}^{A}_{\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}}(\mathcal{C}) \cong \mathcal{C}_{/\mathrm{Hoch}(A)}.$$

Proof The adjunction isomorphism puts operad maps $\mathsf{SC}^{\mathsf{h1}}_{\mathsf{d}} \to \mathrm{End}^{\mathsf{h1}}(B, A)$ in one-to-one correspondence with operad maps $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}} \to \mathrm{End}^{\mathsf{h}}(B, A)$. This gives an isomorphism between $\mathrm{Alg}^{A}_{\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}}(\mathcal{C})$ and $\mathrm{Alg}^{A}_{\mathsf{SC}^{\mathsf{h1}}_{\mathsf{d}}}(\mathcal{C})$. Now apply proposition 4.1.9. □

4.1.12 Definition For each $n \ge 0$, let $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}(n, \bullet)$ denote the operad in $\mathsf{Op}(\mathsf{Top}^{\Sigma_{\le 1}})$ whose arity m component is the degree 0-1 symmetric sequence $(\mathsf{E_{d-1}}(m), \mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}(n, m))$. One may think of $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}(n, \bullet)$ as a bimodule over $\mathsf{E_{d-1}}$. Let $\mathrm{End}(\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}})$ be the symmetric sequence whose n^{th} space is $\mathsf{E_{d-1}}$ bimodule maps

$$\mathrm{End}(\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}})(n) \coloneqq \mathrm{map}_{\mathsf{E_{d-1}}}(\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}(1, \bullet), \mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}(n, \bullet)).$$

4.1.13 Lemma *Operadic composition in* $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}$ *induces the structure of an operad on the symmetric sequence* $\mathrm{End}(\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}})$.

Proof Define an operad $\mathcal{E} \in \mathsf{Op}(\mathsf{Top}^{\Sigma})$ by setting

$$\mathcal{E}(n, m) = \mathrm{map}_{\mathsf{Top}^{\Sigma}}\left(\mathrm{End}(\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}})(\bullet; n),\ \mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}(\bullet, m)\right).$$

Let $n_0 + \cdots + n_m = n$ and $\ell_1 + \cdots + \ell_m = \ell$. The monoidal structure $\otimes$ on symmetric sequences gives a map from $\mathcal{E}(n_0, m) \times \prod_{i=1}^{m} \mathcal{E}(n_i, \ell_i)$ to

$$\operatorname{map}_{\mathsf{Top}^{\Sigma}}\left(\bigotimes_{i=0}^{m} \mathsf{End}(\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty})(\bullet; n_i),\ \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty}(\bullet, m) \otimes \left(\bigotimes_{i=1}^{m} \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty}(\bullet, \ell_i)\right)\right). \tag{27}$$

Now push forward from (27) via the operad structure on $\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty}$,

$$\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty}(\bullet, m) \otimes \left(\bigotimes_{i=1}^{m} \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty}(\bullet, \ell_i)\right) \to \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty}(\bullet, \ell),$$

and pull back from (27) by

$$\mathsf{End}(\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty})(\bullet; n) \to \bigotimes_{i=0}^{m} \mathsf{End}(\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty})(\bullet; n_i). \tag{28}$$

This defines

$$\mathcal{E}(n_0, m) \times \prod_{i=1}^{m} \mathcal{E}(n_i, \ell_i) \to \mathcal{E}(n, \ell).$$

The morphism in (28) comes from the sequence of maps

$$\begin{aligned}
\mathsf{End}(\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty})(k; n) &= \coprod_{f\colon \underline{k} \to \underline{n}} \prod_{i=1}^{n} \mathsf{End}(\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty})(f^{-1}(i)) \\
&= \coprod_{f\colon \underline{k} \to \underline{n}} \prod_{j \in \underline{0} \sqcup \underline{m}} \prod_{i \in g^{-1}(j)} \mathsf{End}(\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty})(f^{-1}(i)) \\
&\to \coprod_{\tilde{f}\colon \underline{k} \to \underline{0} \sqcup \underline{m}} \prod_{j \in \underline{0} \sqcup \underline{m}} \mathsf{End}(\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty})(\tilde{f}^{-1}(j); n_j).
\end{aligned}$$

The first equality holds by definition, the second is a regrouping. The decomposition $\sum_{i=0}^{m} n_i = n$ defines a map $g\colon \underline{n} \to \underline{0} \sqcup \underline{m}$ where $|g^{-1}(i)| = n_i$. The third map sends the component corresponding to $f\colon \underline{k} \to \underline{n}$ to the component corresponding to $fg\colon \underline{k} \to \underline{0} \sqcup \underline{m}$. For each n, m there is a map,

$$\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty}(n, m) \to \mathcal{E}(n, m). \tag{29}$$

When $n = 0, 1$, the map (29) is canonical. Restricting to degrees 0 and 1 gives a map of $\mathsf{Op}(\mathsf{Top}^{\Sigma_{\leq 1}})$ operads $\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}1} \to \mathcal{E}^1$, where $\mathcal{E}^1$ is the degree 0-1 part of $\mathcal{E}$. Since $\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty}$ is freely generated by its degree 0 and 1 pieces, we get (29) for all n, assembling into a map of operads in $\mathsf{Op}(\mathsf{Top}^{\Sigma})$. This guarantees that (30) can be used to define an operadic composition law on $\mathsf{End}(\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty})$,

$$\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty}(n, m) \times \mathsf{End}(\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty})(k_1) \times \cdots \times \mathsf{End}(\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty})(k_n) \to \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty}(k, m). \tag{30}$$

□

4.1.14 Definition Let $\mathcal{O}$ be any 1-colored topological operad, and let $\rho\colon \mathcal{O} \to \mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})$ be a map of operads. Define the K-colored operad $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \rtimes_\rho \mathcal{O}$ by setting

$$(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \rtimes_\rho \mathcal{O})^\mathsf{h}(n,m) = \mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}(n,m) \qquad (\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \rtimes_\rho \mathcal{O})^\mathsf{f}(n,m) = \begin{cases} \mathcal{O}(n) & m = 0 \\ \emptyset & m > 0 \end{cases}$$

Composition in $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \rtimes_\rho \mathcal{O}$ uses composition in $\mathcal{O}$, composition in $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}$, and the action of $\mathcal{O}$ on $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}$ defined by ρ,

$$\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}(n,m) \times \mathcal{O}(k;n) \to \mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}(n,m) \times \mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})(k;n) \to \mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}(k,m),$$

where the right arrow above is the one in (30).

4.1.15 Lemma *Let $\mathcal{O}$ be a topological operad and let $\rho\colon \mathcal{O} \to \mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})$ be a map of operads, then the $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}$ structure on (H,A) naturally extends to a $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \rtimes_\rho \mathcal{O}$ structure on (H,A). In particular, $H = \mathrm{Hoch}(A)$ inherits an $\mathcal{O}$ algebra structure.*

Proof We only need to show there is a map of operads $\mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}) \to \mathsf{End}(H)$ compatible with the action of $\mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})$ on $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}$ and the action of $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}$ on H. Indeed, the map

$$\mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})(n) \otimes H^{\otimes n} \to H \tag{31}$$

is adjoint to the maps, for all $m \geq 0$,

$$\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}(1,m) \otimes \mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})(n) \otimes H^{\otimes n} \otimes A^{\otimes m} \to \mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}(n,m) \otimes H^{\otimes n} \otimes A^{\otimes m} \to A,$$

where the first arrow is (30) and the second arrow is the $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}$ structure on (H,A). To check that (31) is compatible with composition in $\mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})$ observe that there are two morphisms of operads in $\mathsf{Op}(\mathsf{Top}^\Sigma)$,

$$\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \rightrightarrows \mathsf{End}^\mathsf{h}(\mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})(H), A),$$

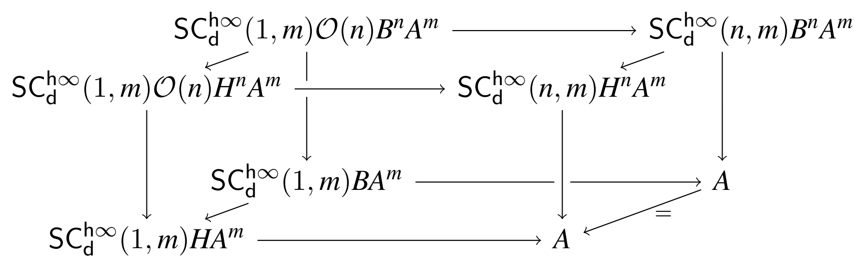
where $\mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})(H)$ is the free $\mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})$ algebra generated by H. One of the arrows uses the action of $\mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})$ on $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}$, while the other uses the map $\mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})(H) \to H$ defined by (31). To check that these arrows agree, we only need to check that they agree out of $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}(n,m)$ when $n = 0, 1$. This is because $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}$ is freely generated in degrees 0 and 1. When $n = 0$, the maps are obviously the same. When $n = 1$, the maps are the same by definition of the $\mathsf{SC}_\mathsf{d}^{\mathsf{h}1}$ structure on (H,A). $\square$

4.1.16 Proposition (The universal cheese theorem) *Let $\mathcal{O}$ be a topological operad and let $\rho\colon \mathcal{O} \to \mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})$ be a map of operads. Then using the induced $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \rtimes_\rho \mathcal{O}$ structure on $(\mathsf{Hoch}(A), A)$ from lemma 4.1.15 gives an isomorphism of categories*

$$\mathrm{Alg}^A_{(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \rtimes_\rho \mathcal{O})}(\mathcal{C}) \cong \mathrm{Alg}_\mathcal{O}(\mathcal{C})_{/\mathrm{Hoch}(A)}.$$

Proof Given any $\mathcal{C}$ morphism $B \to H$ we can form the following diagram. For brevity, we have deleted appearances $\otimes$.

(32)



Let (B,A) be a $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \rtimes_\rho \mathcal{O}$ algebra extending the given $\mathsf{E}_{\mathsf{d}-1}$ structure on A, then by corollary 4.1.11 we get a $\mathcal{C}$ morphism $B \to H = \mathsf{Hoch}(A)$ making the right face of the cube (32) commute. The front face commutes by lemma 4.1.15. The back face commutes by assumption. The bottom face commutes by definition, and the top face commutes trivially. This implies that, after composition with the maps whose codomain is A, the left face of the cube commutes. By adjointness, the two maps $\mathcal{O}(B) \rightrightarrows H$ agree, implying that $B \to H$ is indeed an $\mathcal{O}$ algebra morphism.

On the other hand, given an $\mathcal{O}$ algebra B together with an $\mathcal{O}$ algebra map $B \to H$, we get an $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}$ structure on (B,A) from the underlying $\mathcal{C}$ morphism. We only need to check that the $\mathcal{O}$ structure on B and the $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}$ structure on (B,A) are compatible via ρ. Indeed, since $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}$ is freely generated in degrees 0 and 1, it is enough to check that the back face of the cube commutes. But this holds because all other faces commute. Most importantly, the left face commutes because $B \to H$ is an $\mathcal{O}$ algebra map.

It is easy to see that each of these constructions are natural in B and are inverse to one another. □

5 The homotopy E_d structure on $\mathrm{Hoch}(A)$

In light of proposition 4.1.16, to prove the swiss cheese theorem, 2.2.1, we need to construct $\mathsf{E} \simeq E_d$ and an operad morphism $\mathsf{E} \to \mathsf{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})$ in such a way that the corresponding K-colored operad $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}} \rtimes \mathsf{E}$ is equivalent to SC_d. While $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}$ has no obvious action of E_d, it is equivalent to something that does have an E_d action. The following theorem is proven in section 6.

5.1.17 Theorem *The natural map $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}} \to \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}$ is an acyclic cofibration of operads in $\mathsf{Op}(\mathsf{Top}^{\Sigma})$.*

In this section, we define the precise sense in which theorem 5.1.17 gives us our E_d action on $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}$ up to homotopy. First, we have a lift p in the following diagram,

$$\begin{array}{ccc} \mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}} & \xrightarrow{\mathsf{id}} & \mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}} \\ {\scriptstyle\iota}\downarrow & \overset{p}{\nearrow} & \\ \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}} & & \end{array}$$

We know that $\mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}$ is cofibrant since it is obtained as the W construction applied to a Σ-cofibrant, well-pointed operad SC^{h}_d so it fits into the context covered by Berger-Moerdijk [1, 4]. In Spitzweck [21] we see that the corner axiom (or Quillen's SM7) for monoidal model categories tensored over topological spaces applies to categories of operads in topological spaces. Thus we have an acyclic fibration

$$\mathrm{map}(\mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}, \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}) \xrightarrow{\iota^*} \mathrm{map}(\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}, \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}})$$

given by pre-composing with ι. Since both ιp and id live over ι, they must be homotopic. Let $h\colon [0,\infty] \otimes \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}} \to \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}$ be a homotopy with $h_0 = \mathsf{id}$ and $h_\infty = \iota p$.

We will use this h to define a homotopy right E_d module structure on $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}$. For this we will use a homotopy equivalent version of E_d which sits inside the W construction. For simplicity we denote it by E. First we define the category LE_d. This category is not monoidal, but will be used to build E. The letter L stands for *level trees*. The objects of the topological category LE_d are finite sets, and the morphism space $LE_d(n, n')$ is defined to be a quotient of

$$\coprod_{\substack{k\geq 0\\ n_1,\ldots,n_k}} E_d(n_1, n') \times E_d(n_2, n_1) \times \cdots \times E_d(n, n_k) \times [0,\infty]^k$$

A point of the space above is given by a sequence $\alpha_i \in E_d(n_i, n_{i-1})$ for $1 \leq i \leq k+1$ and $t_i \in [0,\infty]$ for $1 \leq i \leq k$. For convenience of notation, we set $n_0 = n'$, $n_{k+1} = n$, $t_0 = \infty$, and $t_{k+1} = \infty$. We impose the following relations

5.1.18 Relations If $t_i = 0$, then we can delete t_i and replace $(\dots, \alpha_i, \alpha_{i+1}, \dots)$ by the composition $(\dots, \alpha_i \circ \alpha_{i+1}, \dots)$. If $n_i = n_{i-1}$ and α_i is the identity, and $t_{i-1} = \infty = t_i$, then we can delete α_i from the sequence and delete t_i from the sequence.

5.1.19 Remark In the W construction, we could always delete the appearance of an identity and sum the lengths of the surrounding edges. We do not allow that here since we do not have $h_{s+t} = h_s \circ h_t$.

Composition in the category LE_d is given by concatenating sequences, setting the new coordinate in the factor $[0, \infty]$ between the two sequences to be ∞.

We can use the action of E_d on $\mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}$ as well as the maps h_t, p, ι to define

$$LE_d(n, n') \to \mathrm{map}(\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}(n', m), \mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}(n, m)). \tag{33}$$

To do this, represent $\alpha \in LE_d(n, n')$ with a sequence $n' = n_0, n_1, \dots, n_k, n_{k+1} = n$ together with $\alpha_i \in E_d(n_i, n_{i-1})$ for $1 \le i \le k+1$ and $t_i \in [0, \infty]$ for $1 \le i \le k$. This gives a chain of maps

$$\begin{aligned} \mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}(n', m) \xrightarrow{\iota} \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_0, m) \xrightarrow{\alpha_1} \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_1, m) \xrightarrow{h_{t_1}} \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_1, m) \xrightarrow{\alpha_2} \cdots \\ \cdots \xrightarrow{h_{t_k}} \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_k, m) \xrightarrow{\alpha_{k+1}} \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_{k+1}, m) \xrightarrow{p} \mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}(n, m). \end{aligned} \tag{34}$$

The maps $\mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_i, m) \xrightarrow{\alpha_i} \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_{i+1}, m)$ are defined by the action of E_d on $\mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}$,

$$\mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_i, m) \times E_d(n_{i+1}, n_i) \to \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_{i+1}, m).$$

Let us check that the relations 5.1.18 in LE_d are satisfied and that composition in E_d corresponds to composition of maps of $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}$. Suppose $t_i = 0$ for some i. Then $h_0 = \mathsf{id}$ so our chain of arrows contains

$$\mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_{i-1}, m) \xrightarrow{\alpha_i} \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_i, m) \xrightarrow{\alpha_{i+1}} \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_{i+1}, m).$$

The composition of these two is equal to the map given by $\alpha_i \alpha_{i+1} \in E_d(\underline{n}_{i+1}, \underline{n}_{i-1})$. This is because $\mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(-, m)$ is a right E_d module.

If $n_i = n_{i-1}$, α_i is identity, and $t_{i-1} = t_i = \infty$, then the composition $h_{t_{i-1}} \circ \alpha_i \circ h_{t_i}$ is equal to $h_{t_{i-1}} = h_\infty$, so we are justified in deleting α_i and t_i from the sequence.

Now suppose we have some $t_i = \infty$, so that $\alpha \in LE_d(n, n')$ decomposes as $\beta_1 \beta_2$ for some $\beta_1 \in \mathsf{E}(n_i, n')$ and $\beta_2 \in \mathsf{E}(n, n_i)$. The chain of compositions defining the action of α from $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}(1, m)$ to $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}(n, m)$ contains the following segment.

$$\cdots \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_i, m) \xrightarrow{h_\infty} \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_i, m) \to \cdots$$

The composite of the actions of β_1 and β_2 is computed by joining the chains for β_1 and for β_2. This joined chain agrees with the chain for α except for the segment above, which is replaced with the segment

$$\cdots \to \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}(n_i, m) \xrightarrow{p} \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty}(n_i, m) \xrightarrow{\iota} \mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}(n_i, m) \to \cdots$$

Since $h_\infty = \iota p$, these chains of maps have the same composition.

The maps (33) define a functor

$$LE_d \to \mathrm{End}(\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}\infty}). \tag{35}$$

There is no obvious operad structure on LE_d so we take the smallest operad containing LE_d. More precisely, (35) is a morphism of topological categories whose objects are finite sets. There is a forgetful functor from operads to the category of such topological categories. The operad E is defined to be the result of applying the left adjoint of this forgetful functor to the category LE_d.

5.1.20 Definition Let $F(LE_d)$ be the free one-colored operad generated by the symmetric sequence $n \mapsto LE_d(n, 1)$. For each $n, n' \geq 0$ let $\mathsf{E}(n, n')$ be the topological space given by the coequalizer

$$\coprod_{n''} LE_d(n'', n') \times LE_d(n, n'') \rightrightarrows F(LE_d)(n, n') \xrightarrow{\text{eq}} \mathsf{E}(n, n'),$$

where the two maps are given by composition in either $F(LE_d)$ or LE_d and the inclusion of LE_d into $F(LE_d)$.

5.1.21 Lemma *The category E is an operad and is equivalent to E_d.*

Proof Given a tree with its internal edges labeled by lengths $[0, \infty]$, call it a *level tree* if edges equidistant from the root vertex have the same length. Every morphism in $LE_d(n, 1)$ can be represented by a level tree with vertices labeled by E_d. We can represent a point of $F(LE_d)$ with a tree whose vertices are labeled by level trees in LE_d. The relation defining $F(LE_d) \to \mathsf{E}$ allows us to break up a level tree with at least one level of length ∞ into several level trees all of whose levels have finite length. We conclude that E consists of trees labeled by E_d on the vertices, and $[0, \infty]$ on the internal edges, satisfying the condition that every maximal finite subtree is level.

There is an operad morphism $\mathsf{E} \to E_d$ which collapses all edge lengths to 0. On the level of symmetric sequences, there is a homotopy inverse $E_d \to \mathsf{E}$. The homotopy $g_t \colon \mathsf{E} \to \mathsf{E}$ first collapses lengths of the edges furthest from the root to zero. This preserves the condition that every maximal finite subtree is level. Continuing in this way, we collapse all edge lengths to zero. □

The adjoint to (35) is an operad morphism $\mathsf{E} \to \mathrm{End}(\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty})$, which by definition 4.1.14 we can use to define the K-colored operad $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \rtimes \mathsf{E}$.

5.1.22 Lemma *The* $\{\mathsf{f},\mathsf{h}\}$*-colored operad* $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \rtimes \mathsf{E}$ *is weakly equivalent to the swiss cheese operad.*

Proof First, note that SC_d is equivalent to the semi-direct product of $\mathsf{SC}_\mathsf{d}^\mathsf{h}$ and E_d where the action of E_d factors through the map $\mathsf{E}_\mathsf{d} \to E_d$ which sends all lengths of internal edges to zero. This is because the map $\mathsf{SC}_\mathsf{d}^\mathsf{h} \to SC_d^\mathsf{h}$ which collapses trees is a weak equivalence and respects the action of E_d.

The action of E on $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}$ can be extended to an action on all of $\mathsf{SC}_\mathsf{d}^\mathsf{h}$. The sequence $\alpha_1, t_1, \cdots, t_k, \alpha_{k+1}$ acts via the composition

$$\begin{aligned}(36)\quad \mathsf{SC}_\mathsf{d}^\mathsf{h}(n_0,m) \xrightarrow{\alpha_1} \mathsf{SC}_\mathsf{d}^\mathsf{h}(n_1,m) \xrightarrow{h_{t_1}} \mathsf{SC}_\mathsf{d}^\mathsf{h}(n_1,m) \to \cdots \to \mathsf{SC}_\mathsf{d}^\mathsf{h}(n_{l-1},m) \\ \xrightarrow{h_{t_l}} \mathsf{SC}_\mathsf{d}^\mathsf{h}(n_{l-1},m) \xrightarrow{\alpha_l} \mathsf{SC}_\mathsf{d}^\mathsf{h}(n_l,m) \xrightarrow{h_\infty} \mathsf{SC}_\mathsf{d}^\mathsf{h}(n_l,m).\end{aligned}$$

Define for each $s \in [0,\infty]$ a homotopy $h^{[0,s]} \colon [0,\infty] \otimes \mathsf{SC}_\mathsf{d}^\mathsf{h} \to \mathsf{SC}_\mathsf{d}^\mathsf{h}$ by setting $h_t^{[0,s]} = h_{\min(s,t)}$. We have $h_0^{[0,s]} = \mathsf{id}$ and $\mathsf{h}_\infty^{[0,s]} = h_s$, therefore we can define an action of E on $[0,\infty]_s \otimes \mathsf{SC}_\mathsf{d}^\mathsf{h}$ by replacing h_{t_i} in (36) with $h_{t_i}^{[0,s]}$. Then, when $s = 0$ each $h_{t_i}^{[0,0]}$ is the identity, so the action factors through the map $\mathsf{E} \to E_d$ collapsing all edges to 0. When $s = \infty$ we have $h_{t_i}^{[0,\infty]} = h_{t_i}$ so the action of E on $\mathsf{SC}_\mathsf{d}^\mathsf{h}$ is 36. Thus we have a diagram of equivalences

$$SC_d \leftarrow \mathsf{SC}_\mathsf{d}^\mathsf{h} \rtimes_{s=0} \mathsf{E} \to ([0,1] \otimes \mathsf{SC}_\mathsf{d}^\mathsf{h}) \rtimes \mathsf{E} \leftarrow \mathsf{SC}_\mathsf{d}^\mathsf{h} \rtimes_{s=\infty} \mathsf{E} \to \mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \rtimes \mathsf{E},$$

where the map on the left collapses all edge lengths to 0. □

6 The equivalence $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \to \mathsf{SC}_\mathsf{d}^\mathsf{h}$

This section is dedicated to proving theorem 5.1.17. The proof uses a recasting (6.1.4) of definition 2.1.1 which is equivalent when considering operads in Top [10]. First we set the context for this new definition, then we prove that $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \to \mathsf{SC}_\mathsf{d}^\mathsf{h}$ is a cofibration. Finally, we show that it is a weak equivalence.

6.1 The category of Forests

The following definition is an amalgamation of those found in [17], [7] and [10].

6.1.1 Definition Fix a set K. A *K-colored young forest* is an uncolored map of finite K-colored sets $x\colon I_x \to J_x$. A *K-colored forest* $f\colon x \to y$ is a color-preserving isomorphism $f\colon I_y \sqcup J_x \to J_y \sqcup I_x$ such that for every $i \in I_x$ there is a $k \geq 0$ such that $(f \circ x)^k(i) \in J_y$ and for every $i \in I_y$ there is an $\ell \geq 0$ such that $(f \circ x)^\ell \circ f(i) = y(i)$.

6.1.2 Definition Given a K-colored forest $f\colon x \to y$ we call $V(f) \coloneqq J_x$ the set of *internal vertices* of f. We call $\mathsf{in}(f) \coloneqq I_y$ the set of *input vertices* of f and $\mathsf{rt}(f) \coloneqq J_y$ the set of *root vertices* of f. In addition, $Edge(f) = J_x \sqcup I_y \cong J_y \sqcup I_x$ is called the set of *extended edges* of f and $E(f) \coloneqq J_x \times_f I_x$ is the set of *internal edges* of f.

6.1.3 Definition If $g\colon x \to y$ and $f\colon y \to z$ are forests, we can define a composite forest $fg\colon x \to z$. We use concatenation to denote this composition and $\circ$ to denote composition of maps of finite sets. The forest fg, as a map $I_z \sqcup J_x \to J_z \sqcup I_x$, is defined by the following rule. If $i \in I_z$, then there is a $k \geq 0$ and an $\varepsilon \in \{0, 1\}$ such that $g^\varepsilon(f \circ g)^k(i) \in J_z \sqcup I_x$. Similarly, if $i \in J_x$ there is a k and ε such that $f^\varepsilon(g \circ f)^k(i) \in J_z \sqcup I_x$. In [23], it is shown that forest composition is associative. This, together with disjoint union, makes young forests the objects and forests the morphisms of a symmetric monoidal category denoted For.

6.1.4 Definition A K-colored operad $\mathcal{O}$ is a strong symmetric monoidal functor $(\mathsf{For}_K, \sqcup) \to (\mathsf{Top}, \times)$. The category of operads is the category $\mathsf{Fun}^{\otimes}(\mathsf{For}_K, \mathsf{Top})$ of symmetric monoidal functors and natural transformations. We denote this category by Op_K just as in definition 2.3.3.

6.1.5 Remark The category of K-colored operads Op_K as defined in 2.3.3 is naturally isomorphic to the functor category $\mathsf{Fun}^{\otimes}(\mathsf{For}_K, \mathsf{Top})$. Indeed, given $\mathcal{O}$ from 2.1.1 as in definition 2.3.3 we define $\mathcal{O}(x)$ for a young forest x to be $\bigotimes_{j \in J_x} \mathcal{O}(x^{-1}(j); j)$.

6.1.6 Remark When the set of colors K is understood, we often drop it from the notation. Forests and young forests are always K-colored, for some set K. The category For_K will be abbreviated For, and Op_K will be denoted Op.

6.1.7 Definition Let $(C, \otimes)$ be a symmetric monoidal category. Call an object $c \in C$ *indecomposable* if it cannot be written as a tensor product $c \cong c_1 \otimes c_2$ for any $c_1, c_2 \in C$. Let $\mathsf{Fun}^{\otimes}(C, \mathsf{Top})$ denote the category of strong symmetric monoidal functors $(C, \otimes) \to (\mathsf{Top}, \times)$. We call a morphism $\psi \colon \mathcal{O} \to \mathcal{P}$ in $\mathsf{Fun}^{\otimes}(C, \mathsf{Top})$ a *fibration* (respectively *weak equivalence*) if $\psi(c) \colon \mathcal{O}(c) \to \mathcal{P}(c)$ is a fibration (respectively weak equivalence) for every indecomposable $c \in C$. Define the class of *cofibrations* in the usual manner (see Hovey [12]).

For every symmetric monoidal category C we consider in this paper we will use 6.1.7 to define cofibrations, fibrations, and weak equivalences regardless of whether or not these form a model structure.

6.1.8 Remark A young forest x is indecomposable (definition 6.1.7) if $J_x \simeq *$. In this case we say x is a young *tree*. We say a forest $f \colon y \to x$ is a *tree* if x is a young tree.

6.1.9 Definition For a category C, let $C^{\times}$ denote the category with the same objects as those of C, but only invertible morphisms. Let C_{indec} denote the full subcategory (not monoidal) of C consisting indecomposable objects. Let $C^{\times}_{\text{indec}} = (C_{\text{indec}})^{\times}$. The functor category $\mathsf{Top}^{C^{\times}_{\text{indec}}}$ is called the category of *C-symmetric sequences*, denoted C-sSeq.

6.1.10 Remark A forest $f \colon x \to y$ is invertible if and only if $f(J_x) = J_y$ and $f(I_y) = I_x$. Thus an invertible forest gives a pair of isomorphisms of K-colored finite sets $J_f \colon J_x \to J_y$ and $I_f \colon I_y \to I_x$ which are compatible with the maps $x \colon I_x \to J_x$ and $y \colon I_y \to J_y$. We conclude that the category $\mathsf{For}^{\times}_{\text{indec}}$ is isomorphic to the *opposite* of the category of K-colored finite sets and bijections. In the case $K \simeq \{*\}$ we get $\mathsf{For}^{\times}_{\text{indec}} \cong \Sigma$ where Σ is as in definition 2.3.1. Moreover, in this case the category of For-symmetric sequences is isomorphic to the category Top^{Σ} of symmetric sequences.

6.1.11 Notation If $C \to \mathsf{For}$ is any symmetric monoidal functor, we will denote the category $\mathsf{Fun}^{\otimes}(C, \mathsf{Top})$ of strong symmetric monoidal functors by C-Op unless we say otherwise (for example, we do not use this notation in definition 6.3.4). We call the objects of C-Op *C-operads*. In all cases we consider, the functor $C \to \mathsf{For}$ will be apparent from the category C so we leave the functor out of the notation. If $C \to D \to \mathsf{For}$ is a pair of symmetric monoidal functors we denote the forgetful functor D-$\mathsf{Op} \to C$-Op by U^D_C, and denote its left adjoint by F^C_D.

6.2 The W construction

We show how the W construction of Boardman and Vogt [5] can be realized as a coend construction using For. In nice situations the W construction gives a cofibrant replacement for an operad, as shown in [1]. We use $[0,\infty]$ as our edge labels as in [16].

Given any young forest z there is a contravariant functor $W\colon \mathsf{For}_{/z} \to \mathsf{Top}$ from the over category of z to topological spaces. For any object $g\colon y \to z$ of this over category, set $W(g) = \mathrm{map}(E(g), [0,\infty])$. If $f\colon x \to y$ is a forest, the map $W(g) \to W(gf)$ is denoted $W_\Sigma(f)$. This map uses the sum operation on $[0,\infty]$. This is an extension of $+$ on $[0,\infty)$ such that $t+\infty = \infty = \infty + t$ for all values of t. Concretely, g and gf define maps of sets

$$E(g) \xrightarrow{\tilde{g}} J_z \sqcup I_x \xleftarrow{\widetilde{gf}} E(gf)$$

We can turn a function $t \in W(g)$ to a function $W_\Sigma(t) \in W(gf)$ by pushing forward along $\tilde{g}$, then pulling back along $\widetilde{gf}$. Pushing forward means summing over fibers, which is well-defined since all the sets we are considering are finite.

If $h\colon z \to w$ is a forest there is a morphism $W_\infty(h)\colon W(g) \to W(hg)$ which uses the maps $E(g) \hookrightarrow I_y \hookleftarrow E(hg)$. In this case we do not push forward and pull back functions. Rather we extend a function $t\colon E(g) \to [0,\infty]$ to a function on $E(hg)$ by setting $t(\epsilon) = \infty$ if $\epsilon \notin E(g)$. This defines a natural transformation $W_\infty(h)\colon W \to Wh_*$ where $h_*\colon \mathsf{For}^{op}_{/z} \to \mathsf{For}^{op}_{/w}$ is induced by $h\colon z \to w$. In other words, diagram (37) commutes for all composeable triples h, g, and f.

(37)
$$\begin{array}{ccc} W(g) & \xrightarrow{W_\Sigma(f)} & W(gf) \\ {\scriptstyle W_\infty(h)}\downarrow & & \downarrow{\scriptstyle W_\infty(h)} \\ W(hg) & \xrightarrow{W_\Sigma(f)} & W(hgf) \end{array}$$

Consider an operad $\mathcal{O}$ as a collection of functors $\mathcal{O}_z\colon \mathsf{For}_{/z} \to \mathsf{Top}$ by setting $\mathcal{O}_z(g) := \mathcal{O}(y)$ for $g\colon y \to z$. For a young forest z the topological space $W\mathcal{O}(z)$ is the coend

(38)
$$W\mathcal{O}(z) = W \otimes_{\mathsf{For}_{/z}} \mathcal{O}_z = \left(\coprod_{g\colon y \to z} W(g) \times \mathcal{O}(y) \right) /\!\sim$$

where $(W_\Sigma(f)t, \alpha) \sim (t, \mathcal{O}(f)\alpha)$ for every $f\colon x \to y$, $t \in W(g)$ and $\alpha \in \mathcal{O}(x)$. Now a forest $h\colon z \to w$ with the natural transformations above gives us a map

(39)
$$W\mathcal{O}(z) = W \otimes_{\mathsf{For}_{/z}} \mathcal{O}_z \xrightarrow{W_\infty(h)\otimes \mathrm{id}} Wh_* \otimes_{\mathsf{For}_{/z}} \mathcal{O}_z \to W \otimes_{\mathsf{For}_{/w}} \mathcal{O}_w = W\mathcal{O}(w).$$

This defines $W\mathcal{O}$ as a functor $\mathsf{For} \to \mathsf{Top}$. This functor is symmetric monoidal, so $W\mathcal{O}$ is a K-colored operad.

6.2.1 Remark The space $W\mathcal{O}(z)$ consists of labeled forests $f\colon y \to z$. A labeled forest is one whose internal edges each have a length in $[0,\infty]$ and whose vertices $j \in J_x$ have a corresponding label in $\mathcal{O}(x^{-1}(j);j)$.

In the sequel, we will define several variants on the category For. Each of these variants admits a functor to For and we want a corresponding W construction for each.

6.2.2 Definition Suppose $\mathcal{O}$ is an operad $\mathsf{For} \to \mathsf{Top}$ and C is any symmetric monoidal category equipped with a symmetric monoidal functor $G\colon C \to \mathsf{For}$. From this data we can construct an operad $W_C\mathcal{O} \in C\text{-}\mathsf{Op}$ using formulas analogous to (38) and (39). Specifically, let W_C be the composite $W \circ G_c\colon (C_{/c})^{op} \to (\mathsf{For}_{/G(c)})^{op} \to \mathsf{Top}$ where $c \in C$. We define

$$W_C\mathcal{O}(c) = W_C \otimes_{C_{/c}} (G^*\mathcal{O})_c = \left(\coprod_{f\colon b \to c} W(G(f)) \times \mathcal{O}(G(y)) \right) /\sim \tag{40}$$

We define $W_C\mathcal{O}(c) \to W_C\mathcal{O}(d)$ for a C-morphism $g\colon c \to d$ just as in (39).

6.2.3 Example Define the full subcategory $D \hookrightarrow \mathsf{For}$ to be given by those young forests x where J_x has only color h. Restricting the swiss cheese operad to this full subcategory gives the operad $\mathsf{SC}_\mathsf{d}^\mathsf{h}$ as $W_D SC_d$. To get $\mathsf{SC}_\mathsf{d}^{\mathsf{h}1}$, we use $C \hookrightarrow \mathsf{For}$, the full subcategory of young forests x where J_x has color h and for each $j \in J_x$, there is at most one element of $x^{-1}(j)$ of color f.

It should be clear that $D\text{-}\mathsf{Op} \simeq \mathsf{Op}(\mathsf{Top}^\Sigma)$. Indeed, the forgetful functor $\mathsf{Op}_K \to \mathsf{Op}(\mathsf{Top}^\Sigma)$ from definition 2.3.3 is given by pulling back along the symmetric monoidal functor $D \to \mathsf{For}_K$. Finally, note that $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}$ as defined in definition 4.1.10 is $F_D^C \mathsf{SC}_\mathsf{d}^{\mathsf{h}1}$, where F_D^C is defined in notation 6.1.11

6.3 Weighted Forests

We need to define the category of weighted forests to prove the following half of theorem 5.1.17.

6.3.1 Theorem *The natural map* $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \to \mathsf{SC}_\mathsf{d}^\mathsf{h}$ *is a cofibration in* $\mathsf{Op}(\mathsf{Top}^\Sigma)$.

The proof follows closely the work of Berger-Moerdijk [1]. From an operad $\mathcal{O}$ they construct an increasing chain of symmetric sequences

$$W_0\mathcal{O} \to W_0^+\mathcal{O} \to W_1\mathcal{O} \to W_1^+\mathcal{O} \to W_2\mathcal{O} \to \cdots \tag{41}$$

The symmetric sequence $W_0\mathcal{O}$ is just the underlying symmetric sequence of $\mathcal{O}$. If $\mathcal{O}$ is cofibrant as a symmetric sequence $\mathcal{O}$ is said to be Σ*-cofibrant*. If the operadic unit maps of $\mathcal{O}$ are cofibrations, $\mathcal{O}$ is said to be *well-pointed*. In the case $\mathcal{O}$ is well-pointed and Σ-cofibrant, Berger-Moerdijk show that $W\mathcal{O} \coloneqq \operatorname{colim}_k W_k\mathcal{O}$ is a cofibrant replacement of $\mathcal{O}$ as an operad. In the course of the proof, they show that $W_k\mathcal{O}$ is a k*-operad*, which is a partial operad in a certain sense. This partial operad structure will be encoded here in the category Op_k from definition 6.3.4. Each $W_k^+\mathcal{O}$ is an operad in Op_{k+1}, and in our context, is given by the left adjoint to a forgetful functor $\mathsf{Op}_{k+1} \to \mathsf{Op}_k$ applied to $W_k\mathcal{O}$. Concretely, the points of $W_k\mathcal{O}$ are given by trees with at most k internal edges whose vertices are labeled by $\mathcal{O}$ and whose internal edges are labeled by $[0,\infty]$. This section mimics this work of Berger-Moerdijk to prove theorem 6.3.1.

6.3.2 Definition Let $f\colon x \to y$ be a forest. Let $(f|x)$ denote the endomorphism of $I_y \sqcup J_y \sqcup I_x \sqcup J_x$ which is f on $I_y \sqcup J_x$, x on I_x, and the identity on J_y. By assumption that f is a forest, there is a $k \geq 0$ such that for every i we have $(f|x)^k(i) \in J_y$. Let $[f|x]$ denote $(f|x)^\infty$.

6.3.3 Definition For $I \subset I_y \sqcup I_x \sqcup J_x$ and $j \in J_y$ let $I(j)$ denote $I \cap [f|x]^{-1}(j)$, the set of elements of I living over j. A *weighted young forest* is a pair (x,ω_x) where x is a young forest and $\omega_x\colon J_x \to \mathbb{Z}_{\geq 0}$ is any function, called the *weight of* x. A *weighted forest* $f\colon (x,\omega_x) \to (y,\omega_y)$ is a forest $f\colon x \to y$ such that for all $j \in J_y$,

$$\omega_y(j) \geq \#E(f) + \sum_{i \in J_x(j)} \omega_x(i). \tag{42}$$

If $g\colon (y,\omega_y) \to (z,\omega_z)$ is a weighted forest. then one can show that $gf\colon x \to z$ defines a weighted forest $gf\colon (x,\omega_x) \to (z,\omega_z)$.

6.3.4 Definition Disjoint union of forests extends to disjoint union of weighted forests. Let For_ω denote the symmetric monoidal category of weighted forests. For each $k \geq 0$, let For_k denote the full subcategory of For_ω generated by objects of the form (x,ω_x) such that $\omega_x(j) \leq k$ for every $j \in J_x$. Let Op_k denote the category of *(K-colored) weight k operads*, which are strong symmetric monoidal functors $\mathsf{For}_k \to \mathsf{Top}$. Note that there is a symmetric monoidal functor $\mathsf{For}_\omega \to \mathsf{For}$ which forgets the weights.

6.3.5 Remark If a forest $f\colon x \to y$ has no internal edges then f is a disjoint union of isomorphisms and maps of the form $[\emptyset \to \emptyset] \to [\{\kappa\} \to \{\kappa\}]$, where $\kappa \in K$. The restriction of an operad $\mathcal{O}$ to For_0 remembers only

- the spaces $\mathcal{O}(I;\kappa)$ for each K-colored set I and each color κ, together with the right $\mathsf{aut}(I)$ action on $\mathcal{O}(I;\kappa)$, and
- the operadic unit maps $* \to \mathcal{O}(\kappa;\kappa)$ for each color $\kappa \in K$.

Thus $\mathcal{O}$ is cofibrant (definition 6.1.7) as an object of $\mathsf{Fun}^{\otimes}(\mathsf{For}_0, \mathsf{Top})$ if and only if it is well-pointed and Σ-cofibrant as in [1, section 3].

Proof of theorem 6.3.1 In example 6.2.3 we constructed categories C and D such that $\mathsf{SC}^{\mathsf{h1}}_{\mathsf{d}} = W_C SC_d$, $\mathsf{SC}^{\mathsf{h}}_{\mathsf{d}} = W_D SC_d$ and the map $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}} \to \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}$ is $F^C_D W_C SC_d \to W_D SC_d$. Lemma 6.3.12 applies since C is a full subcategory of D which is a full subcategory of For, and $SC_d\colon \mathsf{For} \to \mathsf{Top}$ is cofibrant as a functor $\mathsf{For}_0 \to \mathsf{Top}$ (see remark 6.3.5). □

6.3.6 Definition Let C be any full subcategory of For. Define C_ω to be the full subcategory of For_ω given by young weighted forests (x, ω_x) such that $x \in C$. The functor $\mathsf{For}_\omega \to \mathsf{For}$ induces a functor $C_\omega \to C$.

If $C \hookrightarrow \mathsf{For}$ is a full subcategory, there is a map of C_ω operads $W_{C_\omega} \to U^C_{C_\omega} W_C \mathcal{O}$. Concretely, for each weighted young forest (x, ω_x) we have a map

$$\coprod_{f\colon (y,\omega_y)\to(x,\omega_x)} W(f) \times \mathcal{O}(y) \to \coprod_{f\colon y\to x} W(f) \times \mathcal{O}(y), \tag{43}$$

defined in the obvious way. This descends to give a morphism

$$W_{C_\omega}\mathcal{O} \to U^C_{C_\omega} W_C \mathcal{O} \tag{44}$$

6.3.7 Lemma *For any full subcategory $C \hookrightarrow \mathsf{For}$, the left adjoint of (44),*

$$F^{C_\omega}_C W_{C_\omega}\mathcal{O} \to W_C\mathcal{O},$$

is an isomorphism.

Proof The left adjoint $F^{C_\omega}_C W_{C_\omega}\mathcal{O}$ can be computed at a young tree x as $\operatorname{colim}_{k\to\infty}(W_{C_\omega}\mathcal{O})(x,k)$. For each k, a point of $(W_{C_\omega}\mathcal{O})(x,k)$ is given by a labeled tree (6.2.1) $f\colon y \to x$ with at most k internal edges. Taking the colimit as k goes to ∞, we get all labeled trees over x, which is $W_C\mathcal{O}(x)$. □

6.3.8 Lemma *Let C be any symmetric monoidal full subcategory of For_ω. Suppose $\psi\colon \mathcal{O} \to \mathcal{P}$ is a morphism of operads in C-Op which is a cofibration of C-symmetric sequences. Further suppose that for every young forest $x \in C$ such that there is a tree $f\colon x \to y$ in C with at least one internal edge we have that $\psi(x)\colon \mathcal{O}(x) \to \mathcal{P}(x)$ is an isomorphism. These conditions imply that ψ is a cofibration of C-operads.*

Proof Suppose $\mathcal{Q} \to \mathcal{Q}'$ is an acyclic fibration of C-operads. By definition 6.1.7 this means that for every young tree $x \in C$, $\mathcal{Q}(x) \to \mathcal{Q}'(x)$ is an acyclic fibration of topological spaces. Suppose we have a commutative diagram of C-operads

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathcal{Q} \\ \downarrow & & \downarrow \\ \mathcal{P} & \longrightarrow & \mathcal{Q}'. \end{array} \tag{45}$$

By assumption, there is a lift $\rho\colon \mathcal{P} \to \mathcal{Q}$ *of C-symmetric sequences*. We claim that ρ is automatically a morphism of C-operads. To prove this claim, it is enough to show that the square on the right in diagram (46) commutes for every tree $f\colon x \to y$ in C.

$$\begin{array}{ccccc} \mathcal{O}(x) & \xrightarrow{\psi(x)} & \mathcal{P}(x) & \xrightarrow{\rho(x)} & \mathcal{Q}(x) \\ {\scriptstyle\mathcal{O}(f)}\downarrow & & {\scriptstyle\mathcal{P}(f)}\downarrow & & \downarrow{\scriptstyle\mathcal{Q}(f)} \\ \mathcal{O}(y) & \xrightarrow{\psi(y)} & \mathcal{P}(y) & \xrightarrow{\rho(y)} & \mathcal{Q}(y) \end{array} \tag{46}$$

Suppose f is an isomorphism. Then we know the square commutes because ρ is a map of C-symmetric sequences. Suppose f is not an isomorphism and has no internal edges. Since f is a tree it must be of the form $[\emptyset \to \emptyset] \to [\{c\} \to \{c\}]$ for some $c \in K$. In this case we have $\mathcal{O}(x) = \mathcal{P}(x) = \mathcal{Q}(x) = *$, in particular $\psi(x)$ is an isomorphism. Thus our assumption shows that if f is not an isomorphism then $\psi(x)$ is an isomorphism. We can deduce that the square on the right commutes in this case from the fact that the square on the left and the outer square commute. □

Given a full subcategory $D \hookrightarrow \mathsf{For}$ and a full subcategory $C \hookrightarrow D$, we can interpolate between C_ω and D_ω with a sequence of subcategories of For_ω. For each $k \geq 0$, let D_k be the full subcategory of For_ω given by disjoint unions of young weighted trees (x, ω_x) where either $x \in C$ or $x \in D$ and $\omega_x \leq k$. Note that $D_{-1} = C_\omega$. We have left the inclusion $C \hookrightarrow D$ implicit in the notation D_k.

We have a commutative diagram of symmetric monoidal functors

$$\begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & & \uparrow \\ C_\omega = D_{-1} \to D_0 \longrightarrow & \cdots & \longrightarrow D_\omega . \end{array} \tag{47}$$

The bottom row consists of successively larger full subcategories of For_ω and the top row consists of full subcategories of For.

From the symmetric monoidal functors in (47) we get the categories and forgetful functors in (48).

$$\begin{array}{ccc} C\text{-}\mathsf{Op} & \longleftarrow & D\text{-}\mathsf{Op} \\ \downarrow & & \downarrow \\ C_\omega\text{-}\mathsf{Op} = D_{-1}\text{-}\mathsf{Op} \longleftarrow D_0\text{-}\mathsf{Op} \longleftarrow & \cdots & \longleftarrow D_\omega\text{-}\mathsf{Op} \end{array} \tag{48}$$

6.3.9 Remark The forgetful functor $D_k\text{-}\mathsf{Op} \to D_\ell\text{-}\mathsf{Op}$ for $k \geq \ell$, where $k, \ell \in \{-1, 0, 1, \ldots\} \sqcup \{\omega\}$ clearly preserves fibrations (definition 6.1.7), thus the left adjoint $F_{D_k}^{D_\ell}$ preserves cofibrations.

6.3.10 Lemma *For each $k \geq 0$ there is a natural map in D_ω-Op,*

$$F_{D_\omega}^{D_k} W_{D_k}\mathcal{O} \to W_{D_\omega}\mathcal{O}.$$

The colimit of these maps as $k \to \infty$ is an isomorphism.

Proof For each young tree x and each weight $\omega_x = \ell \in \mathbb{Z}_{\geq 0}$, the space $(F_{D_\omega}^{D_k} W_{D_k}\mathcal{O})(x, \ell)$ can be described as the subspace of $(W_{D_\omega}\mathcal{O})(x, \ell)$ given by labeled trees $f\colon y \to x$ with at most ℓ internal edges such that, after cutting apart f at the edges labeled ∞, each remaining subtree of f has $\leq k$ internal edges. Taking the colimit as $k \to \infty$ we get all labeled trees $f\colon x \to y$ with at most ℓ internal edges. That is, we get all of $(W_{D_\omega}\mathcal{O})(x, \ell)$. □

6.3.11 Lemma *Suppose $\mathcal{O} \in \mathsf{Op}$ is cofibrant as an object of $\mathsf{Fun}^{\otimes}(\mathsf{For}_0, \mathsf{Top})$ (see remark 6.3.5). Then, for each $k \geq 0$ the natural map $\iota_k\colon F_{D_k}^{D_{k-1}} W_{D_{k-1}}\mathcal{O} \to W_{D_k}\mathcal{O}$ is a cofibration in D_k-Op.*

Proof Observe that if $f\colon x \to y$ is a tree in D_k with at least one internal edge, then $x \in D_{k-1}$, so $\iota_k(x)$ is an isomorphism. By lemma 6.3.8 we only need to show that ι_k is a cofibration of D_k-symmetric sequences.

In the case $k = 0$, if x is a tree in $C = D_{-1}$ then $\iota_0(x, 0)$ is an isomorphism. If x is in $D - C$ then $\iota(x, 0)$ is the map $\emptyset \to \mathcal{O}(x)$. The assumption that $\mathcal{O}$ is cofibrant as a functor $\mathsf{Fun}^{\otimes}(\mathsf{For}_0, \mathsf{Top})$ implies in particular that each $\mathcal{O}(x)$ is $\mathsf{aut}(x)$-cofibrant. That is, $\mathcal{O}$ is cofibrant as a For-symmetric sequence, so ι_0 is a cofibration of D_0-symmetric sequences. We now consider the case $k > 0$. We follow [1, lemma 5.4]. For a young tree $z \in \mathsf{For}_k$ and $g\colon y \to z$ a tree in For_k, let $(W \times \mathcal{O})_k^+(g)$ be $W(g) \times \mathcal{O}(y)$ if g has $\leq k-1$ internal edges. Otherwise let $(W \times \mathcal{O})_k^+(g) \subset W(g) \times \mathcal{O}(y)$ be the set of (t, α) such that $t(\epsilon) = 0$ or $t(\epsilon) = \infty$ for some $\epsilon \in E(g)$ or $\alpha(j) = \mathsf{id}$ for some $j \in V(g)$. Using the techniques of Berger and Moerdijk [1, section 2] one can also show that $\mathcal{O}$ cofibrant as an object of $\mathsf{Fun}^{\otimes}(\mathsf{For}_0, \mathsf{Top})$ (see remark 6.3.5) then $(W \times \mathcal{O})_k^+(g) \to W(g) \times \mathcal{O}(\mathsf{dom}\, g)$ is an $\mathsf{aut}(g)$-cofibration, where $\mathsf{aut}(g)$ is the automorphism group of g as an object of the category $\mathsf{For}_{k/z}$. Define the map $(W \times \mathcal{O})_k^+(g) \to F_{D_k}^{D_{k-1}} W_{D_{k-1}} \mathcal{O}(z)$ by collapsing any edge labeled 0 and deleting any vertex labeled with the identity.

In the diagram below, $\pi_0 \mathsf{For}_{k/z}$ is the set of isomorphism classes of forests $g\colon y \to z$ in For_k. The domain, y, of a forest $g\colon y \to z$ is denoted $\mathsf{dom}\, g$.

$$\begin{array}{ccc} \coprod_{[g] \in \pi_0 \mathsf{For}_{k/z}} ((W \times \mathcal{O})_k^+(g))_{\mathsf{aut}(g)} & \longrightarrow & (F_{D_k}^{D_{k-1}} W_{D_{k-1}} \mathcal{O})(z) \\ \downarrow & & \downarrow \\ \coprod_{[g] \in \pi_0 \mathsf{For}_{k/z}} (W(g) \times \mathcal{O}(\mathsf{dom}\, g))_{\mathsf{aut}(g)} & \longrightarrow & (W_{D_k} \mathcal{O})(z) \end{array} \tag{49}$$

The square in diagram (49) is a pushout. By remarks above, we know the map on the left in (49) is an $\mathsf{aut}(z)$-cofibration. We conclude $F_{D_k}^{D_{k-1}} W_{D_{k-1}} \mathcal{O} \to W_{D_k} \mathcal{O}$ is a cofibration of D_k-symmetric sequences. $\square$

6.3.12 Lemma *If C is a full subcategory of D and D is a full subcategory of For, and $\mathcal{O}$ is an operad $\mathsf{For} \to \mathsf{Top}$ which is cofibrant as a functor $\mathsf{For}_0 \to \mathsf{Top}$, then the natural map $F_D^C W_C \mathcal{O} \to W_D \mathcal{O}$ is a cofibration of D-operads.*

Proof By lemma 6.3.7 we have

$$W_C \mathcal{O} \cong F_C^{C_\omega} W_{C_\omega} \mathcal{O} = F_{C_\omega}^{C} W_{D_{-1}} \mathcal{O}.$$

The commutative diagram (48) shows that $F_D^C F_C^{C_\omega} = F_D^{D_\omega} F_{D_\omega}^{D_{-1}}$. This gives the first equality below.

$$\begin{aligned} F_D^C F_C^{C_\omega} W_{C_\omega} \mathcal{O} &= F_D^{D_\omega} F_{D_\omega}^{D_{-1}} W_{D_{-1}} \mathcal{O} \\ &\to F_D^{D_\omega} \operatorname*{colim}_k F_{D_\omega}^{D_k} W_{D_k} \mathcal{O} \\ &= F_D^{D_\omega} W_{D_\omega} \mathcal{O} \\ &\cong W_D \mathcal{O}. \end{aligned}$$

The equality in the third line above comes from lemma 6.3.10 and the fourth from lemma 6.3.7. The arrow above is a cofibration. Indeed, by repeated application of lemma 6.3.11 together with the fact that each $F_{D_k}^{D_{k-1}}$ preserves cofibrations (remark 6.3.9) we see that $F_{D_k}^{D_{k-1}} \cdots F_{D_0}^{D_{-1}} W_{D_{-1}}\mathcal{O} \to W_{D_k}\mathcal{O}$ is a cofibration in D_k-Op. Again, by remark 6.3.9 $F_{D_\omega}^{D_k}$ and $F_D^{D_\omega}$ preserve cofibrations. Thus the arrow above is a cofibration in D-Op. ☐

6.4 Weak equivalence proof

This section contains a proof of

6.4.1 Theorem *The natural map* $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty} \to \mathsf{SC}_\mathsf{d}^\mathsf{h}$ *is a weak equivalence of operads in* $\mathsf{Op}(\mathsf{Top}^\Sigma)$.

The idea of the proof is to consider the maps $p_1 \colon \mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}(n,m) \to \mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}(n-1,m)$ and $p \colon \mathsf{SC}_\mathsf{d}^\mathsf{h}(n,m) \to \mathsf{SC}_\mathsf{d}^\mathsf{h}(n-1,m)$ given by forgetting the n^{th} disk. By induction, we can suppose $\mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}(n-1,m) \to \mathsf{SC}_\mathsf{d}^\mathsf{h}(n-1,m)$ is a weak equivalence. We continue the induction by showing that $p_1^{-1}(\alpha) \to p^{-1}(\alpha)$ is a weak equivalence for every $\alpha \in \mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}(n-1,m)$.

To make the computation of $p_1^{-1}(\alpha)$ and $p^{-1}(\alpha)$ accessible, we will collapse the n^{th} disk of $\alpha \in \mathsf{SC}_\mathsf{d}^\mathsf{h}(n,m)$ to a point. Our goal in the next section is to make this precise.

6.4.1 Defining $SC_{d,\bullet}(k,\ell|n,m)$

When we collapse the n^{th} disk of $\alpha \in SC_d^\mathsf{h}(n,m)$ to its center, we think of the result $\hat{\alpha}$ as living in a four-colored operad which we denote by $SC_{d,\bullet}$. We add the colors $\mathsf{f}_\bullet$ and $\mathsf{h}_\bullet$. Let $K_\bullet = \{\mathsf{f}_\bullet, \mathsf{h}_\bullet, \mathsf{f}, \mathsf{h}\}$ be the set of colors for this new operad. The color $\mathsf{f}_\bullet$ stands for collapsed full disks. It is convenient to also allow a collapsed half disk, which we color with $\mathsf{h}_\bullet$. Let $(k,\ell|n,m)$ denote the $K_\bullet$-colored finite set with $k,\ell,n,$ and m elements of color $\mathsf{f}_\bullet, \mathsf{h}_\bullet, \mathsf{f},$ and h respectively. Let $D_\bullet$ denote the full sub category of $\mathsf{For}_{K_\bullet}$ with objects isomorphic to disjoint unions of the young trees

(50) $(0,0|n,m) \to \{\mathsf{h}\}\ \ (1,0|n,m) \to \{\mathsf{h}\}\ \ (0,1|n,m) \to \{\mathsf{h}\}\ \ (1,0|0,0) \to \{\mathsf{h}_\bullet\}.$

To define $SC_{d,\bullet} \colon D_\bullet \to \mathsf{Top}$ we need the notion of the geometric realization of $\beta \in SC_d^\mathsf{h}(n,m)$.

6.4.2 Definition Given $\beta \in SC_d^{\mathsf{h}}(n,m)$, let $|\beta|$ be its *geometric realization*. This is the subset of $\mathbb{R}^d$ given by deleting the open discs and half-discs of β from the closed unit half-disk. More precisely, if $\bar{D}_+^d$ is the closed unit half-disk in $\mathbb{R}^d$, $\{(D_{\mathsf{f}}^d)_j\}_{j=1}^m$ are the open discs of β, and $\{(D_{\mathsf{h}}^d)_i\}_{i=1}^n$ are the open half-discs of β considered as open discs in $\mathbb{R}^d$ whose center lies in $\mathbb{R}^{d-1}$, then

$$|\beta| = \bar{D}_+^d - \left(\left(\bigcup_{i=1}^n (D_{\mathsf{h}}^d)_i\right) \cup \left(\bigcup_{j=1}^m (D_{\mathsf{f}}^d)_j\right)\right).$$

Let $\partial_{\mathsf{h}}|\beta| \coloneqq \partial(\bar{D}_+^d - (\cup_i (D_{\mathsf{h}}^d)_i))$ be the h*-colored boundary* of $|\beta|$. Let $\partial_{\mathsf{rt}}(|\beta|)$ be the upper hemisphere $S_+^{d-1} \subset \partial\bar{D}_+^d$ and let $\partial_i|\beta|$ be the upper hemisphere of $\partial(D_{\mathsf{h}}^d)_i$ for $1 \le i \le n$.

Now we can set

$$\begin{aligned}
SC_{d,\bullet}^{\mathsf{h}}(0,0|n,m) &= SC_d^{\mathsf{h}}(n,m)\\
SC_{d,\bullet}^{\mathsf{h}}(1,0|n,m) &= \{(\alpha,q) \mid \alpha \in SC_d^{\mathsf{h}}(n,m), q \in |\alpha|\}\\
SC_{d,\bullet}^{\mathsf{h}}(0,1|n,m) &= \{(\alpha,q) \mid \alpha \in SC_d^{\mathsf{h}}(n,m), q \in |\alpha| \cap \mathbb{R}^{d-1}\}\\
SC_{d,\bullet}^{\mathsf{h}\bullet}(k,\ell|n,m) &= *
\end{aligned}$$

We think of the point $q \in |\alpha|$ as a collapsed disk and the point $q \in |\alpha| \cap \mathbb{R}^{d-1}$ as a collapsed half-disk. Composition in $SC_{d,\bullet}$ takes place in the half-discs and collapsed half-discs only. The un-collapsed discs play no part in composition. However the collapsed half-discs and collapsed discs only play a part in composition when we plug a collapsed disk into a collapsed half-disk. The result is a collapsed disk which happens to live on the boundary of the geometric realization. See figure 6.4.1.

6.4.3 Definition Let $\boldsymbol{C}_\bullet$ denote the full sub category of $\boldsymbol{D}_\bullet$ with objects isomorphic to disjoint unions of the young forests from (50) with $n \le 1$. Recall from example 6.2.3 that we used $\boldsymbol{D}$ to denote the full subcategory of For_K whose young trees are isomorphic to the trees in (50) with $k = \ell = 0$. Likewise $\boldsymbol{C}$ is the full sub category of $\boldsymbol{D}$ given by trees isomorphic to disjoint unions of forests from (50) with $k = \ell = 0$ and $n \le 1$. We write $x \le 1$ if $x \in \boldsymbol{C}$ or $x \in \boldsymbol{C}_\bullet$. We have that $\mathsf{SC}_{\mathsf{d}}^{\mathsf{h1}}$ is the restriction of SC_d to $\boldsymbol{C}$, and $\mathsf{SC}_{\mathsf{d}}^{\mathsf{h}}$ is the restriction of SC_d to $\boldsymbol{D}$. Let $SC_{d,\bullet}^{\mathsf{h1}}$ denote the restriction of $SC_{d,\bullet}$ to $\boldsymbol{C}_\bullet$.

Let $\mathsf{SC}_{\mathsf{d},\bullet}^{\mathsf{h}}$ and $\mathsf{SC}_{\mathsf{d},\bullet}^{\mathsf{h1}}$ denote the W construction applied to the four-colored operads $SC_{d,\bullet}$ and $SC_{d,\bullet}^{\mathsf{h1}}$ via definition 6.2.2. Let F denote Kan extension along $\boldsymbol{C}_\bullet \to \boldsymbol{D}_\bullet$, that is, $F = F_{\boldsymbol{D}_\bullet}^{\boldsymbol{C}_\bullet}$ using the notation from 6.1.11.

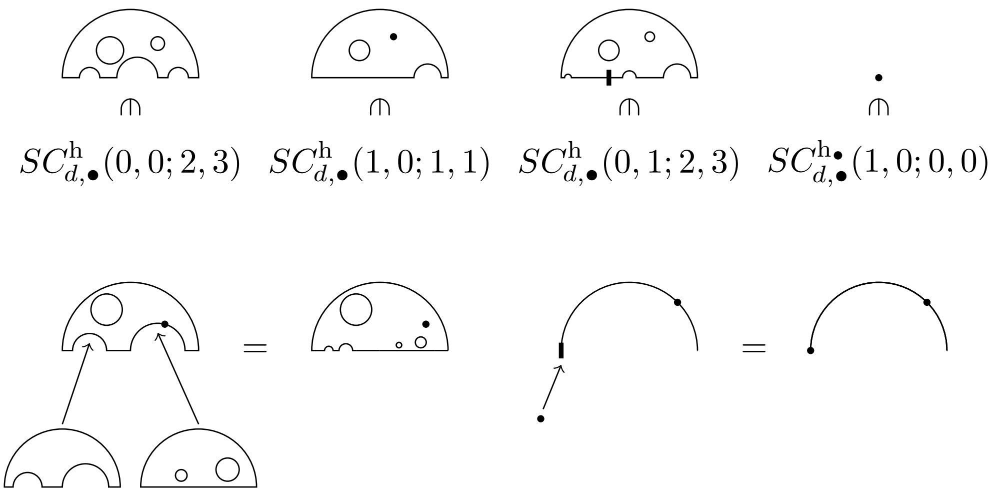


Figure 2: The collapsed discs are denoted by dots and the collapsed half-discs by tick marks. Collapsed discs are color $\mathsf{f}_\bullet$ input edges and collapsed half-discs are color $\mathsf{h}_\bullet$ input edges. To keep the collapsed discs and half-discs from coinciding, we only allow one or the other in any composition. Composition in $SC_{d,\bullet}$ takes place only in the half-discs and collapsed half discs. The only composition we can do in a collapsed half-disk is given by plugging in a collapsed disk. The result is a collapsed disk replacing the collapsed half-disk.

6.4.4 Definition Define $p\colon D_\bullet \to D$ by sending the $K_\bullet$-colored young forest $x\colon I_x \to J_x$ to the K-colored forest px with

$$I_{px} = (I_x)_{\mathsf{f},\mathsf{h}} \quad J_{px} = (J_x)_{\mathsf{f},\mathsf{h}},$$

where for $K' \subset K_\bullet$ we set $I_{K'} = \mathrm{col}_I^{-1}(K')$, $\mathrm{col}_I\colon I \to K_\bullet$. In (50) we see that $x(I_{px}) \subset J_{px}$ so that we can define px as the restriction of x to I_{px}. Observe that $p(1,0|n,m) = (n,m)$. If $f\colon y \to x$ is a forest, then $pf\colon py \to px$ is defined using f. Since f preserves the colorings pf is indeed a forest from py to px. If f is a morphism in $C_\bullet$ then pf is a morphism in C.

If $\beta \in SC_{d,\bullet}(z)$ for a $K_\bullet$-colored young forest z, then we get $p\beta \in SC_d(pz)$. To define $p\beta$ write $\beta = (\beta_j)_{j\in J_z}$ where $\beta_j \in SC_{d,\bullet}(z^{-1}(j);j)$. Each β_j is of the form (γ_j, q_j) with $q_j \in |\gamma_j|$ or of the form $\gamma_j \in SC_d(z^{-1}(j);j)$. Set $p\beta = (\gamma_j)_{j\in J_{pz}}$.

If $t \in W(f)$ and $f \in D_\bullet$, then $E(pf) \subset E(f)$ and $pt \in W(pf)$ is defined to be the pullback of $t\colon E(f) \to [0,\infty]$.

If $(f\colon y \to x, t \in W(f), \alpha \in SC_{d,\bullet}(x))$ represents a point in $\mathsf{SC}^{\mathsf{h}}_{\mathsf{d},\bullet}(x)$ then $(pf, pt, p\alpha)$ represents a point in $\mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(x)$. This defines the map $p\colon \mathsf{SC}^{\mathsf{h}}_{\mathsf{d},\bullet}(1,0|n,m) \to$

$\mathsf{SC^h_d}(n,m)$. The restriction of p to $F(\mathsf{SC^{h1}_{d,\bullet}})(1,0|n,m)$ factors through the inclusion $\mathsf{SC^{h\infty}_d}(n,m) \to \mathsf{SC^h_d}(n,m)$. Let p_1 be the induced map $F(\mathsf{SC^{h1}_{d,\bullet}})(1,0|n,m) \to \mathsf{SC^{h\infty}_d}(n,m)$.

Consider the commutative diagram of topological spaces where the horizontal arrows do not assemble to operad maps,

$$\begin{array}{ccccc} \mathsf{SC^{h\infty}_d}(n+1,m) & \xrightarrow{\sim} & F(\mathsf{SC^{h1}_{d,\bullet}})(1,0|n,m) & \xrightarrow{p_1} & \mathsf{SC^{h\infty}_d}(n,m) \\ \downarrow & & \downarrow & & \downarrow \iota \\ \mathsf{SC^h_d}(n+1,m) & \xrightarrow{\sim} & \mathsf{SC^h_{d,\bullet}}(1,0|n,m) & \xrightarrow{p} & \mathsf{SC^h_d}(n,m). \end{array}$$

The maps p_1 and p delete the collapsed disk and, if necessary, a left over collapsed half-disk. By induction on n we assume the right vertical arrow is an equivalence. We will show that for each $\alpha \in \mathsf{SC^{h\infty}_d}(n,m)$ the inclusion $p_1^{-1}(\alpha) \to p^{-1}(\iota\alpha)$ is an equivalence. Then by the long exact sequence of homotopy groups we conclude that the middle vertical arrow is an equivalence. The top left and bottom right maps collapse the n^{th} full disk. One can show that these are equivalences. We conclude that the left vertical arrow is also an equivalence. This will prove theorem 6.4.1.

6.4.2 Computing $p^{-1}(\iota\alpha)$ and $p_1^{-1}(\alpha)$.

We have shown that the proof rests on proposition 6.4.5 below. This section is dedicated to the proof of this proposition.

6.4.5 Proposition *Fix $\alpha \in \mathsf{SC^{h\infty}_d}(n,m)$. The inclusion of the fiber $p_1^{-1}(\alpha)$ into the fiber $p^{-1}(\iota\alpha)$ is a weak equivalence.*

Combining the W construction (38) and the left adjoint C-$\mathsf{Op} \to D$-Op (i.e. $\mathsf{Op}(\mathsf{Top}^{\Sigma_{\leq 1}}) \to \mathsf{Op}(\mathsf{Top}^{\Sigma})$) we get

$$\mathsf{SC^{h\infty}_d}(n,m) = \left(\coprod_{\substack{g\colon z \to y \\ f\colon y \to (n,m)}} W(g) \times SC_d(z) \right) /\sim ,$$

where $y \leq 1$ (definition 6.4.3) . If $\alpha \in \mathsf{SC^{h\infty}_d}(n,m)$ is represented by $(f,g,t,\tilde{\alpha})$ where $f\colon y \to (n,m)$, $g\colon z_\alpha \to y$, $t \in W(g)$, and $\tilde{\alpha} \in SC_d(z)$, then $\iota\alpha \in \mathsf{SC^h_d}(n,m)$ is represented by $(fg, W_\infty(f)t, \tilde{\alpha})$. Let $T_\alpha = fg\colon z_\alpha \to (n,m)$ and $t_\alpha = W_\infty(f)t$. Without loss of generality, we may assume $t_\alpha(i) > 0$ for every $i \in E(T_\alpha)$ and that $\tilde{\alpha}(j) \neq \mathsf{id}_{SC_d}$ for any $j \in J_{z_\alpha}$.

6.4.6 Definition Let $\mathsf{Trees}(1,0|n,m)$ denote the over-category $(D_\bullet)_{/(1,0|n,m)}$. Similarly, let $\mathsf{Trees}(n,m) = D_{/(n,m)}$. Let $p\colon \mathsf{Trees}(1,0|n,m) \to \mathsf{Trees}(n,m)$ denote the functor induced by p from definition 6.4.4.

Note that $T_\alpha \in \mathsf{Trees}(n,m)$. Let $(S,\nu) \in \mathsf{Trees}(1,0|n,m)_{/T_\alpha}$ where $S\colon x \to (1,0|n,m)$ is any $K_\bullet$-colored tree and $\nu\colon z_\alpha \to px$ is a forest such that $(pS)\nu = T_\alpha$. Define functors $W_\alpha\colon \mathsf{Trees}(1,0|n,m)^{op}_{/T_\alpha} \to \mathsf{Top}$ and $\mathsf{SC}_\alpha\colon \mathsf{Trees}(1,0|n,m)_{/T_\alpha} \to \mathsf{Top}$ via the pullbacks

$$\begin{array}{ccccc} \mathsf{SC}_\alpha(S) & \longrightarrow & & & SC_{d,\bullet}(x) \\ \downarrow & & & & \downarrow p \\ * & \xrightarrow{\tilde\alpha} & SC_d(z) & \xrightarrow{SC_d(\nu)} & SC_d(px) \end{array} \qquad \begin{array}{ccccc} W_\alpha(S) & \to & W(S) & \xrightarrow{p} & W(pS) \\ \downarrow & & & & \downarrow W_\Sigma(\nu) \\ * & & \xrightarrow{t_\alpha} & & W(T_\alpha). \end{array} \tag{51}$$

We want to replace $\mathsf{Trees}(1,0|n,m)$ by a much smaller category. First we need the wedge operation on forests.

6.4.7 Definition Let $f\colon x \to y$ be a K_f-colored forest and let $g\colon z \to w$ be an K_g-colored forest for some finite sets K_f, K_g. Let $\tau\colon J_w \to J_x$ be any map. Define $x \vee_\tau z$ to be the young $K_f \sqcup K_g$-colored forest $(x,\tau,z)\colon I_x \sqcup J_w \sqcup I_z \to J_x \sqcup J_z$ and define $y \vee_\tau w$ to be the young forest $(y,[f|x]\tau w)\colon I_y \sqcup I_w \to J_y$ (see definition 6.3.2). Finally, set $f \vee_\tau g\colon x \vee_\tau z \to y \vee_\tau w$ to be the forest $(f,g,f,\tau)\colon I_y \sqcup I_w \sqcup J_x \sqcup J_z \to I_x \sqcup J_w \sqcup I_z \sqcup J_y$.

6.4.8 Definition Let Γ_0 be the tree with with no internal vertices and a single input vertex of color $\mathsf{f}_\bullet$. Let Γ_1 be the tree with a single internal vertex of color $\mathsf{h}_\bullet$ and a single input vertex of color $\mathsf{f}_\bullet$.

For any edge $i \in Edge(T_\alpha)_\mathsf{h}$ define $\nu(i)\colon T_\alpha \to T_\alpha(i)$ to be the morphism in $\mathsf{Trees}(n,m)$ which inserts a unary vertex along i. Call this new vertex i_v. Let $S_{i,k} = T_\alpha(i) \vee_{i_v} \Gamma_k$. For any internal vertex $j \in J_{z_\alpha}$ let $S_{j,k} = T_\alpha \vee_j \Gamma_k$. Note that $pS_{i,k} = T_\alpha(i)$ and $pS_{j,k} = T_\alpha$.

Let Trees_α be the full subcategory of $\mathsf{Trees}(1,0|n,m)_{/T_\alpha}$ given by the objects $S_{i,k} = (S_{i,k}, \nu(i))$ and $S_{j,k} = (S_{j,k}, \mathsf{id}_{T_\alpha})$ where $i \in (I_{z_\alpha})_\mathsf{h} \sqcup \{\mathsf{rt}\}$, $j \in J_{z_\alpha}$ and $k \in \{0,1\}$.

6.4.9 Remark The advantage of T_α is that it is easy to understand and computes the space $p^{-1}\alpha$ (lemma 6.4.10). There is a unique morphism $S_{\ell,1}$ to $S_{\ell,0}$ for every ℓ and unique morphisms $S_{i,k} \to S_{T_\alpha^{-1}(i),k}$ and $S_{i,k} \to S_{z_\alpha(i),k}$. See figure 6.4.2 for an illustration.

6.4.10 Lemma *The fiber $p^{-1}(\alpha)$ is given by the coend*

$$W_\alpha \otimes_{\mathsf{Trees}_\alpha} \mathsf{SC}_\alpha.$$

Proof Let $\gamma = [S, s, \tilde{\gamma}] \in \mathsf{SC}^{\mathsf{h}}_{\mathsf{d},\bullet}(1,0|n,m)$ where $S\colon x \to (1,0|n,m)$ is a forest in $\boldsymbol{D}_\bullet$, $s \in W(S)$, and $\tilde{\gamma} \in SC_{d,\bullet}(x)$. Let us assume that $\tilde{\gamma}(j) \neq \mathsf{id}$ for all $j \in J_x$ and $s(i) > 0$ for all $i \in E(S)$. Observe that $p\gamma \in \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n,m)$ is given by $[pS, ps, p\tilde{\gamma}]$. If $p\gamma = \alpha$ there must be some $\nu\colon T_\alpha \to pS$ in $\mathsf{Trees}(n,m)$ such that $SC_d(\nu)\tilde{\alpha} = p\tilde{\gamma}$ and $W(\nu)ps = t_\alpha$. The condition $t_\alpha(i) > 0$ for all $i \in E(T_\alpha)$ implies that $t_\alpha \neq W_\Sigma(\nu)(t')$ for any t' and any ν which collapses any edges. Moreover the condition $\tilde{\gamma}(j) \neq \mathsf{id}$ for all j implies that $p\tilde{\gamma}(j) \neq \mathsf{id}$ for all $j \in J_x$ such that $x^{-1}(j)_{\mathsf{f}_\bullet,\mathsf{h}_\bullet}$ is not empty. We conclude that either $\nu = \mathsf{id}$ or ν is the insertion of the unique unary (in pS, not in S) vertex j such that $x^{-1}(j)_{\mathsf{f}_\bullet,\mathsf{h}_\bullet}$ is not empty. In the former case we must have $S = S_{j,k}$ for some vertex $j \in J_x$ and some $k \in \{0,1\}$ In the latter case we have $S = S_{i,k}$ for some edge i of T_α and some k. This defines the map $p^{-1}(\alpha) \to W_\alpha \otimes_{\mathsf{Trees}_\alpha} \mathsf{SC}_\alpha$. The map in the other direction is clear and the verification that they are inverses is left to the reader. □

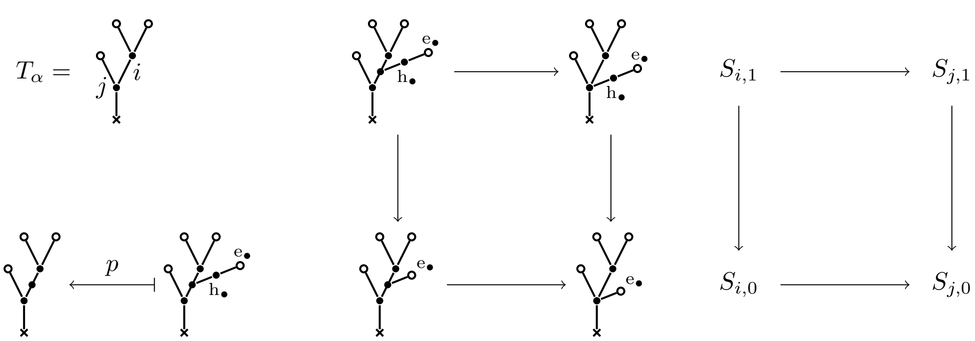


Figure 3: The edge i and vertex j of T_α give a commutative square in Trees_α. The input vertices are circles. The output vertex ends in an ×. The internal vertices are filled dots. The input and internal vertices of Γ_0 and Γ_1 are labeled with their colors. In addition, the image of $S_{i,1}$ under the functor p is shown. This makes it clear that the map $T_\alpha \to pS_{i,1}$ is given by inserting a single vertex.

In diagram (52) we have h-colored edges i_1, i_2 of T_α with $z_\alpha(i_1) = j = T_\alpha^{-1}(i_2)$. Thus we get the commutative diagram on the left. The image of this diagram under SC_α is shown on the right.

$$(52)\qquad \begin{array}{ccccc} S_{i_1,1} & \to & S_{j,1} & \leftarrow & S_{i_2,1} \\ \downarrow & & \downarrow & & \downarrow \\ S_{i_1,0} & \to & S_{j,0} & \leftarrow & S_{i_2,0} \end{array} \qquad \begin{array}{ccccc} |\mathsf{id_h}| \cap \mathbb{R}^{d-1} & \to & |\tilde\alpha(j)| \cap \mathbb{R}^{d-1} & \leftarrow & |\mathsf{id_h}| \cap \mathbb{R}^{d-1} \\ \downarrow & & \downarrow & & \downarrow \\ |\mathsf{id_h}| & \longrightarrow & |\tilde\alpha(j)| & \longleftarrow & |\mathsf{id_h}| \end{array}$$

The geometric realization of the identity $\mathsf{id_h}$ is just S^{d-1}_+, the top half of the $(d-1)$-sphere. The input of $\tilde\alpha(j)$ corresponding to i_1 is a half disk and the map $|\mathsf{id_h}| \to |\tilde\alpha(j)|$ corresponding to $S_{i_1,0} \to S_{j,0}$ is just $\partial_{i_1} |\tilde\alpha(j)| \to |\tilde\alpha_j|$ (see definition 6.4.2). On the other hand the image of $S_{i_2,0} \to S_{j,0}$ is the inclusion of the output boundary $\partial_{\mathsf{rt}} |\tilde\alpha(j)| \to |\tilde\alpha(j)|$.

6.4.11 Definition Let $\epsilon_\bullet \in E(S_{\ell,1})$ be the unique internal edge of color $\mathsf{h}_\bullet$. If $i \in \mathit{Edge}(T_\alpha)$ (6.1.2), let i_ν denote the vertex inserted by $\nu\colon T_\alpha \to pS_{i,k}$. Let i_{in} and i_{out} respectively denote the incoming and outgoing edges of i_ν considered as internal edges of $S_{i,k}$. For any object $S_{\ell,k}$ of Trees_α, let $E_\alpha(S_{\ell,k}) = \{\epsilon_\bullet\}^k \sqcup (\{i_{\mathrm{in}}, i_{\mathrm{out}}\} \cap E(S_{\ell,k}))$. This defines a functor $E_\alpha\colon \mathsf{Trees}_\alpha^{op} \to \mathsf{Set}$.

The image under W_α of the square in diagram (52) is in diagram (53).

$$(53)\qquad \begin{array}{ccccc} [0,\infty]^2 & \xleftarrow{(\mathsf{id},0)} & [0,\infty] & \xrightarrow{(\mathsf{id},\infty)} & [0,\infty]^2 \\ {\scriptstyle (0,\mathsf{id})}\uparrow & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle (0,\mathsf{id})} \\ [0,\infty] & \xleftarrow{\quad 0 \quad} & * & \xrightarrow{\quad \infty \quad} & [0,\infty] \end{array}$$

More precisely,

$$(54)\qquad W_\alpha(S) = \{s\colon E_\alpha(S) \to [0,\infty] \mid s(i_{\mathrm{in}}) + s(i_{\mathrm{out}}) = t_\alpha(i)\},$$

and $W_\alpha(S) \to W_\alpha(S')$ for a map $S' \to S$ in Trees_α is given by push forward of functions along the map of finite sets $E_\alpha(S) \hookrightarrow E_\alpha(S')$. There is no condition on $s(\epsilon_\bullet)$, the length of the edge of color $\mathsf{h}_\bullet$. The isomorphism $W_\alpha(S_{i,1}) \to [0,\infty]^2$ sends s to $(s(\epsilon_\bullet), r(s(i_{\mathrm{out}}), s(i_{\mathrm{in}})))$ where

$$r(s_o, s_i) = \frac{1 - e^{-s_o}}{1 - e^{-s_i}},$$

which lands in $[0,\infty]$ because $s_o + s_i = t_\alpha > 0$. Note that $s_o = 0$ if and only if $r(s_o, s_i) = 0$ and $s_o = t_\alpha$ if and only if $r(s_o, s_i) = \infty$. Since the morphism $S_{i_1,1} \to S_{j,1}$ from diagram (52) collapses the edge $(i_1)_{\mathrm{out}}$ we get $W_\alpha(S_{j,1}) \cong \{(r_\bullet, r) \in W_\alpha(S_{i_1,1}) \mid r = 0\}$. In the same diagram, the morphism $S_{i_2,1} \to S_{j,1}$ collapses the edge $(i_2)_{\mathrm{in}}$, so we have $W_\alpha(S_{j,1}) \cong \{(r_\bullet, r) \in W_\alpha(S_{i_2,1}) \mid r = \infty\}$. The unique morphism $S_{i,1} \to S_{i,0}$ collapses the edge $i_\bullet$ so that $W_\alpha(S_{i,0}) \cong \{(r_\bullet, r) \in W_\alpha(S_{i,1}) \mid r_\bullet = 0\}$. The rest can be deduced from these cases.

6.4.12 Lemma *For any functor $F\colon \mathsf{Trees}_\alpha \to \mathsf{Top}$ the coend $W_\alpha \otimes_{\mathsf{Trees}_\alpha} F$ is the homotopy colimit of F over Trees_α.*

Proof It is clear from diagrams (53) and (52) that $W_\alpha(S)$ is the geometric realization of the nerve of the under category of S for each object $S \in \mathsf{Trees}_\alpha$. In addition the maps $W_\alpha(S) \to W_\alpha(S')$ for $S' \to S$ agree with the maps obtained from the nerves of under categories. □

6.4.13 Lemma *We can explicitly compute $p^{-1}(\iota\alpha)$ as*

$$p^{-1}(\iota\alpha) \simeq |SC_d(T_\alpha)\tilde{\alpha}| \simeq (S^{d-1})^{\vee n},$$

where $SC_d(T_\alpha)\tilde{\alpha}$ is the composition of all vertex labels from $\iota\alpha$.

Proof Let $\mathsf{Trees}_{\alpha,0}$ denote the full subcategory of Trees_α consisting of objects $S_{j,0}$ and $S_{i,0}$ for internal vertices j and internal edges i. This category is homotopy terminal, so by lemma 6.4.12 and lemma 6.4.10 we have $p^{-1}(\iota\alpha) = \operatorname{hocolim}_{\mathsf{Trees}_{\alpha,0}} \mathsf{SC}_\alpha$. This is the same as the homotopy colimit of the coequalizer diagram

$$\coprod_{i\in E(T_\alpha)} |\mathsf{id}_{\mathsf{h}}| \rightrightarrows \coprod_{j\in V(T_\alpha)} |\tilde{\alpha}(j)|\,,$$

where one arrow is given by including into output parts of the boundaries of $|\tilde{\alpha}(j)|$'s, and the other arrow is given by including into input boundaries. These maps are cofibrations with disjoint images. Each space in the coequalizer diagram is cofibrant. Thus the coequalizer diagram is already cofibrant as a functor $(\cdot \rightrightarrows \cdot) \to \mathsf{Top}$. Thus we can compute the normal colimit. It is clear that this is the same as composing the $\tilde{\alpha}(j)$'s via T_α then taking the realization of the result. In addition $|\beta|$ is equivalent to a wedge of n spheres of dimension $d-1$ if $\beta \in SC_d^{\mathsf{h}}(n,m)$. □

6.4.14 Definition Let $\mathsf{Trees}_{\alpha,1}$ denote the full subcategory of Trees_α where we discard the objects $S_{j,0}$ and $S_{i,0}$ for $j \in J_{z_\alpha}$ and $i \in E(T_\alpha)$. Define a functor $W_{\alpha,1}\colon \mathsf{Trees}_{\alpha,1}^{op} \to \mathsf{Top}$ by setting

$$W_{\alpha,1}(S) = \{s\colon E_\alpha(S) \to [0,\infty] \mid \sum_{i\in E_\alpha(S)} s(i) = \infty\}.$$

6.4.15 Lemma *Suppose $t_\alpha < \infty$, and $n = 1$, then $p_1^{-1}(\alpha)$ is given by the coend*

$$W_\alpha \otimes_{\mathsf{Trees}_{\alpha,1}} \mathsf{SC}_\alpha,$$

where SC_α *is the functor in definition 6.4.6 restricted to* $\mathsf{Trees}_{\alpha,1}$ *and* $W_{\alpha,1}$ *is defined in 6.4.14.*

Proof Let $\gamma \in F(\mathsf{SC}^{\mathsf{h}1}_{\mathsf{d},\bullet})(1,m)$ such that $p_1(\gamma) = \alpha$. Pick a representative $(f,g,s,\tilde{\gamma})$ where $f\colon y \to (1,0|1,m)$, $y \le 1$, $g\colon z \to y$, $s \in W(g)$ and $\tilde{\gamma} \in SC_{d,\bullet}(z)$. Consider $\iota\gamma \in \mathsf{SC}^{\mathsf{h}}_{\mathsf{d},\bullet}(1,m)$, which is represented by $(fg, W_\infty(f)s, \tilde{\gamma})$. Recall that the condition $y \le 1$ means that each connected component of the young forest y has at most one input whose color lives in $\{\mathsf{f}, \mathsf{f}_\bullet\}$. This implies that f has at least one internal edge $i \in E(f)$. Thus $W_\infty(f)s(i) = \infty$ when i is viewed as an internal edge in fg.

We know $p\iota\gamma = \iota\alpha$, so $\iota\gamma$ is represented by some triple $(S, s', \tilde{\gamma})$ with $S \in \mathsf{Trees}_\alpha$, $s' \in W_\alpha(S)$, and $\tilde{\gamma} \in \mathsf{SC}_\alpha(S)$. The relations in $\mathsf{SC}^{\mathsf{h}}_{\mathsf{d},\bullet}$ preserve edges of length ∞, so we must have $s'(i) = \infty$ for some $i \in E(S)$. We are assuming $t_\alpha(i) < \infty$ for all $i \in E(T_\alpha)$, so the infinite edge in S must be in $E_\alpha(S)$. This implies $s' \in W_{\alpha,1}(S)$. Moreover we cannot have such an infinite edge if $S = S_{j,0}$ for some vertex j or $S = S_{i,0}$ for some internal edge i. Thus $S \in \mathsf{Trees}_{\alpha,1}$. This defines the map from $p^{-1}(\alpha)$ to the coend. We leave the remainder to the reader. □

6.4.16 Lemma *For any functor* $F\colon \mathsf{Trees}_{\alpha,1} \to \mathsf{Top}$ *the coend* $W_{\alpha,1} \otimes_{\mathsf{Trees}_{\alpha,1}} F$ *is the homotopy colimit of* F *over* $\mathsf{Trees}_{\alpha,1}$*.*

Proof The argument here is similar to the proof of lemma 6.4.12. □

6.4.17 Corollary *If* $t_\alpha < 0$ *and* $n = 1$*, then the fiber* $p_1^{-1}(\alpha)$ *is equivalent to* $\partial_{\mathsf{h}} |SC_d(T_\alpha)\tilde{\alpha}| \simeq S^{d-1}$*.*

Proof By the same argument as in lemma 6.4.13, $\operatorname{hocolim}_{\mathsf{Trees}_{\alpha,1}} \mathsf{SC}_\alpha$ is equivalent to $\operatorname{colim}_{\mathsf{Trees}_{\alpha,1}} \mathsf{SC}_\alpha$. This is easily computed as the h-colored boundary of the composite of $\tilde{\alpha}$. □

Proof of proposition 6.4.5 Recall α is represented by $f\colon y \to (n,m)$, $y \le 1$, $g\colon z_\alpha \to y$, $t \in W(g)$ and $\tilde{\alpha} \in SC_d(z_\alpha)$. By applying relations in $\mathsf{SC}^{\mathsf{h}\infty}_{\mathsf{d}}$ we may assume $0 < t < \infty$. We may think of (g,t) as representing an element of $\mathsf{SC}^{\mathsf{h}1}_{\mathsf{d}}(y)$ which we can write as $(\alpha(j))_{j \in J_y}$. If $\alpha(j) \in \mathsf{SC}^{\mathsf{h}}_{\mathsf{d}}(n_j, m_j)$ then $n_j \le 1$. Clearly $p_1^{-1}(\alpha(j)) \simeq p^{-1}(\alpha(j)) \simeq *$ when $n_j = 0$. Since $t_{\alpha(j)} < \infty$ we can use corollary 6.4.17 to conclude $p_1^{-1}(\alpha(j)) \simeq \partial_{\mathsf{h}} |(SC_d(g)(\alpha))(j)|$. The fiber $p_1^{-1}(\alpha)$ is equal to the colimit of the diagram

$$\coprod_{i \in E(f)} |1_{\mathsf{h}}| \rightrightarrows \coprod_{j \in V(f)} p_1^{-1}(\alpha(j)),$$

where one arrow is given by $|1_\mathsf{h}| \simeq \partial_i |\alpha(y(i))| \to \partial_\mathsf{h} |(SC_d(g)(\alpha))(y(i))|$ and the other by $|1_\mathsf{h}| \simeq \partial_\mathsf{rt} |\alpha(f(i))| \to \partial_\mathsf{h} |(SC_d(g)(\alpha))(y(i))|$. This colimit is clearly $(S^{d-1})^{\vee n} \simeq p^{-1}(\iota\alpha)$. □

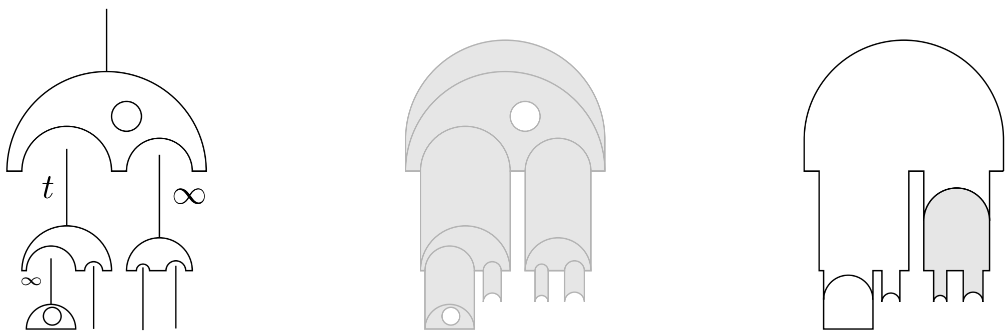


Figure 4: On the left is $\alpha \in \mathsf{SC}_\mathsf{d}^{\mathsf{h}\infty}(2,3)$. In the middle is $p^{-1}(\alpha)$, and on the right is $p_1^{-1}(\alpha)$. Both $p^{-1}(\alpha)$ and $p^{-1}(\alpha)$ have the homotopy type of a wedge of spheres, one for each disk in α.

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Department of Mathematics, University of Notre Dame, 255 Hurley Hall, Notre Dame, IN, 46556-4618

jthoma20@nd.edu

http://www.nd.edu/~jthoma20/