

## TATE COHOMOLOGY LOWERS CHROMATIC BOUSFIELD CLASSES

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ABSTRACT. Let  $G$  be a finite group. We use recent results of J. P. C. Greenlees and H. Sadofsky to show that the Tate homology of  $E(n)$  local spectra with respect to  $G$  produces  $E(n-1)$  local spectra. We also show that the Bousfield class of the Tate homology of  $L_n X$  (for  $X$  finite) is the same as that of  $L_{n-1} X$ .

To be precise, recall that Tate homology is a functor from  $G$ -spectra to  $G$ -spectra. To produce a functor  $P_G$  from spectra to spectra, we look at a spectrum as a naive  $G$ -spectrum on which  $G$  acts trivially, apply Tate homology, and take  $G$ -fixed points. This composite is the functor we shall actually study, and we'll prove that  $\langle P_G(L_n X) \rangle = \langle L_{n-1} X \rangle$  when  $X$  is finite.

When  $G = \Sigma_p$ , the symmetric group on  $p$  letters, this is related to a conjecture of Hopkins and Mahowald (usually framed in terms of Mahowald's functor  $\mathbf{R}P_{-\infty}(-)$ ).

### 1. INTRODUCTION

We briefly recall the spectra that occur in Lin's proof of the Segal conjecture for the group  $\mathbf{Z}/(2)$ . Embed  $\mathbf{Z}/(p)$  into  $S^1$  and look at the pullback of the tautological (complex) line bundle over  $BS^1$  as a bundle over  $B\mathbf{Z}/(p)$ . Call this bundle  $\xi$ .

We denote by  $P_{-2k}$  the spectrum given by the Thom spectrum  $(B\mathbf{Z}/(p))^{-k\xi}$  when  $p = 2$ , or when  $p$  is odd the summand of that spectrum corresponding to  $B\Sigma_p$ . (We refer the reader to [20] for the definition of a Thom spectrum associated to a virtual bundle.)  $P_{-2k}$  has a cell in every dimension  $\geq -2k$  and is the spectrum frequently called  $\mathbf{R}P_{-2k}^\infty$  when  $p = 2$ . When  $p$  is odd,  $P_{-2k}$  has a cell in every dimension congruent to 0 or  $-1$  modulo  $q = 2p - 2$  and  $\geq -2k$ .  $P_{-2k}$  is the same as the spectrum denoted  $P_{-2k}^\infty$  in [19], and constructed there by James periodicity rather than via Thom spectra.

Lin's theorem [10] (Gunawardena's theorem when  $p > 2$  [1]) states that

$$(1) \quad \lim_{\leftarrow k} (P_{-2k} \wedge X) = \Sigma^{-1} X_p^\wedge$$

when  $X$  is a finite spectrum. This inverse system of spectra (with some minor alterations) is also what is used to define the root invariant (see [11] for  $p = 2$ , or more generally, [13, 19]). As shorthand, we write  $P_{-\infty}(X)$  for  $\lim_{\leftarrow k} (P_{-2k} \wedge X)$ .

Mahowald and Ravenel [12] have conjectured a relationship between chromatic periodicity and Mahowald's root invariant. There is a related conjecture by Hopkins

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and Mahowald that is more closely related to our concerns in this paper. Denote Bousfield localization with respect to  $E(n)$  by  $L_n$ . They conjecture that

$$(2) \quad P_{-\infty}(L_n X) = \Sigma^{-2} L_{n-1} X_p \hat{\vee} \Sigma^{-1} L_{n-1} X_p \hat{\vee}$$

for  $X$  finite. (A word to the experts: this conjecture is connected with Hopkins's chromatic splitting conjecture (see [7]) and at the time (2) was conjectured the chromatic splitting conjecture was in too simple a form. In light of its current corrected form as in [7], (2) is probably also too optimistic, though we expect it is true as stated when  $X$  has type  $n - 1$ .)

Greenlees and May put the  $P_{-\infty}$  construction in a more general context in [4]. There they define the Tate  $G$ -spectrum associated to a  $G$ -spectrum  $X$ ,  $t_G(X)$ . We are not concerned with equivariant spectra here, but we use  $t_G$  to construct a functor from (ordinary) spectra to spectra. We abuse notation and write  $i_*$  for the functor which is the composite of the inclusion of ordinary spectra into the category of naive  $G$ -spectra (as the objects on which  $G$  acts trivially) with the left adjoint of the forgetful functor from  $G$ -spectra to naive  $G$ -spectra. So  $i_*$  is a functor from ordinary spectra to  $G$ -spectra. Our other functor is the  $G$ -fixed point functor,  $(-)^G$ , which goes from  $G$ -spectra to spectra. We refer the reader to [9] for details. We define

$$P_G(X) = t_G(i_* X)^G.$$

Then [4, 16.1] shows that

$$P_{\Sigma_p}(X) = P_{-\infty}(\Sigma X)$$

(the left-hand side needs to be localized at  $p$  when  $p$  is odd).

Henceforth we will only be concerned with the case where  $G$  is a finite group. We now have a family of functors, one for each finite group  $G$ , and we assume (to guarantee non-triviality of our functors) that  $p$  divides the order of  $G$ . Our main theorem is the following.

**Theorem 1.1.** *Let  $X$  be a finite spectrum. Then  $\langle P_G(L_n X) \rangle = \langle L_{n-1} X \rangle$ .*

Here  $\langle X \rangle$  means the Bousfield class of the spectrum  $X$  as given in [2].

We also give a result about complex oriented  $v_n$ -periodic spectra.

**Theorem 1.2.** *If  $E$  is Landweber exact and  $v_n$ -periodic, then  $P_{\mathbf{Z}/(p)} E$  is Landweber exact and  $v_{n-1}$ -periodic. It follows generally that  $\langle P_G(E_{(p)}) \rangle = \langle E(n-1) \rangle$ .*

Our proofs rely on [5, Theorem 1.1], which implies that  $P_G(K(n)) \simeq *$ . We also use two other results that are relatively well known. We use Ravenel's Proposition 1.34 from [16]:

$$(3) \quad \langle X \rangle = \langle C(f) \rangle \vee \langle Tel(f) \rangle$$

where  $f$  is a self-map of  $X$ ,  $C(f)$  is the cofiber and  $Tel(f)$  is the mapping telescope. Finally, we use a theorem of Hopkins and Ravenel from [18] to show that  $P_G(K(n)) \simeq *$  implies  $P_G(L_n X) \simeq *$  when  $X$  is finite type  $n$ .

Using the interesting results of Mahowald and Shick in [15] one can show that  $P_{\mathbf{Z}/(2)}(Tel(X)) \simeq *$  where  $X$  is finite type  $n$  and  $Tel(X)$  is the mapping telescope of  $X$  under a  $v_n$  map. One can use this to deduce our theorems in the special case  $G = \mathbf{Z}/(2)$ . Chun-Nip Lee has done this independently [8].

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2. THERE IS A FINITE TYPE  $n$  SPECTRUM  $F$  WITH  $L_n F$   $K(n)$ -NILPOTENT

Recall that  $X$  is said to be  $E$ -prenilpotent if  $L_E X$  is  $E$ -nilpotent; that is if  $L_E X$  can be built up in finitely many stages from spectra of the form  $E \wedge Z$  by taking cofibrations and retracts. Note that  $N$  is  $E$ -nilpotent implies  $N \wedge M$  is  $E \wedge M$ -nilpotent. By [18, 8.3] there is a finite type 0 spectrum  $Y$  that is  $L_n BP$ -prenilpotent.

By [18, Lemma 8.1.4]

$$\langle L_n BP \rangle = \langle v_n^{-1} BP \rangle,$$

which is in turn equal to  $\langle E(n) \rangle$  by [16]. So  $L_n Y$  is  $L_n BP$ -nilpotent. Now let  $M$  be a finite type  $n$  spectrum with

$$BP_* M = BP_*/(p^{i_0}, \dots, v_{n-1}^{i_{n-1}})$$

and such that  $M$  is a ring spectrum. (See [3] for the existence of such ring spectra.) Then  $BP \wedge M = BP/(p^{i_0}, \dots, v_{n-1}^{i_{n-1}})$ , so by [17, Theorem 1],

$$L_n BP \wedge M = v_n^{-1} BP \wedge M.$$

But  $v_n^{-1} BP \wedge M$  is made out of finitely many cofibrations with cofiber  $v_n^{-1} BP/I_n = B(n)$ . Now by [21, Remark 6.19],  $B(n) = K(n) \wedge B$  for some  $B$ , so it follows that  $L_n BP \wedge M$  is  $K(n)$ -nilpotent.

Since  $L_n Y \wedge M$  is  $L_n BP \wedge M$ -nilpotent, it follows that  $L_n Y \wedge M$  is  $K(n)$ -nilpotent.

**Lemma 2.1.** *There is a finite type  $n$  spectrum  $F$  with  $L_n F$   $K(n)$ -nilpotent.*

*Proof.* Take  $F = Y \wedge M$ . □

3.  $P_G(L_n X \wedge F) \simeq *$  IF  $F$  IS TYPE  $n$

We recall from [5] that  $t_G(i_* K(n)) \simeq *$  as a  $G$ -spectrum. It follows that  $P_G(K(n)) \simeq *$ . We record the following lemma.

**Lemma 3.1.** *If  $X$  is  $K(n)$ -nilpotent, then  $P_G(X) \simeq *$ .*

*Proof.* First note that since  $P_G$  takes cofibrations to cofibrations, it suffices to prove that  $P_G(K(n) \wedge Z) \simeq *$  for any  $Z$ . But  $P_G(R)$  is a ring spectrum when  $R$  is a ring spectrum, and  $P_G(N)$  is a module spectrum over  $P_G(R)$  if  $N$  is a module spectrum over  $R$  [4, Proposition 3.5]. It follows that  $P_G(K(n) \wedge Z) \simeq *$ . □

*Remark.* The same proof shows that  $t_G(X) \simeq *$  equivariantly if  $X$  is  $i_* K(n)$ -nilpotent in the category of  $G$ -spectra.

**Corollary 3.2.** *If  $F$  is finite type  $n$ , then  $P_G(L_n X \wedge F) \simeq *$  for any spectrum  $X$ .*

*Proof.* Let  $\mathcal{C}$  be the category of finite spectra  $F$  such that  $P_G(L_n X \wedge F) \simeq *$  for all spectra  $X$ .  $\mathcal{C}$  is a thick subcategory in the sense of [6]. It follows that if  $\mathcal{C} \cap \mathcal{C}_n \neq \emptyset$ , then  $\mathcal{C}_n \subseteq \mathcal{C}$ . We recall that if a spectrum  $Y$  is  $K(n)$ -nilpotent, so is  $X \wedge Y$  for any spectrum  $X$ . Since  $L_n X \wedge F = X \wedge L_n F$  ( $L_n$  is smashing) by Lemma 3.1 and Lemma 2.1,  $\mathcal{C} \cap \mathcal{C}_n \neq \emptyset$ . □

*Remark.* Since  $L_n F = L_{K(n)} F$  when  $F$  is finite type  $n$ , one might ask when  $P_G(L_{K(n)} X) \simeq *$ . While we don't know the most general answer, this does not hold in general for  $X$  finite. Using the methods of this paper, one can easily check that if  $X$  is finite,  $\langle P_G(L_{K(n)} X) \rangle = \langle L_{n-1} X \rangle$ .

4.  $P_G(L_n X)$  IS  $E(n - 1)$ -LOCAL

We use equation (3) inductively. We get

$$\begin{aligned} \langle P_G L_n X \rangle &= \langle p^{-1} P_G L_n X \rangle \vee \langle P_G L_n X \wedge M(p^{i_0}) \rangle \\ &= \langle p^{-1} P_G L_n X \rangle \vee \langle v_1^{-1} P_G L_n X \wedge M(p^{i_0}) \rangle \vee \cdots \\ &\quad \vee \langle v_{n-1}^{-1} P_G L_n \wedge M(p^{i_0}, \dots, v_{n-2}^{i_{n-2}}) \rangle \vee \langle P_G L_n X \wedge M(p^{i_0}, \dots, v_{n-1}^{i_{n-1}}) \rangle. \end{aligned}$$

Now since the  $P_G L_n X \wedge M(p^{i_0}, \dots, v_{n-1}^{i_{n-1}}) \simeq *$  by Corollary 3.2, we get

$$\begin{aligned} \langle P_G L_n X \rangle &= \langle p^{-1} P_G L_n X \rangle \vee \langle v_1^{-1} P_G L_n X \wedge M(p^{i_0}) \rangle \vee \cdots \\ &\quad \vee \langle v_{n-1}^{-1} P_G L_n \wedge M(p^{i_0}, \dots, v_{n-2}^{i_{n-2}}) \rangle. \end{aligned}$$

Since  $P_G L_n X$  is an  $L_n S^0$  module, it follows that

$$\begin{aligned} v_j^{-1} P_G L_n X \wedge M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}}) &= P_G L_n X \wedge v_j^{-1} L_n M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}}) \\ &= (P_G L_n X) \wedge L_j M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}}). \end{aligned}$$

(The last equality follows from [14, Proposition 6.1].) Since  $\langle L_j M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}}) \rangle = \langle K(j) \rangle$ , we see that

$$(4) \quad \langle P_G L_n X \rangle \leq \langle K(0) \rangle \vee \cdots \vee \langle K(n - 1) \rangle = \langle E(n - 1) \rangle.$$

Now since  $L_n X$  is an  $L_n S^0$  module,  $P_G L_n X$  is a  $P_G L_n S^0$  module [4, Proposition 3.5]. But  $P_G L_n S^0$  is self-local since it is a ring spectrum [16], so by equation (4)  $P_G L_n S^0$  is  $E(n - 1)$ -local, hence so is  $P_G L_n X$ .

To finish the proof of Theorem 1.1 it remains to show the inequality in equation (4) is actually an equality when  $X = S^0$ . In section 6 we use Theorem 1.2 to do this.

5.  $P_G$  OF LANDWEBER EXACT  $v_n$ -PERIODIC THEORIES

In this section we prove Theorem 1.2. We will suppose that  $E$  is a complex oriented homology theory. We also assume that  $E$  is  $p$ -local ( $P_G(E) = P_G(E_{(p)})$  if  $G$  happens to be a  $p$ -group). Then we can assume  $E$  is oriented by a map from  $BP$ , so that we can consider  $v_i$  as an element of  $E_*$ . We remind the reader that  $I_j = (p, v_1, \dots, v_{j-1})$ , and that for  $BP$  (and hence for any spectrum oriented from  $BP$ ),

$$[p](x) = px +_F v_1 x^p +_F v_2 x^{p^2} +_F \cdots +_F v_i x^{p^i} +_F \cdots$$

where  $+_F$  is the sum in the formal group law on  $E_*$ .

We begin by remarking that for complex oriented  $E$  in which the leading coefficient of  $[p](x)$  is not a 0-divisor,

$$\pi_* P_{\mathbf{Z}/(p)} E = E_*((x))/([p](x))$$

where  $|x| = -2$ ,  $E_*((x))$  denotes the ring of Laurent series over  $E_*$  which have only finitely many terms involving negative powers of  $x$ , and  $[p](x)$  is the  $p$ -series. It follows that when  $[p](x)$  is not a zero divisor, we have a short exact sequence

$$E_*((x)) \xrightarrow{[p](x)} E_*((x)) \rightarrow \pi_* P_{\mathbf{Z}/(p)} E.$$

We now assume that  $E$  is  $v_n$ -periodic Landweber exact. We define  $v_n$ -periodic almost as in [5, Definition 1.3];  $E$  is  $v_n$ -periodic if  $v_n$  is a unit on  $E_*/I_n$  and in addition  $E_*/I_n \neq 0$ .

For each  $j \leq n$ , we know that  $E_*/I_j \rightarrow v_j^{-1}E_*/I_j$  is injective by the hypothesis of Landweber exactness. It follows that

$$E_*((x))/I_j = E_*/I_j((x)) \rightarrow (v_j^{-1}E_*/I_j)((x))$$

is injective also. Now  $[p](x)$  is a unit in  $(v_j^{-1}E_*/I_j)((x))$  since it is a power series with leading term  $v_j x^{p^j}$ , which is a unit. It follows that

$$E_*((x))/I_j \rightarrow (v_j^{-1}E_*/I_j)((x)) \xrightarrow{\cdot[p](x)} (v_j^{-1}E_*/I_j)((x))$$

is injective, hence

$$E_*((x))/I_j \xrightarrow{\cdot[p](x)} E_*((x))/I_j$$

is also.

We examine the diagram of short exact sequences below, in which the bottom row is the cokernel of the map between the top two rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_*((x))/I_j & \xrightarrow{\cdot[p](x)} & E_*((x))/I_j & \longrightarrow & (\pi_*P_{\mathbf{Z}/(p)}E)/I_j \longrightarrow 0 \\ & & \cdot v_j \downarrow & & \downarrow \cdot v_j & & \downarrow \cdot v_j \\ 0 & \longrightarrow & E_*((x))/I_j & \xrightarrow{\cdot[p](x)} & E_*((x))/I_j & \longrightarrow & (\pi_*P_{\mathbf{Z}/(p)}E)/I_j \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_*((x))/I_{j+1} & \xrightarrow{\cdot[p](x)} & E_*((x))/I_{j+1} & \longrightarrow & (\pi_*P_{\mathbf{Z}/(p)}E)/I_{j+1} \longrightarrow 0 \end{array}$$

By the snake lemma applied to the first two rows (together with the observation that the first two vertical maps are injective) we see that  $v_j$  is injective on  $(\pi_*P_{\mathbf{Z}/(p)}E)/I_j$ .

We can also see that  $[p](x)$  is not a unit in  $E_*((x))/I_j$  unless  $v_j$  is a unit in  $E_*/I_j$ , therefore  $(\pi_*P_{\mathbf{Z}/(p)}E)/I_j \neq 0$  unless  $j = n$ , and this last observation tells us that  $v_{n-1}$  is a unit on  $(\pi_*P_{\mathbf{Z}/(p)}E)/I_{n-1}$ .

We conclude that  $P_{\mathbf{Z}/(p)}E$  is Landweber exact, and that

$$\pi_*(P_{\mathbf{Z}/(p)}E)/I_{n-1} \neq 0$$

while  $\pi_*(P_{\mathbf{Z}/(p)}E)/I_n = 0$ . It follows by using (3) as before (see [7, Corollary 1.12]) that

$$\langle P_{\mathbf{Z}/(p)}E \rangle = \langle E(n-1) \rangle.$$

By using the maps of complex oriented ring spectra

$$E \rightarrow P_G E \rightarrow P_{\mathbf{Z}/(p)} E$$

(when  $\mathbf{Z}/(p) \subseteq G$ ) we also deduce that

$$\langle P_G E \rangle = \langle E(n-1) \rangle.$$

### 6. PROOF OF THEOREM 1.1

We recall from section 4 that  $v_j^{-1}P_G L_n S^0 \wedge M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}})$  has the Bousfield class of either a point or of  $K(j)$ . So to show it has the class of  $K(j)$ , we need only show that it is not contractible.

We pick  $i_0, \dots, i_{j-1}$  so that  $M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}})$  is a ring spectrum. Then we observe that the map

$$\begin{aligned} S^0 &\rightarrow v_j^{-1}P_G L_n S^0 \wedge M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}}) \rightarrow v_j^{-1}P_{\mathbf{Z}/(p)} L_n S^0 \wedge M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}}) \rightarrow \\ &v_j^{-1}P_{\mathbf{Z}/(p)} E(n) \wedge M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}}) \xrightarrow{=} v_j^{-1}P_{\mathbf{Z}/(p)} E(n)/(p^{i_0}, \dots, v_{j-1}^{i_{j-1}}) \rightarrow \\ &v_j^{-1}P_{\mathbf{Z}/(p)} E(n)/(p, \dots, v_{j-1}) \end{aligned}$$

is the unit of the ring spectrum  $v_j^{-1}P_{\mathbf{Z}/(p)} E(n)/(p, \dots, v_{j-1})$ . This is non-zero if  $j < n$  by Theorem 1.2. So none of the intervening spectra are contractible either.

For arbitrary finite  $X$  (instead of  $S^0$ ) just smash with  $X$ . Note that  $\langle - \rangle$ ,  $P_G(-)$ , and localization commute with smashing with a finite spectrum.

*Remark.* The same proof can be iterated to draw the obvious conclusions about  $P_G^k(L_n S^0)$ .

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