TATE COHOMOLOGY LOWERS CHROMATIC BOUSFIELD CLASSES

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Abstract. Let G be a finite group. We use the results of [5] to show that the Tate homology of $E(n)$ local spectra with respect to G produces $E(n-1)$ local spectra. We also show that the Bousfield class of the Tate homology of $L_n X$ (for X finite) is the same as that of $L_{n-1}X$.

To be precise, recall that Tate homology is a functor from G-spectra to G-spectra. To produce a functor P_G from spectra to spectra, we look at a spectrum as a naive G-spectrum on which G acts trivially, apply Tate homology, and take G-fixed points. This composite is the functor we shall actually study, and we'll prove that $\langle P_G(L_n X)\rangle = \langle L_{n-1}X\rangle$ when X is finite.

When $G = \Sigma_p$, the symmetric group on p letters, this is related to a conjecture of Hopkins and Mahowald (usually framed in terms of Mahowald's functor $RP_{-\infty}(-)$).

1. Introduction

We briefly recall the spectra that occur in Lin's proof of the Segal conjecture for the group $Z/(2)$. Embed $Z/(p)$ into S^1 and look at the pullback of the tautological (complex) line bundle over BS^1 as a bundle over $BZ/(p)$. Call this bundle ξ.

We denote by P_{-2k} the spectrum given by the Thom spectrum $(BZ/(p))^{-k\xi}$ when $p = 2$, or when p is odd the summand of that spectrum corresponding to $B\Sigma_p$. (We refer the reader to [18] for the definition of a Thom spectrum associated to a virtual bundle). P_{-2k} has a cell in every dimension $\geq -2k$ and is the spectrum freqeuently called RP^{∞}_{-2k} when $p = 2$. When p is odd, P_{-2k} has a cell in every dimension congruent to 0 or -1 modulo $q = 2p-2$ and $\geq -2k$. P_{-2k} is the same as the spectrum denoted P^{∞}_{-2k} in [17], and constructed there by James periodicity rather than via Thom spectra.

Lin's theorem [10] (Gunarwardena's theorem when $p > 2$ [1]) states that

$$\lim_{\overleftarrow{k}} (P_{-2k} \wedge X) = \Sigma^{-1} X^{\hat{}}_p \tag{1}$$

when X is a finite spectrum. This inverse system of spectra (with some minor alterations) is also what is used to define the root invariant (see [11] for $p = 2$, or more generally, [17]). As shorthand, we write $P_{-\infty}(X)$ for $\underleftarrow{\lim}_k (P_{-2k} \wedge X)$.

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Mahowald and Ravenel [12] have conjectured a relationship between chromatic periodicity and Mahowald's root invariant. There is a related conjecture by Hopkins and Mahowald that is more closely related to our concerns in this paper. Denote Bousfield localization with respect to $E(n)$ by L_n. They conjecture that

$$P_{-\infty}(L_nX) = \Sigma^{-2}L_{n-1}X_p^{\hat{}} \vee \Sigma^{-1}L_{n-1}X_p^{\hat{}} \tag{2}$$

for X finite. (A word to the experts: this conjecture is connected with Hopkins's chromatic splitting conjecture (see [7]) and at the time (2) was conjectured the chromatic splitting conjecture was in too simple a form. In light of its current corrected form as in [7], (2) is probably also too optimistic, though We expect it is true as stated when X has type $n-1$.)

Greenlees and May put the $P_{-\infty}$ construction in a more general context in [4]. There they define the Tate G spectrum associated to a G spectrum X, $t_G(X)$. We are not concerned with equivariant spectra here, but we use t_G to construct a functor from (ordinary) spectra to spectra. We abuse notation and write i_* for the functor which is the composite of the inclusion of ordinary spectra into the category of naive G-spectra (as the objects on which G acts trivially) with the left adjoint of the forgetful functor from G-spectra to naive G-spectra. So i_* is a functor from ordinary spectra to G-spectra. Our other functor is the G-fixed point functor, $(-)^G$ which goes from G-spectra to spectra. We refer the reader to [9] for details. We define

$$P_G(X) = t_G(i_*X)^G.$$

Then [4, 16.1] shows that

$$P_{\Sigma_p}(X) = P_{-\infty}(\Sigma X).$$

(the left hand side needs to be localized at p when p is odd).

Henceforth we will only be concerned with the case where G is a finite group. We now have a family of functors, one for each finite group G. Our main theorem is the following.

Theorem 1.1. *Let X be a finite spectrum. Then $\langle P_G(L_nX)\rangle = \langle L_{n-1}X\rangle$*

Here $\langle X\rangle$ means the Bousfield class of the spectrum X as given in [2].

We also give a result about complex oriented v_n-periodic spectra.

Theorem 1.2. *If E is Landweber exact and v_n-periodic then $P_{Z/(p)}E$ is Landweber exact and v_{n-1}-periodic. It follows generally that $\langle P_G(E)\rangle = \langle E(n-1)\rangle$.*

Our proofs rely on [5, Theorem 1.1] which implies that $P_G(K(n)) \simeq *$. We also use two other results that are relatively well known. We use Ravenel's Proposition 1.34 from [14]:

$$\langle X\rangle = \langle C(f)\rangle \vee \langle Tel(f)\rangle \tag{3}$$

where f is a self map of X, $C(f)$ is the cofiber and $Tel(f)$ is the mapping telescope. Finally, we use a theorem of Hopkins and Ravenel from [16] to show that $P_G(K(n)) \simeq *$ implies $P_G(L_nX) \simeq *$ when X is finite type n.

Using the interesting results of Mahowald and Shick in [13] one can show that $P_{Z/(2)}(Tel(X)) \simeq *$ where X is finite type n and $Tel(X)$ is the infinite mapping telescope of X under a v_n map. One can use this to deduce our theorems in the special case $G = Z/(2)$. Chun-Nip Lee has done this independently [8].

2. There is a finite type n spectrum X with L_nX $K(n)$-nilpotent

Recall that X is said to be E-prenilpotent if L_EX is E nilpotent; that is if L_EX can be built up from cofibrations in a finite number of stages with cofibers retracts of spectra of form $E \wedge Z$. By [16, 8.3] there is a finite type 0 spectrum Y that is L_nBP-prenilpotent.

By [16, Lemma 8.1.4]

$$\langle L_nBP\rangle = \langle v_n^{-1}BP\rangle$$

which is in turn equal to $\langle E(n)\rangle$ by [14]. So L_nY is L_nBP-nilpotent, which gives a sequence of spectra

$$* = Y_0 \to Y_1 \to Y_2 \to \cdots \to Y_r = L_nY$$

such that cofiber$(Y_{i-1} \to Y_i)$ is a retract of $L_nBP \wedge Z_i$ for some Z_i. Now let M be a finite type n spectrum with

$$BP_*M = BP_*/(p^{i_0}, \dots , v_{n-1}^{i_{n-1}})$$

and such that M is a ring spectrum. (See [3] for the existence of such ring spectra.) Then $BP \wedge M = BP/(p^{i_0}, \dots , v_{n-1}^{i_{n-1}})$, so by [15, Theorem 1],

$$L_nBP \wedge M = v_n^{-1}BP \wedge M.$$

But $v_n^{-1}BP \wedge M$ is made out of finitely many cofibrations with cofiber $v_n^{-1}BP/I_n = B(n)$. Now by [19, Remark 6.19], $B(n) = K(n) \wedge B$ for some B, so it follows that $L_nBP \wedge M$ is $K(n)$-nilpotent.

From this we see that $L_nBP \wedge Z_i \wedge M$ is $K(n)$-nilpotent, and deduce the following lemma.

Lemma 2.1. *There is a finite type n spectrum F with L_nF $K(n)$-nilpotent.*

Proof. Take $F = Y \wedge M$. □

3. $P_G(L_nX \wedge F) \simeq *$ if F is type n.

We recall from [5] that $t_G(i_*K(n)) \simeq *$ as a G-spectrum. It follows that $P_G(K(n)) \simeq *$. We record the following lemma.

Lemma 3.1. *If X is $K(n)$-nilpotent then $P_G(X) \simeq *$.*

Proof. First note that since P_G takes cofibrations to cofibrations, it suffices to prove that $P_G(K(n) \wedge Z) \simeq *$ for any Z. But $P_G(R)$ is a ring spectrum when R is a ring spectrum, and $P_G(N)$ is a module spectrum over $P_G(R)$ if N is a module spectrum over R [4, Proposition 3.5]. It follows that $P_G(K(n) \wedge Z) \simeq *$. □

Remark: The same proof shows that $t_G(X) \simeq *$ equivariantly if X is $i_*K(n)$-nilpotent in the category of G-spectra.

Corollary 3.2. *If F is finite type n, then $P_G(L_nX \wedge F) \simeq *$ for any spectrum X.*

Proof. Let $\mathcal{C}$ be the category of finite spectra F such that $P_G(L_nX \wedge F) \simeq *$ for all spectra X. $\mathcal{C}$ is a thick subcategory in the sense of [6]. It follows that if $\mathcal{C} \cap \mathcal{C}_n \neq \phi$ then $\mathcal{C}_n \subseteq \mathcal{C}$. We recall that if a spectrum Y is $K(n)$-nilpotent, so is $X \wedge Y$ for any spectrum X. Since $L_nX \wedge F = X \wedge L_nF$ (L_n is smashing) by Lemma 3.1 and Lemma 2.1, $\mathcal{C} \cap \mathcal{C}_n \neq \phi$. □

Remark: Since $L_nF = L_{K(n)}F$ when F is finite type n, one might ask when $P_G(L_{K(n)}X) \simeq *$. While we don't know the most general answer, this does not hold in general for X finite. Using the methods of this paper, one can easily check that if X is finite, $\langle P_G(L_{K(n)}X)\rangle = \langle L_{n-1}X\rangle$.

4. $P_G(L_nX)$ is $E(n-1)$-local.

We use equation (3) inductively. We get

$$\begin{aligned}\langle P_G L_n X\rangle &= \langle p^{-1}P_G L_n X\rangle \vee \langle P_G L_n X \wedge M(p^{i_0})\rangle \\ &= \langle p^{-1}P_G L_n X\rangle \vee \langle v_1^{-1}P_G L_n X \wedge M(p^{i_0})\rangle \vee \cdots \\ &\quad \vee \langle v_{n-1}^{-1}P_G L_n \wedge M(p^{i_0},\dots,v_{n-2}^{i_{n-2}})\rangle \vee \langle P_G L_n X \wedge M(p^{i_0},\dots,v_{n-1}^{i_{n-1}})\rangle.\end{aligned}$$

Now since the $P_G L_n X \wedge M(p^{i_0},\dots,v_{n-1}^{i_{n-1}}) \simeq *$ by Corollary 3.2, we get

$$\begin{aligned}\langle P_G L_n X\rangle &= \langle p^{-1}P_G L_n X\rangle \vee \langle v_1^{-1}P_G L_n X \wedge M(p^{i_0})\rangle \vee \cdots \\ &\quad \vee \langle v_{n-1}^{-1}P_G L_n \wedge M(p^{i_0},\dots,v_{n-2}^{i_{n-2}})\rangle.\end{aligned}$$

Since $P_G L_n X$ is an $L_n S^0$ module, it follows that

$$\begin{aligned}v_j^{-1}P_G L_n X \wedge M(p^{i_0},\dots,v_{j-1}^{i_{j-1}}) &= P_G L_n X \wedge v_j^{-1}L_n M(p^{i_0},\dots,v_{j-1}^{i_{j-1}}) \\ &= (P_G L_n X) \wedge L_j M(p^{i_0},\dots,v_{j-1}^{i_{j-1}}).\end{aligned}$$

Since $\langle L_j M(p^{i_0},\dots,v_{j-1}^{i_{j-1}})\rangle = \langle K(j)\rangle$, we see that

$$\langle P_G L_n X\rangle \leq \langle K(0)\rangle \vee \cdots \vee \langle K(n-1)\rangle = \langle E(n-1)\rangle. \tag{4}$$

Now since L_nX is a L_nS^0-module, P_GL_nX is a $P_GL_nS^0$ module. But $P_GL_nS^0$ is self local since it is a ring spectrum [14], so by equation (4) $P_GL_nS^0$ is $E(n-1)$-local, hence so is P_GL_nX.

To finish the proof of Theorem 1.1 it remains to show the inequality in equation (4) is actually an equality when $X = S^0$. In section 6 we use Theorem 1.2 to do this.

5. P_G Landweber exact v_n-periodic theories.

In this section we prove Theorem 1.2. We will suppose that E is a complex oriented homology theory. Without loss of generality, we assume that E is p-local, since we will be applying the functor $P_{Z/(p)}$ which gives the same value on X as on $X_{(p)}$. Then we can assume E is oriented by a map from BP, so that we can consider v_i as an element of E_*. We remind the reader that $I_j = (p, v_1, \dots, v_{j-1})$, and that for BP (and hence for any spectrum oriented from BP),

$$[p](x) = px +_F v_1x^p +_F v_2x^{p^2} +_F \cdots +_F v_ix^{p^i} +_F \cdots$$

where $+_F$ is the sum in the formal group law on E_*.

We begin by remarking that for complex oriented E

$$\pi_* P_{Z/(p)}E = E_*((x))/([p](x))$$

where $|x| = -2$, $E_*((x))$ denotes the ring of Laurent series over E_* which have only finitely many terms involving negative powers of x, and $[p](x)$ is the p-series. It follows that when $[p](x)$ is not a zero divisor, we have a short exact sequence

$$E_*((x)) \xrightarrow{\cdot [p](x)} E_*((x)) \to \pi_* P_{Z/(p)} E.$$

We now assume that E is v_n-periodic Landweber exact. We define v_n-periodic almost as in [5, Definition 1.3]; E is v_n-periodic if v_n is a unit on E_*/I_n and in addition $E_*/I_n \neq 0$.

For each $j \leq n$, we know that $E_*/I_j \to v_j^{-1} E_*/I_j$ is injective by the hypothesis of Landweber exactness. It follows that $E_*((x))/I_j \to v_j^{-1} E_*((x))/I_j$ is injective also. Now $[p](x)$ is a unit in $v_j^{-1} E_*((x))/I_j$ since it is a power series with leading term $v_j x^{p^j}$, which is a unit. It follows that

$$E_*((x))/I_j \to v_j^{-1} E_*((x))/I_j \xrightarrow{\cdot [p](x)} v_j^{-1} E_*((x))/I_j$$

is injective, hence

$$E_*((x))/I_j \xrightarrow{\cdot [p](x)} E_*((x))/I_j$$

is also.

We examine the diagram of short exact sequences below, in which the bottom row is the cokernel of the map between the top two rows.

$$\begin{tikzcd}
0 \arrow[r] & E_*((x))/I_j \arrow[r, "\cdot [p](x)"] \arrow[d, "\cdot v_j"'] & E_*((x))/I_j \arrow[r] \arrow[d, "\cdot v_j"] & (\pi_* P_{Z/(p)} E)/I_j \arrow[r] \arrow[d, "\cdot v_j"] & 0 \\
0 \arrow[r] & E_*((x))/I_j \arrow[r, "\cdot [p](x)"] \arrow[d] & E_*((x))/I_j \arrow[r] \arrow[d] & (\pi_* P_{Z/(p)} E)/I_j \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E_*((x))/I_{j+1} \arrow[r, "\cdot [p](x)"] & E_*((x))/I_{j+1} \arrow[r] & (\pi_* P_{Z/(p)} E)/I_{j+1} \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma applied to the first two rows (together with the observation that the first two vertical maps are injective) we see that v_j is injective on $(\pi_* P_{Z/(p)} E)/I_j$.

We can also see that $\cdot [p](x)$ is not a unit on $E_*((x))/I_j$ unless v_j is a unit on E_*, therefore $(\pi_* P_{Z/(p)} E)/I_j \neq 0$ unless $j = n$, and this last observation tells us that v_{n-1} is a unit on $(\pi_* P_{Z/(p)} E)/I_{n-1}$.

We conclude that $P_{Z/(p)} E$ is Landweber exact, and that $\pi_*(P_{Z/(p)} E)/I_{n-1} \neq 0$ while $\pi_*(P_{Z/(p)} E)/I_n = 0$. It follows by using (3) as before (see [7, Corollary 1.12], that

$$\langle P_{Z/(p)} E \rangle = \langle E(n-1) \rangle.$$

By using the maps of complex oriented ring spectra

$$E \to P_G E \to P_{Z/(p)} E$$

(when $Z/(p) \subseteq G$) we also deduce that

$$\langle P_G E \rangle = \langle E(n-1) \rangle.$$

6. Proof of Theorem 1.1.

We recall from section 4 that $v_j^{-1} P_G L_n S^0 \wedge M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}})$ has the Bousfield class of either a point or of $K(j)$. So to show it has the class of $K(j)$, we need only show that it is not contractible.

We pick $i_0, \dots, i_{j-1}$ so that $M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}})$ is a ring spectrum. Then we observe that the map

$$S^0 \to v_j^{-1} P_G L_n S^0 \wedge M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}}) \to v_j^{-1} P_{Z/(p)} L_n S^0 \wedge M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}}) \to$$

$$v_j^{-1} P_{Z/(p)} E(n) \wedge M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}}) \xrightarrow{=} v_j^{-1} P_{Z/(p)} E(n)/(p^{i_0}, \dots, v_{j-1}^{i_{j-1}}) \to$$

$$v_j^{-1} P_{Z/(p)} E(n)/(p, \dots, v_{j-1})$$

is the unit of the ring spectrum $v_j^{-1} P_{Z/(p)} E(n)/(p, \dots, v_{j-1})$. This is non-zero if $j < n$ by Theorem 1.2. So none of the intervening spectra are contractible either.

For arbitrary finite X (instead of S^0) just smash with X. Note that $\langle - \rangle$, $P_G(-)$ and localization commute with smashing with a finite spectrum.

Remark: The same proof can be iterated to draw the obvious conclusions about $P_G^k(L_n S^0)$.

References

1. J. F. Adams, J. H. Gunarwardena, and H. R. Miller. The Segal conjecture for elementary abelian p-groups. *Topology*, 24:435–460, 1985.
2. A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18:257–281, 1979.
3. E. S. Devinatz. Small ring spectra. *Journal of Pure and Applied Algebra*, 81:11–16, 1992.
4. J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. To appear.
5. J. P. C. Greenlees and H. Sadofsky. The Tate spectrum of v_n-periodic complex oriented theories. submitted.
6. M. J. Hopkins and J. H. Smith. Nilpotence and stable homotopy theory II. To appear.
7. M. Hovey. Bousfield localization functors and Hopkins' chromatic splitting conjecture. In *Proceedings of the Čech centennial homotopy theory conference, 1993*, 1993. to appear.
8. C.-N. Lee. Bousfield equivalence classes and the Lin tower. preprint, 1994.
9. L. G. Lewis, J. P. May, and M. Steinberger. *Equivariant Stable Homotopy Theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer–Verlag, New York, 1986.
10. W. H. Lin. On conjectures of Mahowald, Segal and Sullivan. *Proc. Cambridge Phil. Soc.*, 87:449–458, 1980.
11. M. E. Mahowald. *The Metastable Homotopy of S^n*, volume 72 of *Memoirs of the American Mathematical Society*. American Mathematical Society, Providence, Rhode Island, 1967.
12. M. E. Mahowald and D. C. Ravenel. Toward a global understanding of the homotopy groups of spheres. In Samuel Gitler, editor, *The Lefschetz Centennial Conference: Proceedings on Algebraic Topology*, volume 58 II of *Contemporary Mathematics*, pages 57–74, Providence, Rhode Island, 1987. American Mathematical Society.
13. M. E. Mahowald and P. Shick. Root invariants and periodicity in stable homotopy theory. *Bull. London Math. Soc.*, 20:262–266, 1988.
14. D. C. Ravenel. Localization with respect to certain periodic homology theories. *American Journal of Mathematics*, 106:351–414, 1984.
15. D. C. Ravenel. The geometric realization of the chromatic resolution. In W. Browder, editor, *Algebraic topology and algebraic K–theory*, pages 168–179, 1987.
16. D. C. Ravenel. *Nilpotence and Periodicity in Stable Homotopy Theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, 1992.
17. H. Sadofsky. The root invariant and v_1-periodic families. *Topology*, 31:65–111, 1992.
18. R. Switzer. *Algebraic Topology — Homotopy and Homology*. Springer–Verlag, New York, 1975.

19. U. Würgler. Cobordism theories of unitary manifolds with singularities and formal group laws. *Mathematische Zeitschrift*, 150:239–260, 1976.

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